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VCE Mathematical Methods ¾
Application of Differentiation [0.12]
Workshop Solutions

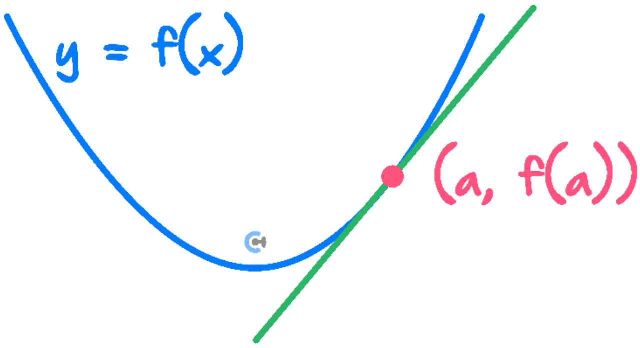
Error Logbook:

New Ideas/Concepts	Didn't Read Question
Pg / Q #: ____________ Notes:	Pg / Q #: ____________ Notes:
Algebraic/Arithmetic/ Calculator Input Mistake	**Working Out Not Detailed Enough**
Pg / Q #: ____________ Notes:	Pg / Q #: ____________ Notes:

Section A: Recap

Tangents

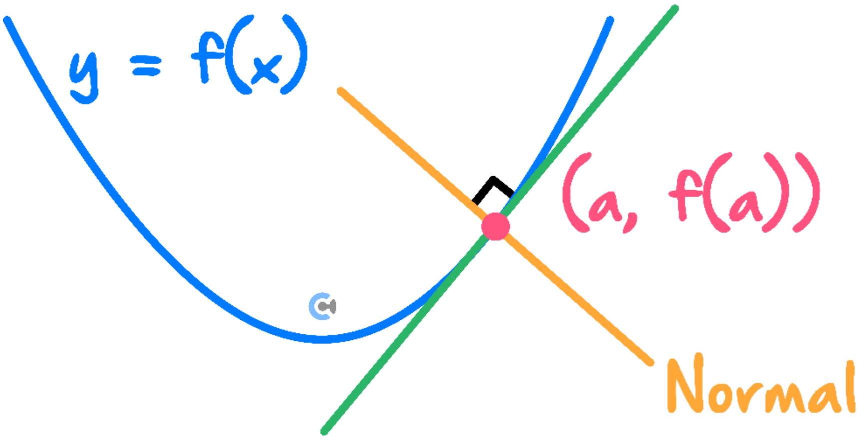
- A **tangent** is a linear line which **just touches** the curve.
- The gradient of a tangent line has to be equal to the gradient of the curve at the intersection.



$$m_{tangent} = f'(a)$$

Normals

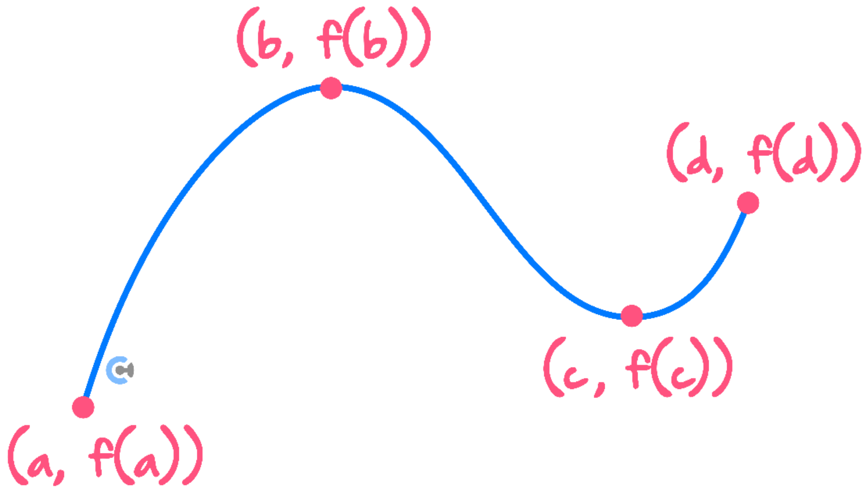
- A **normal** is a linear line which is **perpendicular** to the tangent.
- The gradient of a normal line has to be equal to the **negative reciprocal** of the gradient of the curve at the intersection.



$$m_{normal} = -\frac{1}{f'(a)}$$

Absolute Maximum and Minimum

- **Absolute Maxima/Minima** are the overall **largest/smallest** y-values for the given domain.
- **They occur at either an endpoint** or a **turning point.**



Absolute Min: $f(a)$

Absolute Max: $f(b)$

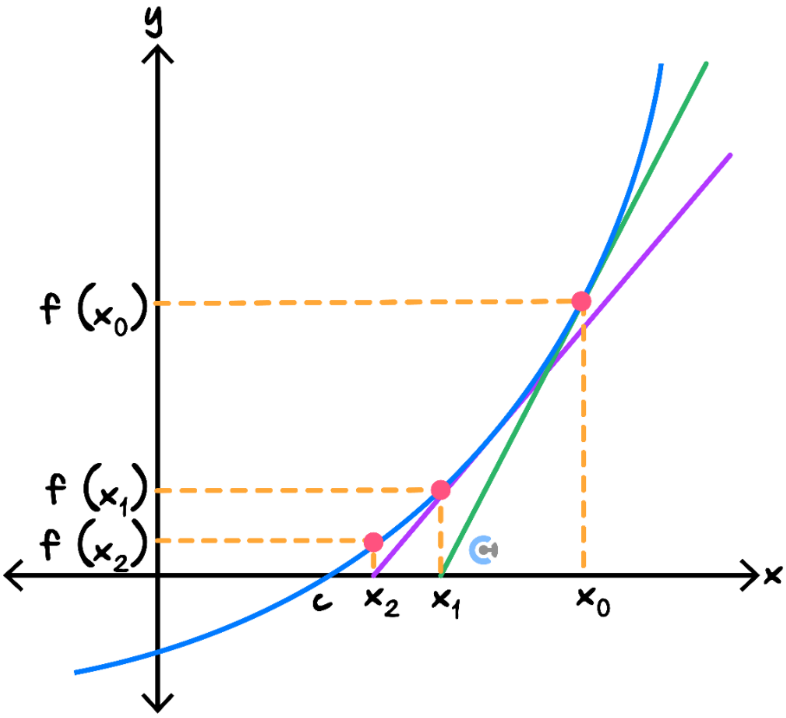
- **Steps**
  1. Find stationary points and endpoints.
  2. Find the largest/lowest y-value for max/min.

Optimisation Problems

- Applying absolute maxima and minima in a real-world setting.
- **Steps:**
  1. Construct a function for the subject you want to find the maximum or minimum of.
  2. Find its domain if appropriate.
  3. Find its endpoints and turning points.
  4. Identify the maximum or minimum y-value.

Newton's Method

- It is a method of approximating the x-intercept using **tangents.**



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- **Steps**

1. Find the tangent at the x-value given.
2. Find the x-intercept of the tangent using an iterative formula.
3. Find the next tangent at the $x = x$-intercept of the previous tangent.
4. Repeat until the value doesn't change by much.

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Definition

<u>Tolerance</u>

- **Definition**: The maximum difference between x_n and x_{n+1} we can have when we stop the iteration.

$$\textit{We stop when } |x_{n+1} - x_n| < \textit{tolerance}.$$

- The question will give us the tolerance level.

Definition

<u>Limitation of Newton's Method</u>

- **Terminating Sequence**: Occurs when we hit a stationary point.
- **Approximating a Wrong Root**: Occurs when we start on the wrong side.
- **Oscillating Sequence**: Occurs when we oscillate between two values without getting closer to the real root.

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Section B: Warm Up (15 Marks)

INSTRUCTION:

- Regular: 15 Marks. 15 Minutes Writing.
- Extension: Skip

Question 1 (2 marks)

Find the equation of the line that is normal to $y = x^3 - 2x^2 + 2x + 1$ at $x = 1$.

```
In[9]:=  f[x_] := x^3 - 2 x^2 + 2 x + 1

In[10]:= NormalLine[f[x], {x, 1}]

Out[10]= 3 - x
```

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Question 2 (3 marks)

Consider the function $f: (1, \infty) \to R, f(x) = \frac{2}{x-1}$. Find the equation of the line tangent to f that is parallel to $y = 4 - 2x$.

```
In[13]:= f[x_] := 2/(x - 1)

In[14]:= f'[x]

Out[14]= -2/(-1 + x)^2

In[15]:= Solve[f'[x] == -2, x]

Out[15]= {{x → 0}, {x → 2}}

In[12]:= TangentLine[f[x], {x, 2}]

Out[12]= 6 - 2 x
```

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Question 3 (3 marks)

Find the global maximum and global minimum of the function $f: [-4,0] \to R, f(x) = x^3 + 3x^2 - 9x - 10$.

Global maximum $= 17$ and global minimum -10.

```
In[20]:= f[x_] := x^3 + 3 x^2 - 9 x - 10
In[21]:= f'[x]
Out[21]= -9 + 6 x + 3 x^2
In[22]:= Solve[f'[x] == 0, x]
Out[22]= {{x → -3}, {x → 1}}
In[23]:= f[-4]
Out[23]= 10
In[24]:= f[-3]
Out[24]= 17
In[25]:= f[0]
Out[25]= -10
```

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Question 4 (3 marks)

James is building a rectangular fence around a garden bed. Find the maximum area of the garden bed that he can enclose with 28 metres of fencing.

Let x be the width and y be the length. Then

$$2x + 2y = 28 \Rightarrow y = 14 - x$$

Then, area is given by $A(x) = x(14 - x)$

$$A'(x) = 14 - 2x = 0$$

$$x = 7$$

$$A(7) = 49$$

Maximum area of 49 square metres.

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Question 5 (4 marks)

Consider the function $f(x) = x^2 - 3$. Newton's method is used to approximate the root of this function.

a. Using Newton's method, an expression for x_{n+1} is:

$$x_{n+1} = \frac{x_n^2 + a}{bx_n}$$

Find the values of a and b. (2 marks)

$a = 3, b = 2$

```
In[26]:= f[x_] := x^2 - 3
In[27]:= x - f[x]/f'[x] // Together
Out[27]= (3 + x^2)/(2 x)
```

b. Explain why $x_0 = 0$ will be a bad starting point. (1 mark)

$x = 0$ is a stationary point of f so we get a division by zero, so Newton's method fails.

c. Find x_1 if $x_0 = 2$. (1 mark)

$$\frac{7}{4}$$

Space for Personal Notes

Section C: Exam 1 Questions (17 Marks)

INSTRUCTION:

- Regular: 17 Marks. 5 Minutes Reading. 25 Minutes Writing.
- Extension: 17 Marks. 5 Minutes Reading. 17 Minutes Writing.

Question 6 (4 marks)

Consider the two functions below.

$$f: [-10,3] \to R, f(x) = (x+1)^2 + 5$$

$$g: [-4,10] \to R, g(x) = -2x - 10$$

a. Find the minimum vertical distance between $f(x)$ and $g(x)$. (3 marks)

$$h(x) = f(x) - g(x) = x^2 + 4x + 16$$

$$h'(x) = 2x + 4 = 0 \Rightarrow 2x + 4 = 0 \Rightarrow x = -2$$

$$h(-2) = 12$$

So minimum vertical distance of 12 when $x = -2$.

b. Find the maximum vertical distance between $f(x)$ and $g(x)$. (1 mark)

Check endpoints: $h(-4) = 16, h(3) = 37$

So maximum distance is 37.

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Question 7 (5 marks)

Consider a function $f(x) = \sqrt{x}$.

a. Find the tangent of $f(x)$ at $x = 2$. (2 marks)

$$\text{tangentLine}\left(\sqrt{x}, x, 2\right) \quad \frac{\sqrt{2} \cdot x}{4} + \frac{\sqrt{2}}{2}$$

b. Find the tangent of $f(x)$ which makes the angle of $30°$ measured anticlockwise from the positive side of the x-axis. (3 marks)

$$\text{solve}\left(\frac{d}{dx}\left(\sqrt{x}\right) = \tan(30), x\right) \quad x = \frac{3}{4}$$

$$\text{tangentLine}\left(\sqrt{x}, x, \frac{3}{4}\right) \quad \frac{\sqrt{3} \cdot x}{3} + \frac{\sqrt{3}}{4}$$

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Question 8 (2 marks)

Consider the equation $x^3 = \sin(\pi x)$.

Using Newton's method with $x_0 = 1$, find the value of x_1.

$$f(x) = x^3 - \sin(\pi x)$$
$$f'(x) = 3x^2 - \pi\cos(\pi x)$$
$$f(1) = 1 \text{ and } f'(1) = 3 + \pi$$

Therefore, $x_1 = 1 - \frac{1}{3+\pi} = \frac{2+\pi}{3+\pi}$

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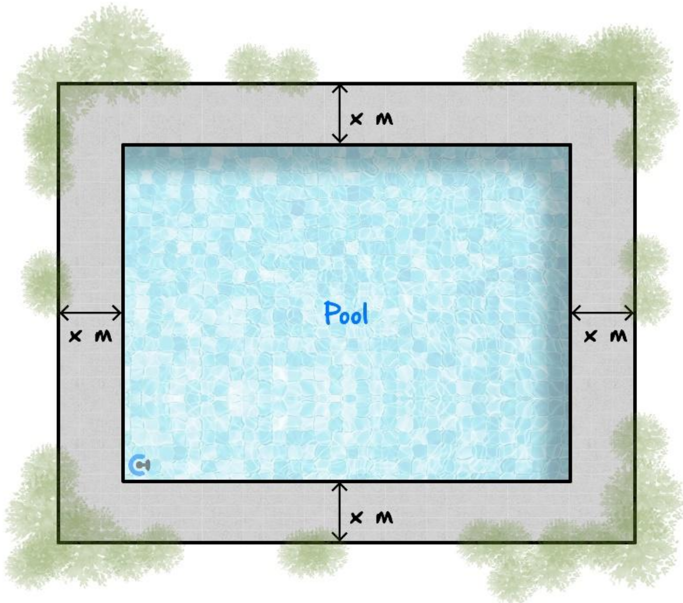
Question 9 (6 marks)

Subu has a rectangular garden. It is 14 metres long and 11 metres wide. He wants to put a rectangular swimming pool in the middle of the garden and a path of width x metres around the edge, as shown below.



a. Show that an expression for the length of the diagonal of the pool in terms of x is $\sqrt{8x^2 - 100x + 317}$. (2 marks)

Let d be the diagonal length of the pool and w and l be its side lengths.
$w = 11 - 2x$ and $l = 14 - 2x$

$$d = \sqrt{w^2 + l^2}$$
$$= \sqrt{121 - 44x + 4x^2 + 196 - 56x + 4x^2}$$
$$= \sqrt{8x^2 - 100x + 317}$$

The pool construction worker says that the diagonal length must not be bigger than 15 metres.

b. For what value(s) of x will the condition be satisfied? (2 marks)

Note, $15^2 = 225$

Thus we require $8x^2 - 100x + 317 \leq 225$

$$8x^2 - 100x + 92 \leq 0$$
$$4x^2 - 50x + 46 \leq 0$$
$$2x^2 - 25x + 23 \leq 0$$
$$(2x - 23)(x - 1) \leq 0$$
$$1 \leq x \leq \frac{23}{2}$$

But we also have the restriction $x < \frac{11}{2}$.

Thus, $1 \leq x < \frac{11}{2}$.

c. Find the maximum possible area of the pool and the value of x for which this occurs. (2 marks)

$$A(x) = (11 - 2x)(14 - 2x)$$

Function we are maximising is a "smiley" parabola. So, maximum will occur at an endpoint.

$$A(1) = 9 \times 12 = 108$$
$$A\left(\frac{11}{2}\right) = 0$$

So maximum area of the pool is 108 square metres.

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Section D: Tech Active Exam Skills

Calculator Commands: Finding Derivatives

Mathematica	TI	Casio

`f'[x]`
 	
`D[f[x], x]`
	Shift Minus
	
	$\frac{d}{dx}(f(x))$
		Math 2
		
		$\frac{d}{dx}(f(x))$

Calculator Commands: Finding Second Derivatives

Mathematica	TI	Casio

`f''[x]`
 	
`D[f[x], {x, 2}]`
	Shift Minus
	
	$\frac{d^2}{dx^2}(f(x))$
		Math 2
		
		$\frac{d^2}{dx^2}(f(x))$

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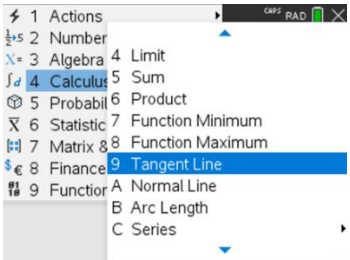
Calculator Commands: Finding tangents on CAS

➤ **Mathematica**

```
<< SuiteTools`
TangentLine[f[x], x, a]
```

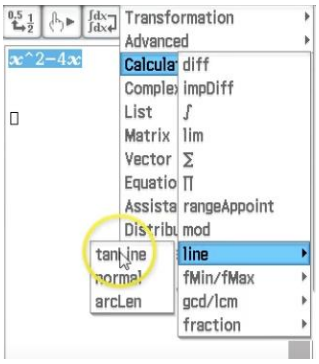
➤ **TI-Nspire**

Menu 4 9



tangentLine(f(x),x,a)

➤ **Casio Classpad**



tangentLine(f(x),x,a)

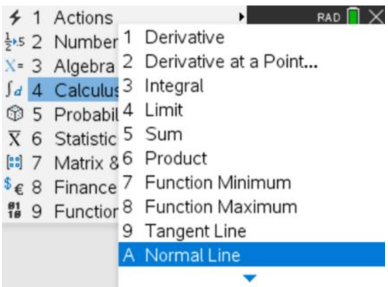
Calculator Commands: Finding normals on CAS

➤ **Mathematica**

```
<< SuiteTools`
NormalLine[f[x], x, a]
```

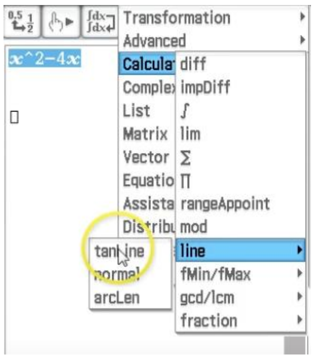
➤ **TI-Nspire**

Menu 4 A



normalLine(f(x),x,a)

➤ **Casio Classpad**



normalLine(f(x),x,a)

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Calculator Commands: Finding Absolute Max and Min for $x \in [a, b]$

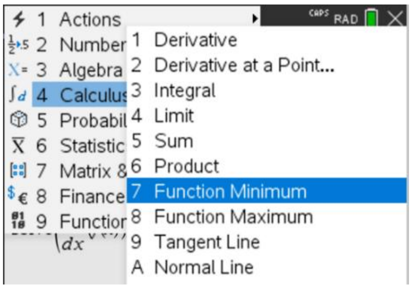
Mathematica

```
Maximize[{f[x], a ≤ x ≤ b}, x]

Minimize[{f[x], a ≤ x ≤ b}, x]
```

TI-Nspire

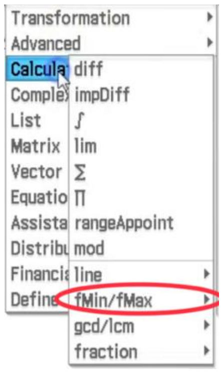
Menu 4 7 and Menu 4 8



$$\mathrm{fMax}(f(x), x, a, b)$$

$$\mathrm{fMin}(f(x), x, a, b)$$

Casio Classpad



$$\mathrm{fMax}(f(x), x, a, b)$$

$$\mathrm{fMin}(f(x), x, a, b)$$

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Calculator Commands: Newton's Method on Technology

- Consider finding a root to $f(x) = x^3 - 2$ with initial value $x_0 = 1$.
- **Mathematica.**

In[531]:= `f[x_] := x^3 - 2`

In[533]:= $n[x\_] := x - \dfrac{f[x]}{f'[x]}$

In[534]:= `n[x] // Together`

Out[534]= $\dfrac{2\left(1 + x^3\right)}{3x^2}$

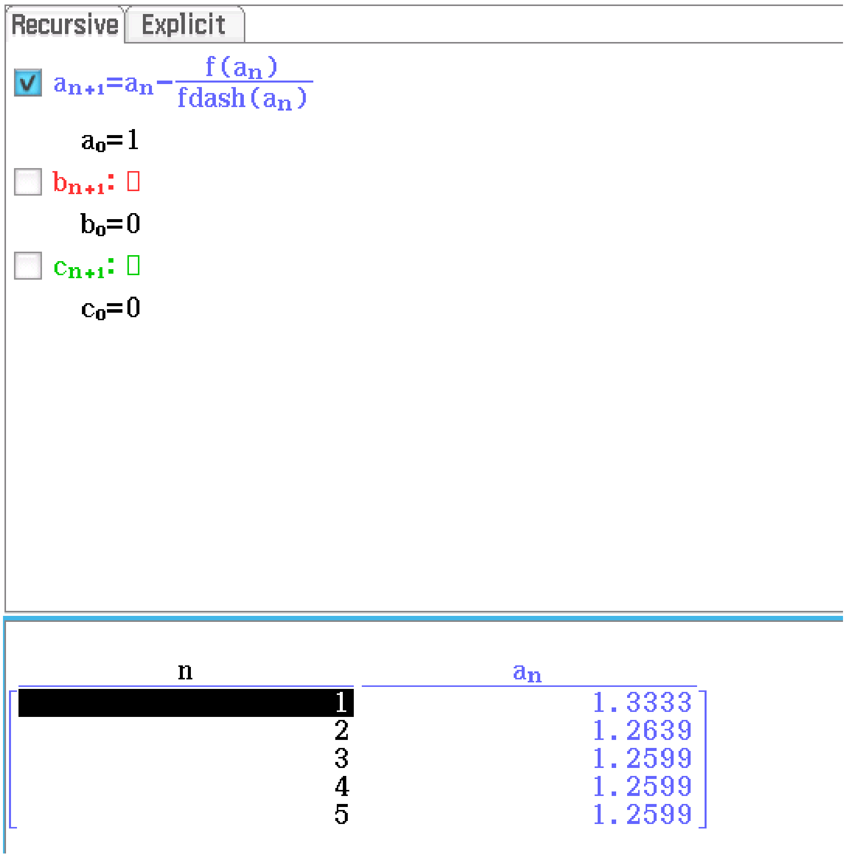
In[537]:= $\text{For}\left[i = 1;\ x = 1,\ i < 5,\ i{+}{+},\ x = \dfrac{2\left(1.0 + x^3\right)}{3x^2};\ \text{Print}[x]\right]$

```
1.33333
1.26389
1.25993
1.25992
```

- **TI.** Define the $n(x)$ function then keep iterating by putting your previous value back into $n(x)$.

Define $f(x)=x^3-2$	*Done*
$x-\dfrac{f(x)}{\frac{d}{dx}(f(x))}$	$\dfrac{2\cdot\left(x^3+1\right)}{3\cdot x^2}$
Define $n(x)=\dfrac{2\cdot\left(x^3+1\right)}{3\cdot x^2}$	*Done*
$n(1)$	1.33333
$n(1.3333333333333)$	1.26389
$n(1.2638888888889)$	1.25993

- **Classpad.**
  - Under Sequences.



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Calculator Commands: Stationary Point

- ALWAYS sketch the graph first to get an idea of the nature of the stationary point.
- The turning points for a function $f(x)$ can be found by solving $f'(x) = 0$ and subbing the result into f.
- **Example:** Find the turning point for $f(x) = e^{-x^2+2x}$.
- **TI:**

$$\text{Define } f(x) = e^{-x^2+2\cdot x} \qquad \textit{Done}$$

$$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right) \qquad x=1$$

$$f(1) \qquad e$$

- **Casio:**

$$\text{define } f(x) = e^{-x^2+2x} \qquad \text{done}$$

$$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right) \qquad \{x=1\}$$

$$f(1) \qquad e$$

- **Mathematica:**

```
In[4]:= f[x_] := Exp[-x^2 + 2 x]
In[5]:= Solve[f'[x] == 0 && y == f[x], Reals]
Out[5]= {{x → 1, y → e}}
```

Section E: Exam 2 Questions (22 Marks)

INSTRUCTION:

- Regular: 22 Marks. 5 Minutes Reading. 33 Minutes Writing.
- Extension: 22 Marks. 5 Minutes Reading. 22 Minutes Writing.

Question 10 (1 mark)

Consider a function $f(x) = \sin(x)$.

The tangent of $f(x)$ at $x = \frac{\pi}{3}$ is given by:

A. $y = \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

B. $y = \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

C. $y = \frac{1}{2}x + \frac{\sqrt{3}}{2}$

D. $y = \frac{1}{2}x + \frac{1}{2} + \frac{\pi}{3}$

Question 11 (1 mark)

Consider the graph of $f: R \to R, f(x) = -x^2 - x + 12$. Find the tangent to the graph of f which is parallel to the line connecting the negative x-intercept with the y-intercept.

A. $y = 3x + 12$

B. $y = 3x + 16$

C. $y = -4x + 3$

D. $y = -3x + 13$

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Question 12 (1 mark)

The minimum value of the function $h: [0,2] \to R, h(x) = (x-2)e^{2x}$ is:

A. $-e^3$

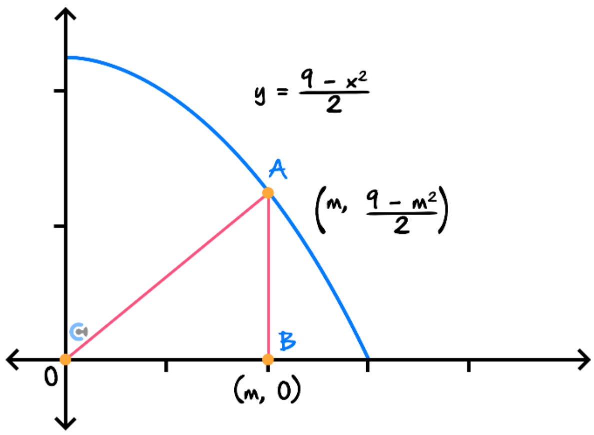
B. $-\frac{e^3}{2}$

C. $-e^2$

D. $-\frac{e^2}{2}$

Question 13 (1 mark)

A right-angled triangle, OAB, is formed using the horizontal axis and the point $A\left(m, \frac{9-m^2}{2}\right)$, where $m \in (0,3)$, on the parabola $y = \frac{9-x^2}{2}$, as shown below. The maximum area of the triangle OAB is:



A. $3\sqrt{3}$

B. $3\sqrt{\frac{3}{2}}$

C. $\frac{3\sqrt{3}}{2}$

D. $6\sqrt{3}$

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Question 14 (1 mark)

Consider an equation $\sin(x^2) - 1 = 0$.

Given that $x_0 = 2$, the value of x_3 correct to two decimal places using the Newton's method is equal to:

A. 1.33

B. 1.27

C. 1.26

D. 1.25

Question 15 (1 mark)

The normal line to the function $f(x) = x^2 - 4$ which goes through the origin could be:

A. $y = 2x - 3$

B. $y = x$

C. $y = -\frac{\sqrt{14}}{14}x$

D. $y = -\frac{\sqrt{2}}{6}x + 1$

Question 16 (1 mark)

The tangent to the graph of $y = 2x^3 - ax^2 + 4$ at $x = 2$ passes through the origin. The value of a is:

A. 5

B. -5

C. -7

D. 7

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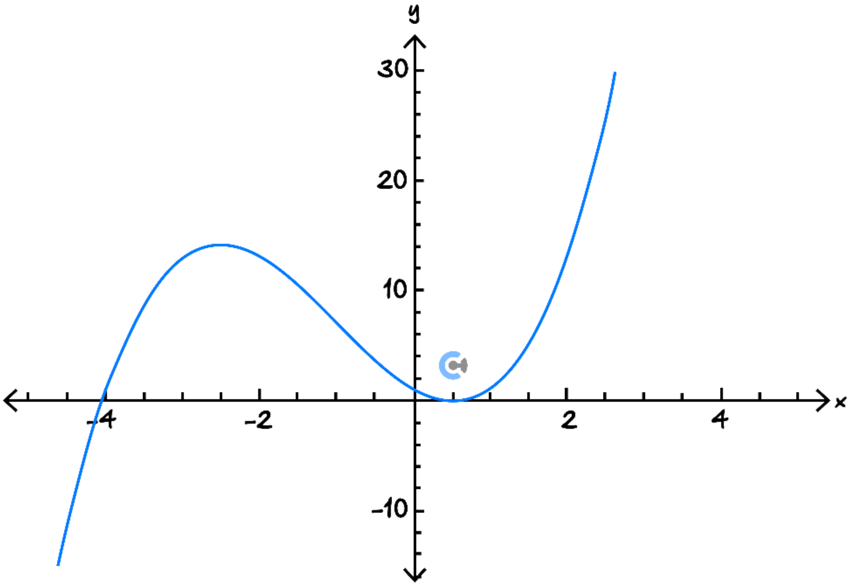
Question 17 (1 mark)

Part of the graph of a polynomial function f is shown below. The graph has turning points at $(-2.53, 14.13)$ and $(0.53, -0.13)$ and a point of inflection at $(-1, 7)$.



$f'(x)$ is strictly decreasing for:

A. $x \in (-\infty, -1]$

B. $x \in (-\infty, -1)$

C. $x \in [-2.53, 0.53]$

D. $x \in (-\infty, -2.53] \cup [0.53, \infty)$

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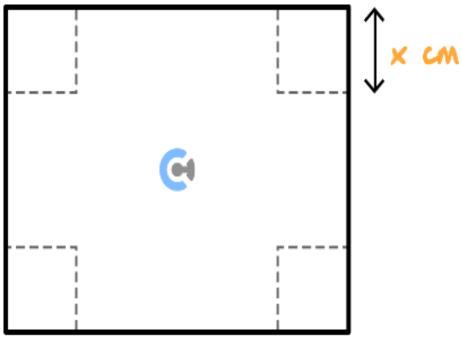
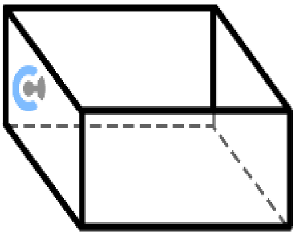
Question 18 (1 mark)

Alicia has a rectangular sheet of cardboard, 15 cm long and 8 cm wide. She cuts squares of side length x cm from each of the corners of the cardboard, as shown in the diagram below.



Alicia turns up the sides to form an open box as shown below.



The value of x for which the volume of the box is a maximum is:

A. $\frac{5}{3}$

B. 3

C. $\frac{5}{6}$

D. 4

Space for Personal Notes

Question 19 (13 marks)

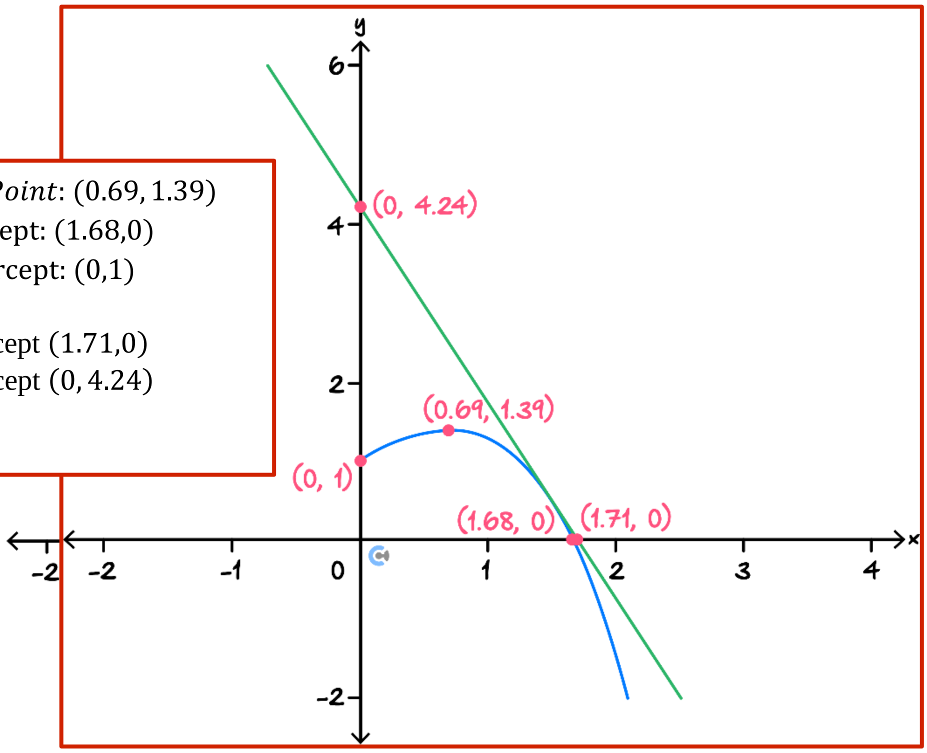
The function f is defined as follows:

$$f:[0,5] \to R, f(x) = -e^x + 2x + 2$$

a. Sketch the graph for $f(x)$ on the axes below. Label all the intercepts and stationary points correct to two decimal places. (2 marks)

$Stationary\ Point$: $(0.69, 1.39)$
x-intercept: (1.68,0)
y-intercept: (0,1)

Tangent: x-intercept (1.71,0)
Tangent: y-intercept $(0, 4.24)$



b. Find the tangent of the graph f at $x = \frac{3}{2}$. (2 marks)

```
Solve[y - f[3/2] == f'[3/2] * (x - 3/2), y] // Expand
풀이 함수                                      확장
```

$$\left\{\left\{y \to 2 + \frac{e^{3/2}}{2} + 2x - e^{3/2} x\right\}\right\}$$

c. Sketch the tangent found in **part b.**, on the set of axes given in **part a.** Label all the axes intercepts. (2 marks)

d. Newton's method is used to find an approximate x-intercept of f.

i. Find the value of x_1, if $x = \frac{3}{2}$. (1 mark)

```
x - f[x]/f'[x] /. x → 3/2 // N
1.70885
```

ii. State the possible value(s) of x_0 such that x_1 equals to the value found in **part d.** Give your answer correct to two decimal places. (2 marks)

```
n[x_] := x - f[x]/f'[x]
Solve[n[x] == n[3/2], x] // N
Solve: Inverse functions are being used by S
{{x → 1.89214}, {x → 1.5}}
```

iii. State an inappropriate choice for x_0, and explain why this choice is not appropriate for Newton's method. (1 mark)

$x = \log_e(2)$ because this is the x-coordinate for the stationary point of f.

e. A tangent is drawn to f at $x = a$, where $a \in [0,5]$. Find the intersection point of this tangent and the tangent drawn to f at $x = \frac{3}{2}$, if the two tangents make an angle of $60°$. Give your answer correct to two decimal places. (3 marks)

$(1.18, 1.32)$

```
In[39]:= f[x]
Out[39]= 2 - e^x + 2 x

In[41]:= TangentLine[f[x], {x, 3/2}] // Expand
Out[41]= 2 + e^(3/2)/2 + 2 x - e^(3/2) x

In[52]:= t[x_] := 2 + e^(3/2)/2 + 2 x - e^(3/2) x

In[53]:= Collect[t[x], x]
Out[53]= 2 + e^(3/2)/2 + (2 - e^(3/2)) x

In[60]:= Solve[Abs[(2 - e^(3/2) - m)/(1 + (2 - e^(3/2)) m)] == √3, m] // N // Quiet
Out[60]= {{m → -0.141484}, {m → 1.27751}}

In[61]:= Solve[f'[x] == -0.14148358512821801`, x] // Quiet
Out[61]= {{x → 0.761499}}

In[62]:= Solve[f'[x] == 1.2775057943142767`, x] // Quiet
Out[62]= {{x → -0.325046}}

In[57]:= TangentLine[f[x], {x, 0.761498852915832}]
Out[57]= 1.48925 - 0.141484 x

In[58]:= t1[x_] := 1.4892537084850035` - 0.1414835851282179` x

In[59]:= Solve[t[x] == t1[x] && y == t[x]]
Out[59]= {{x → 1.17579, y → 1.3229}}
```

Space for Personal Notes

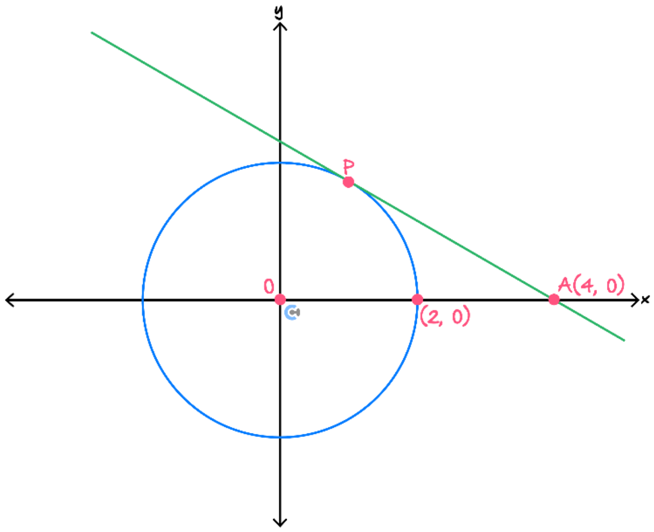
Section F: Extension Exam 1 (10 Marks)

INSTRUCTION:

- Regular: Skip
- Extension: 10 Marks. 5 Minutes Reading. 15 Minutes Writing.

Question 20 (10 marks)

Consider the circle C, with the equation $x^2 + y^2 = 4$ and the tangent to the circle at the point P, shown in the diagram below.



a. The top half of the circle $x^2 + y^2 = 4$, is given by the function $f : [-2,2] \to R, f(x) = \sqrt{4 - x^2}$.

Use this to show that the equation of the line that passes through the points A and P is given by $y = -\frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}}$. (2 marks)

Solution: $f'(x) = -\frac{x}{\sqrt{4-x^2}}$. For line to be tangent we require $f'(x) = -\frac{\sqrt{4-x^2}}{4-x}$.

Thus we solve

$$-\frac{x}{\sqrt{4-x^2}} = -\frac{\sqrt{4-x^2}}{4-x}$$
$$4 - x^2 = 4x - x^2$$
$$4x = 4$$
$$x = 1$$

$f'(1) = -\frac{1}{\sqrt{3}}$. So our line has gradient $-\frac{1}{\sqrt{3}}$ and passes through $(4, 0)$.

$$y - 0 = -\frac{1}{\sqrt{3}}(x - 4)$$
$$y = -\frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}}$$

b. Find the equations of two lines that are tangent to the circle C, and make an angle of $60°$ with the line passing through A and P. (2 marks)

Solution: We find that the required lines have gradient $m = \frac{1}{\sqrt{3}}$, through geometric intuition or solving $\left|\frac{-\frac{1}{\sqrt{3}} - m}{1 - \frac{m}{\sqrt{3}}}\right| = \tan(60°)$ for m.
One line can be obtained by reflecting line through A and P, in the y-axis.

$$y = \frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}}$$

then by the symmetry of the cirlce the other line is $y = \frac{1}{\sqrt{3}}x - \frac{4}{\sqrt{3}}$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x, qy)$, where $q \in \mathbb{R}\backslash\{0\}$, and let the graph of the function g be the transformation of the line that passes through points A and P under T.

c.

i. Find the values of q for which the graph of g intersects with the unit circle at least once. (1 mark)

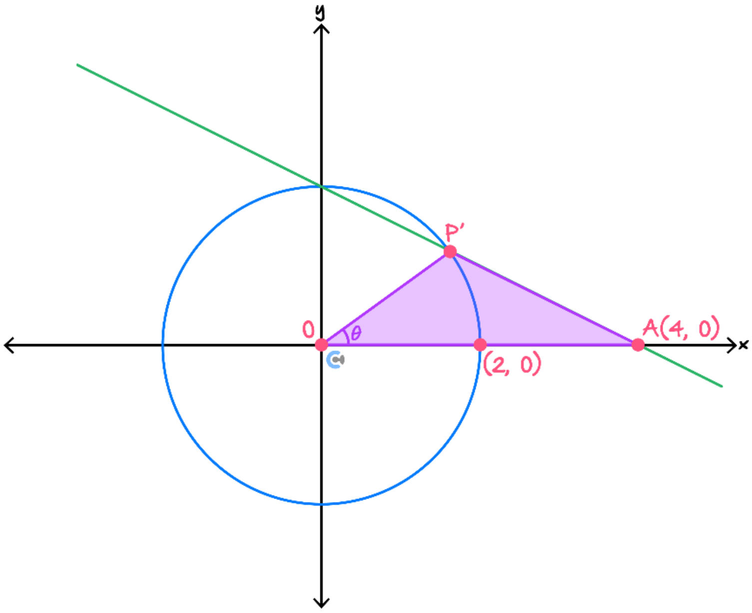
$q \in [-1, 0) \cup (0, 1]$

ii. Let the graph of g intersect the circle C twice.

Find the values of q for which the coordinates of the points of intersection have only positive values. (1 mark)

Solution: Know that $q < 1$ to intersect twice. Line through $(4, 0)$ and $(0, 2)$ is $y = -\frac{1}{2}x + 2 = \frac{\sqrt{3}}{2}\left(-\frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}}\right)$.
Thus $q \in \left(\frac{\sqrt{3}}{2}, 1\right)$

d. For $0 < q \leq 1$, let P' be the point of intersection of the graph of g with the circle C, where P' is always the point of intersection that is closest to A, as shown in the diagram below.



Let h be the function that gives the area of the triangle OAP' in terms of θ.

i. Define the function h. (2 marks)

$$h : \left(0, \frac{\pi}{3}\right] \to \mathbb{R}, h(\theta) = 4\sin(\theta)$$

ii. Determine the maximum possible area of the triangle OAP'. (2 marks)

Solution: Visually we see that the maximum will occur when $P' = P$, so $\theta = \frac{\pi}{3}$, then $A = 2\sqrt{3}$.

Section G: Extension Exam 2 (16 Marks)

INSTRUCTION:

- Regular: Skip
- Extension: 16 Marks. 5 Minutes Reading. 24 Minutes Writing.

Question 21 (1 mark)

A function $g(x)$ is differentiable for all $x \in \mathbb{R}$. The tangent line to $g(x)$ at $x = a$ is given by :

$$y = 2x - 3.$$

If $g(x) = g'(x)$ for all x, what is $g(3)$?

A. $3\sqrt{e}$

B. $\frac{3}{e^2}$

C. $2\sqrt{e}$

D. $\frac{2}{e}$

Question 22 (1 mark)

Let $h(x)$ be a differentiable function satisfying the equation:

$$h(h(x)) = x^2 + 2x.$$

Given that $h(1) = 3$ and $h'(1) = 2$, what is $h'(3)$?

A. 1

B. 2

C. $\frac{1}{4}$

D. $\frac{1}{2}$

Question 23 (1 mark)

A function $f(x)$ satisfies the equation:

$$f'(x) = f(x)(1 - f(x)).$$

If $f(0) = \frac{1}{2}$, which of the following statements is necessarily true?

A. $f(x)$ is always decreasing for all $x \in R$.

B. $f(x)$ is always concave up.

C. $f(x)$ has a horizontal asymptote as $x \to \infty$.

D. $f(x)$ has exactly two inflection points.

Question 24 (13 marks)

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 3x$.

Let $g_a : \mathbb{R} \to \mathbb{R}$ be the function representing the tangent to the graph of f at $x = a$, where $a \in \mathbb{R}$.

Let $(b, 0)$ be the x-intercept of the graph of g_a.

a. Show that $b = \frac{2a^3}{3(a^2-1)}$. (2 marks)

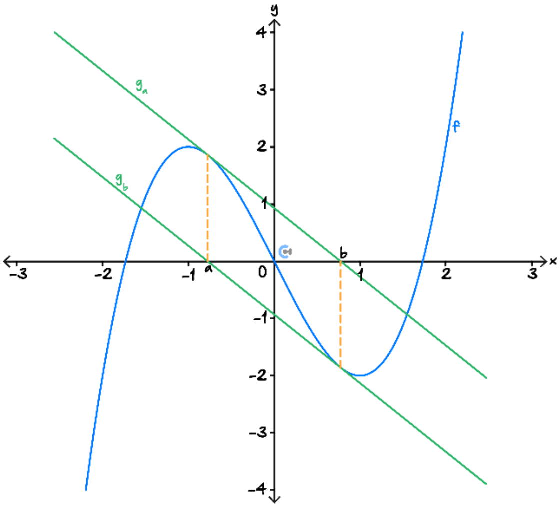
Solution: $f'(a) = 3a^2 - 3$. The tangent line at $x = a$ is $y = (3a^2 - 3)x - 2a^3$. Tangent has an x-intercept when $y = 0$. Solve $(3a^2 - 3)x - 2a^3 = 0 \implies x = \frac{2a^3}{3(a^2-1)}$.

So $b = \frac{2a^3}{3(a^2-1)}$

b. State the values of a for which b does not exist and state the nature of the graph of g_a at these points. (2 marks)

Solution: b does not exist at the stationary points of f. $a = \pm 1$. g_a is a horizontal line at these points.

The coordinates $(b, 0)$ are the horizontal axes intercept of g_a. Let g_b, be the function representing the tangent to the graph of f at $x = b$, as shown in the graph below.



c.

i. Find the values of a for which the graphs of g_a and g_b, where b exists, are parallel and where $b \neq a$. (3 marks)

Solution: We must solve $f'(a) = f'(b)$, make the subtstitution $b = \dfrac{2a^3}{3(a^2-1)}$ and solve on CAS.

$a = 0, \pm\dfrac{\sqrt{15}}{5}, \pm\sqrt{3}$.

Check the condition $b \neq a$ and conclude that $a = \pm\dfrac{\sqrt{15}}{5} = \pm\sqrt{\dfrac{3}{5}}$.

ii. State the values for x_0, which when used in Newton's method to find roots of f, will result in an oscillating sequence. (1 mark)

$$x_0 = \pm\frac{\sqrt{15}}{5}$$

iii. Let $p : \mathbb{R} \to \mathbb{R}, p(x) = x^3 - wx$, where $w > 0$. Newton's method is used to find the roots of p.

Find all initial guesses x_0, in terms of w, which will not converge to a root of p. (2 marks)

Solution: Tangent to p at $x = a$ has x intercept at $x = \dfrac{2a^3}{3a^2 - w}$.

We solve $p'(a) = p'\left(\dfrac{2a^3}{3a^2 - w}\right)$. Oscillating sequence for $x = \pm\sqrt{\dfrac{w}{5}}$.

Stationary point when $x = \pm\sqrt{\dfrac{w}{3}}$.

So no convergence if $x_0 = \pm\sqrt{\dfrac{w}{5}}, \pm\sqrt{\dfrac{w}{3}}$

d. Two parallel tangents are drawn to the graph of f. It is known that the **minimum** distance between the two tangent lines is $\frac{54}{\sqrt{241}}$. Determine possible **rational** x-values that the tangents are drawn at. (3 marks)

Solution: Note that f is an odd function, so parallel tangents occur at $x = \pm a$. We find the two tangent lines in terms of a. Find a line normal to the first tangent at the point $(0, -2a^3)$. Find when this normal line intersects the second tangent. Then solve for when the distance between these two points is $\frac{54}{\sqrt{241}}$.

The rational values that he tangents are drawn to are $x = -\frac{3}{2}$ and $x = \frac{3}{2}$.

In[110]:= TangentLine[f[x], {x, a}]

Out[110]= $-2a^3 + (-3 + 3a^2)\,x$

In[111]:= TangentLine[f[x], {x, -a}]

Out[111]= $2a^3 + (-3 + 3a^2)\,x$

In[109]:= $t1[x\_] := -2a^3 + (-3 + 3a^2)\,x$

In[112]:= $t2[x\_] := 2a^3 + (-3 + 3a^2)\,x$

In[113]:= t1[0]

Out[113]= $-2a^3$

In[114]:= Solve$\left[y - (-2\,a\text{^}3) == \frac{-1}{(-3 + 3a^2)}(x - 0), y\right]$

Out[114]= $\left\{\left\{y \to \frac{6a^3 - 6a^5 - x}{3(-1 + a^2)}\right\}\right\}$

In[115]:= $n1[x\_] := \frac{6a^3 - 6a^5 - x}{3(-1 + a^2)}$

In[116]:= Solve[n1[x] == t2[x] && y == t2[x]]

Out[116]= $\left\{\left\{x \to -\frac{12(-a^3 + a^5)}{10 - 18a^2 + 9a^4},\ y \to -\frac{2(8a^3 - 18a^5 + 9a^7)}{10 - 18a^2 + 9a^4}\right\}\right\}$

In[123]:= Solve$\left[\text{EuclideanDistance}\left[\{0, -2\,a\text{^}3\}, \left\{-\frac{12(-a^3 + a^5)}{10 - 18a^2 + 9a^4}, -\frac{2(8a^3 - 18a^5 + 9a^7)}{10 - 18a^2 + 9a^4}\right\}\right] == \frac{54}{\sqrt{241}}, a, \text{Reals}\right]$ // FullSimplify

Out[123]= $\left\{\left\{a \to -\frac{3}{2}\right\}, \left\{a \to \frac{3}{2}\right\}, \{a \to -0.962\ldots\}, \{a \to 0.962\ldots\}, \left\{a \to -3\sqrt{\frac{1}{241}(61 - \sqrt{1311})}\right\}, \left\{a \to 3\sqrt{\frac{1}{241}(61 - \sqrt{1311})}\right\}\right\}$

Space

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