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VCE Mathematical Methods ¾
Differentiation Exam Skills [0.11]
Workshop Solutions

Error Logbook:

New Ideas/Concepts	Didn't Read Question
Pg / Q #: ____________ Notes:	Pg / Q #: ____________ Notes:
Algebraic/Arithmetic/ Calculator Input Mistake	**Working Out Not Detailed Enough**
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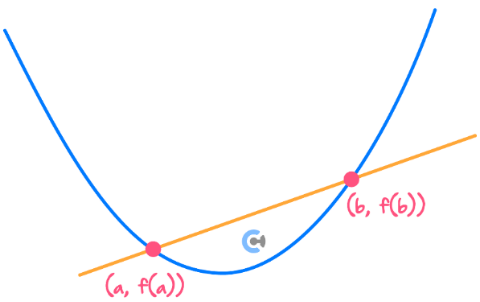
Section A: Recap

Average Rate of Change

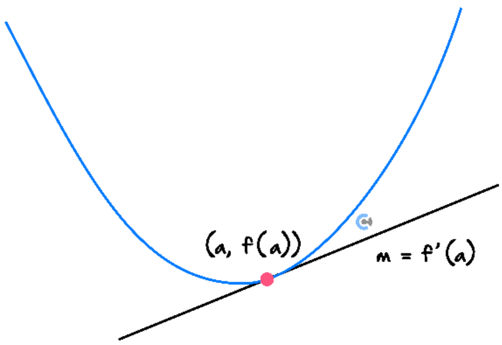


- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:

$$\textbf{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

- It is the gradient of the line joining the two points.

Instantaneous Rate of Change



- **Instantaneous rate of change** is a gradient of a graph at a single point/moment.

$$\textbf{Instantaneous rate of change} = f'(x)$$

- **Differentiation** is the process of finding the derivative of a function.
- **First principles** derivative definition:

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Definition

Alternative Notation for Derivative

$$f'(x) = \frac{dy}{dx}$$

Definition

Derivatives of Functions

- The derivatives of many of the standard functions are in the summary table below:

$f(x)$	$f'(x)$
x^n	$n \times x^{n-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
e^x	e^x
$\log_e(x)$	$\frac{1}{x}$

Definition

The Product Rule

- The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

- Or, in another form:

$$\frac{d}{dx}(u \cdot v) = u'v + v'u$$

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Definition

The Quotient Rule

- The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

- Or, written in another form:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

Always differentiate the top function first.

Definition

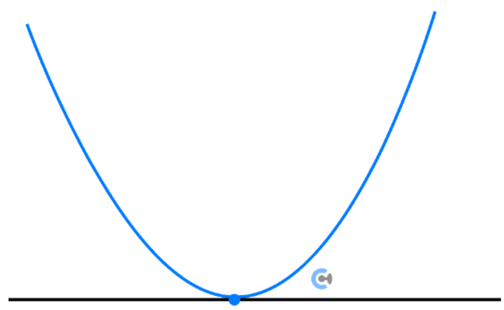
The Chain Rule

$$y = f\left(g(x)\right)$$

$$\frac{dy}{dx} = f'\left(g(x)\right)g'(x)$$

- The process for finding derivatives of **composite functions.**

Stationary Points



- The point where the gradient of the function is zero.

$$f'(x) = 0, \frac{dy}{dx} = 0$$

Calculator Commands: Finding Derivatives

- **Mathematica**

  `f'[x]`

- **TI**

  - Shift Minus

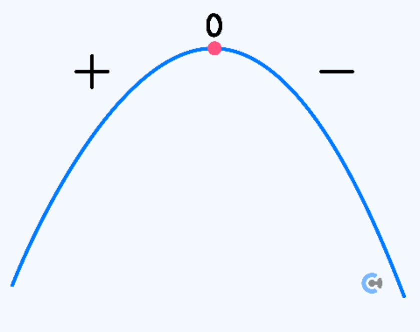
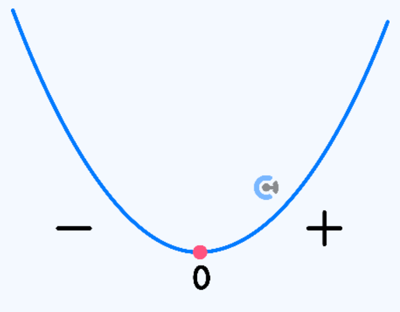
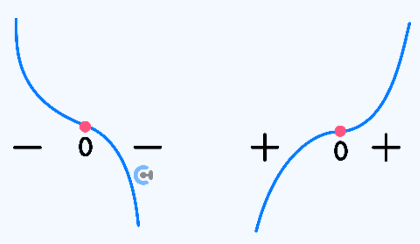
  $$\frac{d}{dx}(f(x))$$

- **Casio**

  - Math 2

  $$\frac{d}{dx}(f(x))$$

Types of Stationary Points

Local Maximum	Local Minimum	Stationary Point of Inflection
		

- Sign test.

- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩- Decreasing curve	Stationary Point	∪- Increasing curve

- Find the gradient of the neighbouring points.

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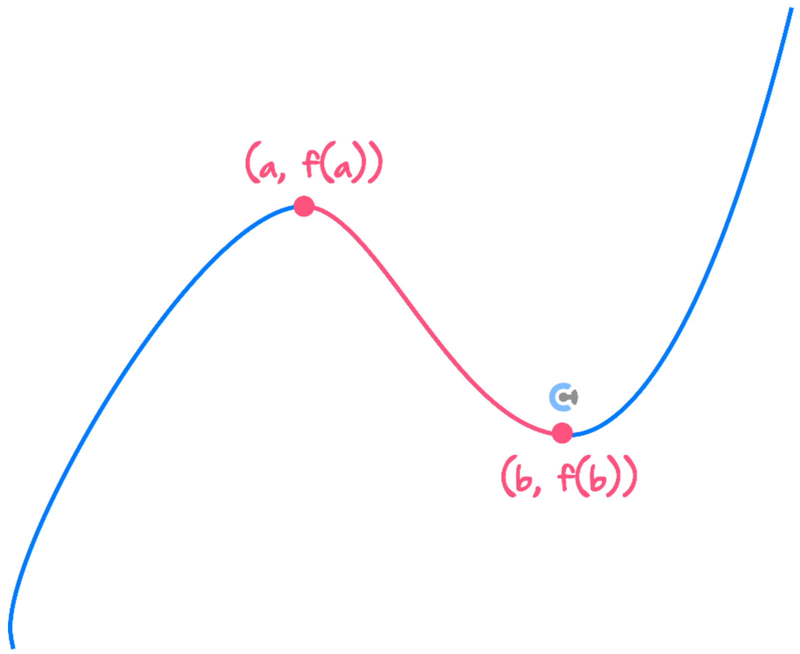
Strictly Increasing and Strictly Decreasing Functions



Strictly Increasing: $x \in (-\infty, a] \cup [b, \infty)$

Strictly Decreasing: $x \in [a, b]$

- Steps:
  1. Find the turning points.
  2. Consider the sign of the derivative between/outside the turning points.

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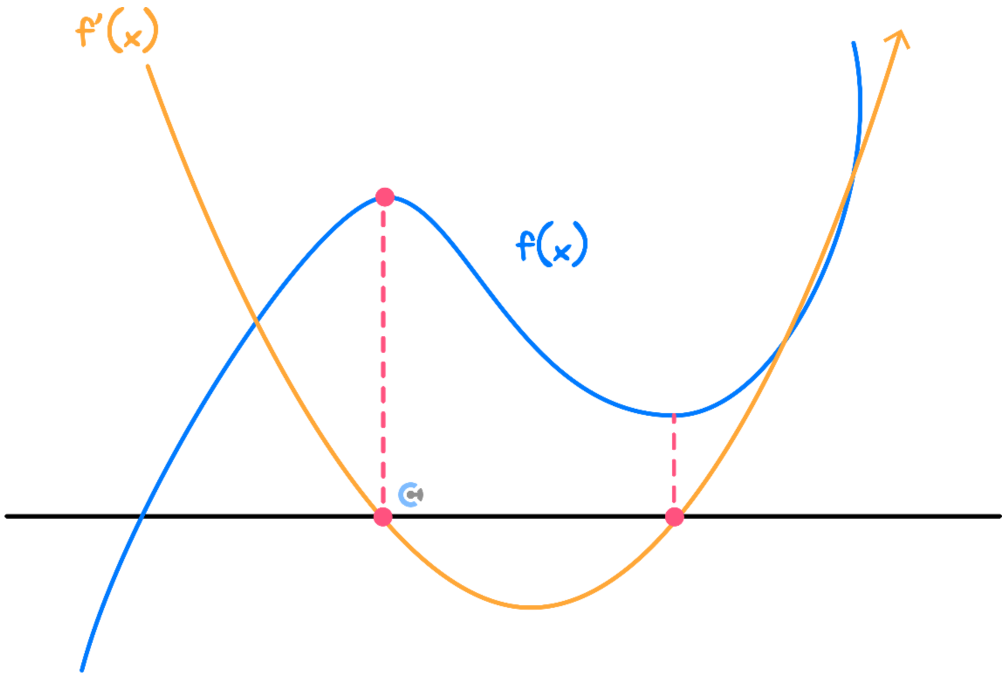
Graphs of the Derivative Function



$f(x)$	$f'(x)$
Stationary Point	x-intercepts
Increasing	Positive
Decreasing	Negative

$$y \text{ value of } f'(x) = \text{Gradient of } f(x)$$

- **Steps**

1. Plot x-intercept at the same x value as the stationary point of the original.
2. Consider the trend of the original function and sketch the derivative.
   - Original is increasing → Derivative is above the x-axis.
   - Original is decreasing → Derivative is below the x-axis.

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Limits

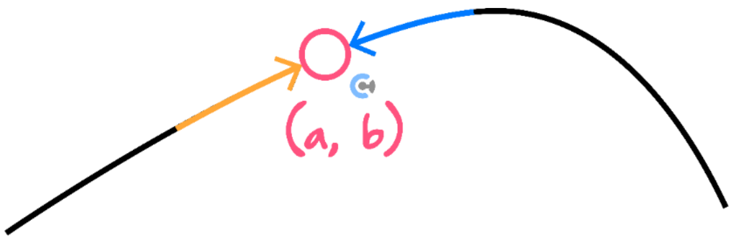
$$\lim_{x \to a} f(x) = L$$

"The function $f(x)$ approaches L as x approaches a."

- Limit is the value that a function (y-value) approaches as the x-value approaches a value.

Definition

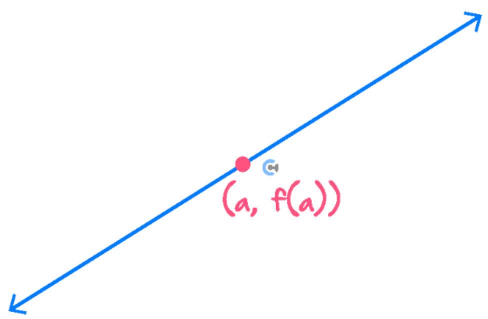
Validity of Limits



$$\lim_{x \to a} f(x) \text{ exists if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

- Limit is defined when the left limit equals the right limit.

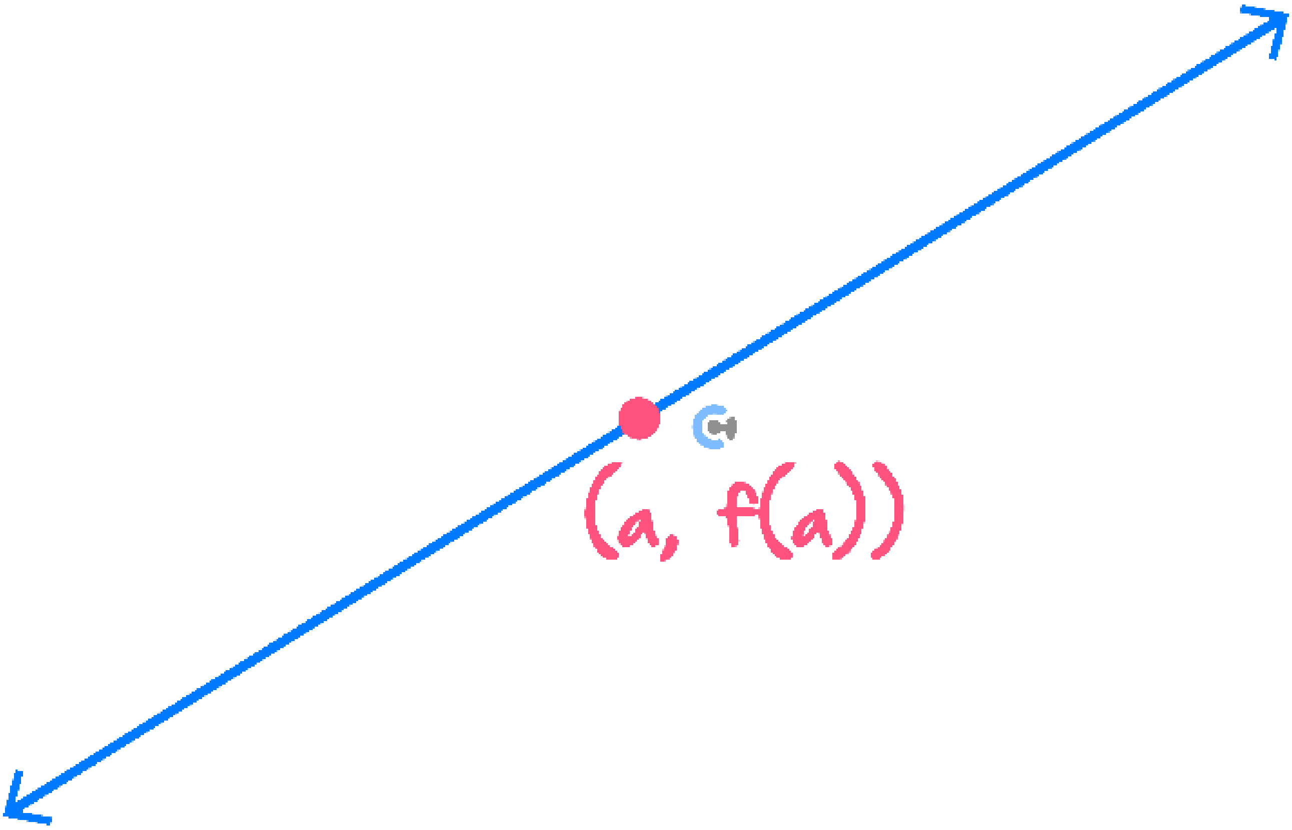
Continuity



- A function f is said to be continuous at a point $x = a$ if:

1. $f(x)$ is defined at $x = a$.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Differentiability



- **A function f is said to be differentiable at a point $x = a$ if:**

  1. **$f(x)$ is continuous at $x = a$.**

  2. **$\lim_{x \to a} f'(x)$ exists.**

     - Limit exists when the left and right limits are the same.

     - Gradient on the LHS and RHS must be the same.

- We **cannot** differentiate:

  1. Discontinuous points.

  2. Sharp points.

  3. Endpoints.

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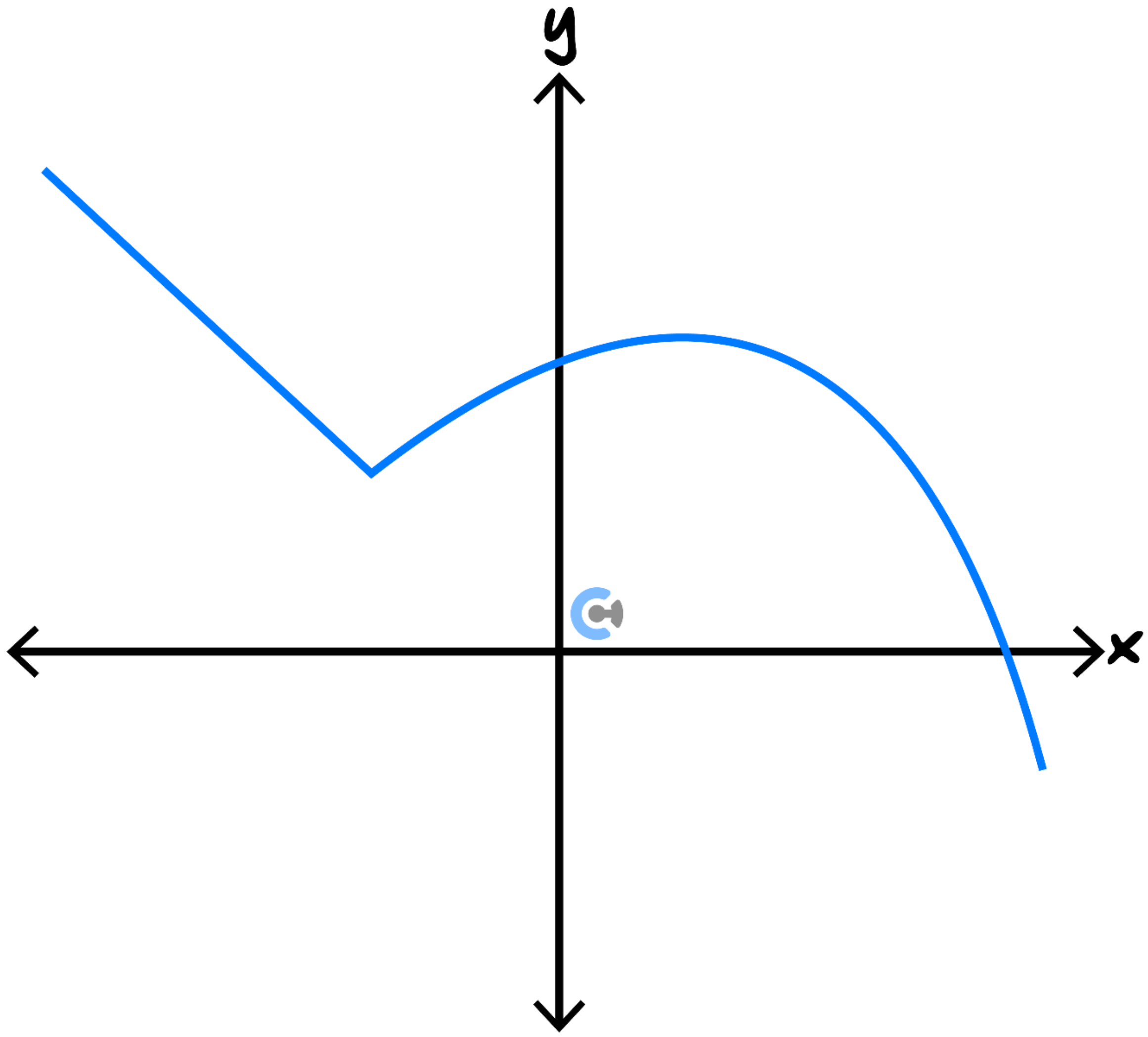
Definition

Finding the Derivative of Hybrid Functions



1. Simply derive each function.

2. Reject the values for x that are not differentiable from the domain.

Definition

Second Derivatives

- The derivative of the derivative.

- To get the second derivative, we can **differentiate the original function twice**.

$$\frac{d^2y}{dx^2} = f''(x)$$

Concavity

- Concave up is when the gradient is increasing.

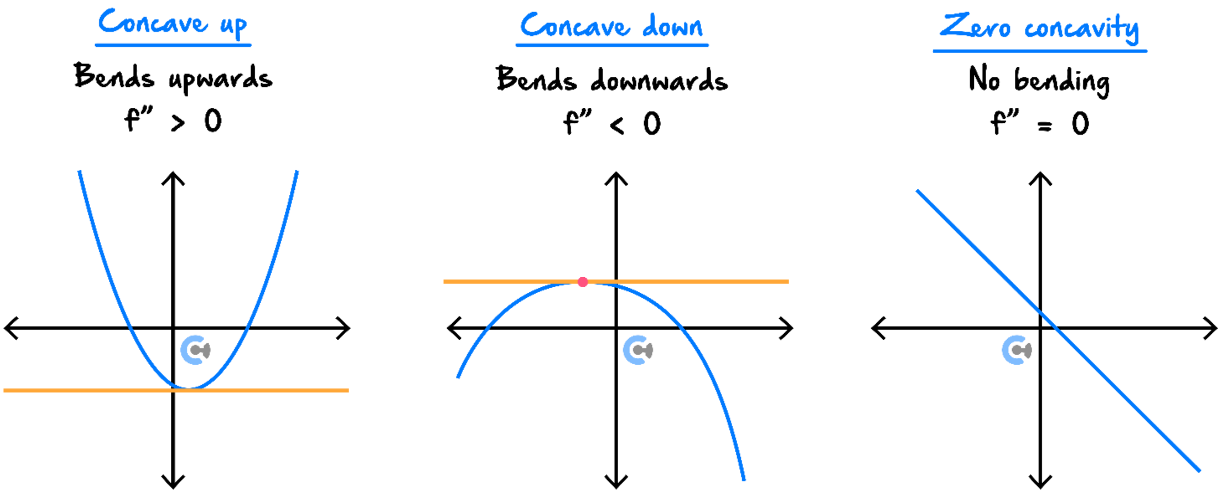
$$f''(x) > 0 \rightarrow \text{Concave Up}$$

- Concave down is when the gradient is decreasing.

$$f''(x) < 0 \rightarrow \text{Concave Down}$$

- Zero concavity is when the gradient is neither increasing nor decreasing.

$$f''(x) = 0 \rightarrow \text{Zero Concavity}$$



Concavity is also linked to how the curve is bent.

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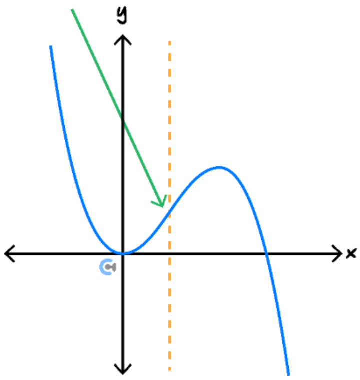
Points of Inflection

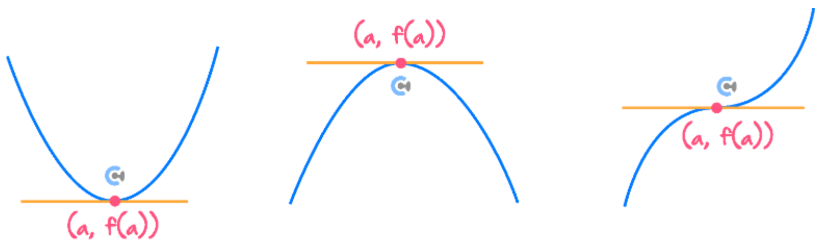
- A point at which a curve **changes concavity** is called a **point of inflection**.



$$f''(x) = 0$$

- Simply, it is when the bending changes.

The Second Derivative Test



- Suppose that $f'(a) = 0$ and hence, f has a stationary point at $x = a$. The second derivative test states:
  - **Concave up gives us a local minimum.**

$$f''(x) > 0 \rightarrow \text{Local Minimum}$$

  - **Concave down gives us a local maximum.**

$$f''(x) < 0 \rightarrow \text{Local Maximum}$$

  - **Zero concavity gives us a stationary point of inflection.**

$$f''(x) = 0 \rightarrow \text{Stationary Point of Inflection}$$

Definition

Joining Smoothly

- Let two different curves be defined as $f(x)$ and $g(x)$. For these two curves to join smoothly at $x = a$, they have to satisfy:
  - $f(a) = g(a)$
  - $f'(a) = g'(a)$
- In other words, the function must be **continuous** and **differentiable** at that point!

Definition

Steps for Finding Strictly Increasing/Decreasing Regions

1. Plot the graph on CAS.
2. Find stationary points.
3. Use a graph to determine which regions are increasing/decreasing.

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Section B: Warmup

Question 1

a. Let $g(x) = e^{\sin(x)}$. Find $g'(x)$.

Solution: $g'(x) = \cos(x)e^{\sin(x)}$

b. Let $h(x)$ be a differentiable function. Find the derivative of $x^2h(x)$, with respect to x.

Solution: Use the product rule.

$$\frac{d}{dx}\left(x^2h(x)\right) = 2xh(x) + x^2h'(x).$$

c. Consider the function f given by:

$$f(x) = \begin{cases} 2 - ax & x < 1 \\ ax^2 + bx + 4 & x \geq 1 \end{cases}$$

Find the integer values of a and b such that the graph of f joins smoothly at $x = 1$.

Solution: Let $g(x) = 2 - ax$ then $g'(x) = -a$. Also let $h(x) = ax^2 + bx + 4$, then $h'(x) = 2ax + b$.
Now $g(1) = h(1) \implies 2 - a = a + b + 4 \implies 2a + b = -2 \implies b = -2 - 2a$.
Also $g'(1) = h'(1) \implies -a = 2a + b \implies b = -3a$.
So we solve $-3a = -2 - 2a \implies a = 2$ then $b = -6$.
$a = 2, b = -6$.

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Section C: Exam 1 Questions (19 Marks)

INSTRUCTION:

- Regular: 19 Marks. 25 Minutes Writing.
- Extension: 19 Marks. 5 Minutes Reading. 19 Minutes Writing.

Question 2 (4 marks)

a. Let $y = \frac{\sin(x)}{x^2+4}$.

Find $\frac{dy}{dx}$. (2 marks)

Solution: $\frac{dy}{dx} = \frac{4\cos(x) + x^2\cos(x) - 2x\sin(x)}{(x^2+4)^2} = \frac{\cos(x)}{x^2+4} - \frac{2x\sin(x)}{(x^2+4)^2}$

b. Let $f(x) = x^2 e^{7x}$. Evaluate $f'(1)$. (2 marks)

Solution: $f'(x) = 2xe^{7x} + 7x^2e^{7x}$.
$f'(1) = 2e^7 + 7e^7 = 9e^7$

Question 3 (4 marks)

Let $f : [-\pi, \pi] \to \mathbb{R}, f(x) = \cos(2x) + 1$.

a. Calculate the average rate of change of f between $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{4}$. (2 marks)

Solution: $f\left(-\frac{\pi}{3}\right) = -\frac{1}{2} + 1 = \frac{1}{2}$ and $f\left(\frac{\pi}{4} = 1\right)$.

Average rate of change $= \dfrac{1 - 1/2}{\pi/4 + \pi/3} = \dfrac{1/2}{7\pi/12} = \dfrac{6}{7\pi}$

b. Find the angle that a tangent to f makes with the positive x-axis when $x = \frac{\pi}{3}$. (2 marks)

Solution: $f'(x) = -2\sin(2x) \implies f'\left(\frac{\pi}{3}\right) = -\sqrt{3}$.

Note that $\tan(120°) = -\sqrt{3}$. So angle is $120°$.

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Question 4 (7 marks)

Let $f : [-2, 1] \to \mathbb{R}, f(x) = (x+1)^2(x-1)$.

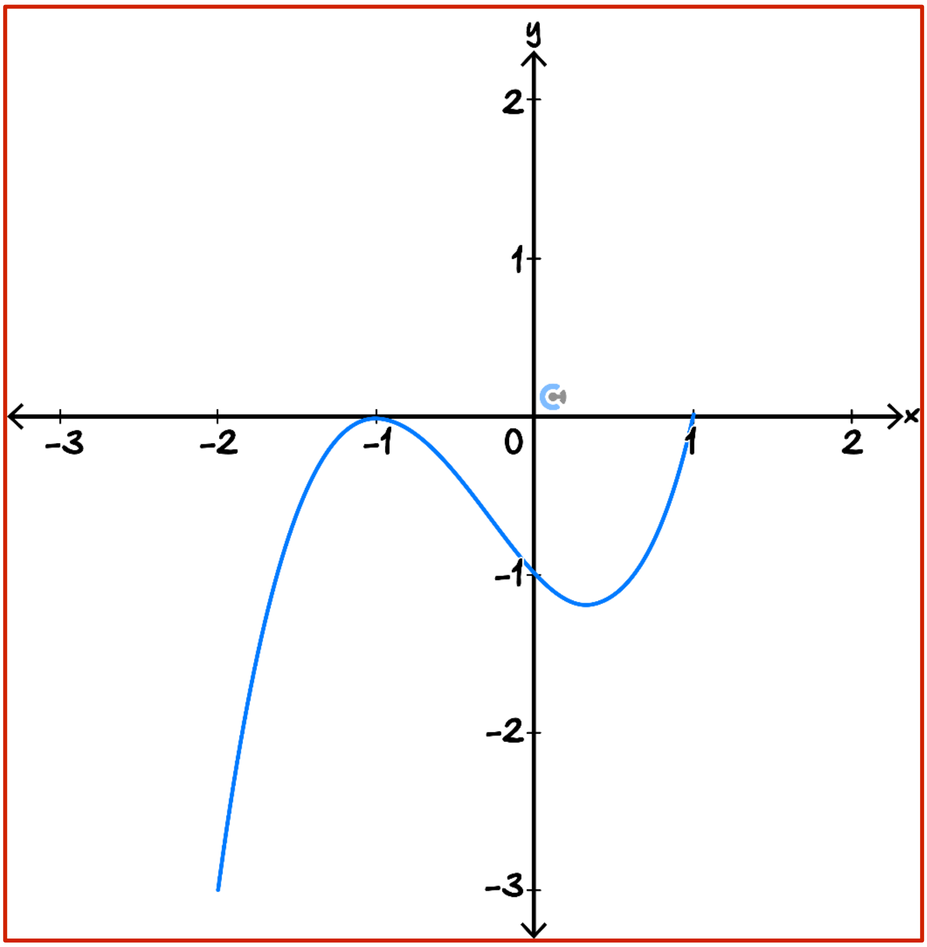
a. Show that $f(x) = x^3 + x^2 - x - 1$. (1 mark)

Solution: $f(x) = (x^2+2x+1)(x-1) = (x^2+2x+1)(x-1) = x^3+2x^2+x-x^2-2x-1 = x^3 + x^2 - x - 1$.

b. Find the x-values for which the graph of $y = f(x)$ has stationary points. (2 marks)

Solution: $f'(x) = 0 \implies 3x^2 + 2x - 1 = 0 \implies (3x-1)(x+1) = 0 \implies x = -1, \frac{1}{3}$.

c. Hence, sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts, stationary points and endpoints with coordinates. (2 marks)



d. The gradient of f at $x = a$ is equal to the average rate of change of f on the interval $x \in [-2,0]$. Determine the possible value(s) of a. (2 marks)

Solution: Average rate of change $= \frac{-1-(-3)}{2} = 1$.
So we solve $f'(x) = 1$.

$$3x^2 + 2x - 1 = 1$$
$$3x^2 + 2x - 2 = 0$$
$$x = \frac{-2 \pm \sqrt{4+24}}{6} = \frac{-2 \pm 2\sqrt{7}}{6} = \frac{-1 \pm \sqrt{7}}{3}$$

Therefore $a = \frac{-1 \pm \sqrt{7}}{3}$.

Question 5 (4 marks)

Consider the function h, where:

$$h(x) = \begin{cases} -x^2 + 2ax + 1 & x < 1 \\ a(x-b)^2 + a & x \geq 1 \end{cases}$$

Find the values of a and b such that the graph of $y = f(x)$ joins smoothly at $x = 1$.

Solution: Let $f(x) = -x^2 + 2ax + 1$ and let $g(x) = a(x-b)^2 + a$.
We have $f(1) = -1 + 2a + 1 = 2a$ and $g(1) = a(1-b)^2 + a$
Also $f'(x) = -2x + 2a \implies f'(1) = 2a - 2$ and $g'(x) = 2ax - 2abx \implies g'(1) = 2a - 2ab$.
Then $2a - 2 = 2a - 2ab \implies ab = 1 \implies b = \frac{1}{a}$.
Then we have

$$2a = a\left(1 - \frac{1}{a}\right)^2 + a$$
$$2a = a\left(1 - \frac{2}{a} + \frac{1}{a^2}\right) + a$$
$$2a = a - 2 + \frac{1}{a} + a$$
$$\frac{1}{a} = 2$$
$$a = \frac{1}{2}$$

So $a = \frac{1}{2}$ and $b = 2$.

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Section D: Tech Active Exam Skills

Calculator Commands: Finding Derivatives

Mathematica	TI	Casio
f'[x] D[f[x], x]	Shift Minus $\frac{d}{dx}(f(x))$	Math 2 $\frac{d}{dx}(f(x))$

Calculator Commands: Finding Second Derivatives

Mathematica	TI	Casio
f''[x] D[f[x], {x, 2}]	Shift Minus $\frac{d^2}{dx^2}(f(x))$	Math 2 $\frac{d^2}{dx^2}(f(x))$

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<u>Calculator Commands:</u> **Stationary Point**

- ALWAYS sketch the graph first to get an idea of the nature of the stationary point.
- The turning points for a function $f(x)$ can be found by solving $f'(x) = 0$ and substituting the result into f.
- **Example:** Find the turning point for $f(x) = e^{-x^2+2x}$.
- **TI:**

Define $f(x)=e^{-x^2+2\cdot x}$	*Done*
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$x=1$
$f(1)$	e

- **Casio:**

define f(x) = e^{-x^2+2x}	
	done
solve($\frac{d}{dx}$(f(x))=0, x)	
	{x=1}
f(1)	
	e

- **Mathematica:**

```
In[4]:= f[x_] := Exp[-x^2 + 2 x]

In[5]:= Solve[f'[x] == 0 && y == f[x], Reals]

Out[5]= {{x → 1, y → e}}
```

Calculator Commands: Using Sliders/Manipulate on CAS

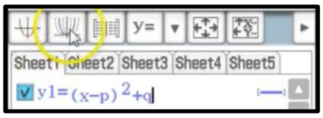
Mathematica

```
Manipulate[Plot[function, {x, xmin, xmax}],
    {unknown, lowerbound, upperbound}]
```

NOTE: The function **must** be typed out instead of using its saved name.

TI-Nspire

Casio Classpad



Calculator Commands: Joining Smoothly

Mathematica

```
f[x_] := One Function
g[x_] := Another Function
Solve[f[x value] == g[x value] && f'[x value] == g'[x value]]
```

TI and Casio

- Define each branch as $f(x)$ and $g(x)$.
- TI: Define its derivative as $df(x)$ and $dg(x)$

  Casio: Define them as different names
- Solve $f(a) = g(a)$ and $df(a) = dg(a)$ simultaneously.

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Section E: Exam 2 Questions (22 Marks)

INSTRUCTION:

- Regular: 22 Marks. 28 Minutes Writing.
- Extension: 22 Marks. 5 Minutes Reading. 22 Minutes Writing.

Question 6 (1 mark)

For the curve with equation $y = -x^3 - x^2 + 2x + 2$, the subset of $\mathbb{R}$ for which the gradient of the curve is positive is closest to:

A. $(-\infty, -1.215)$

B. $(-1.215, 0.548)$

C. $(0.548, \infty)$

D. $(-1.000, 1.414)$

Question 7 (1 mark)

If $y = \frac{\tan x}{x}$, then $\frac{dy}{dx}$ is:

A. $\frac{1}{\cos^2 x}$

B. $\frac{\tan x - \frac{x}{\cos^2 x}}{x^2}$

C. $\frac{\frac{x}{\cos^2 x} - \tan x}{x^2}$

D. $\frac{x}{\cos^2 x} - \frac{\tan x}{x}$

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Question 8 (1 mark)

Let $h(x) = g(x)e^{f(x^2)}$ be a differentiable function. Then $h'(x)$ is equal to:

A. $2e^{f(x^2)}g(x)f'(x^2) + e^{f(x^2)}g(x)$

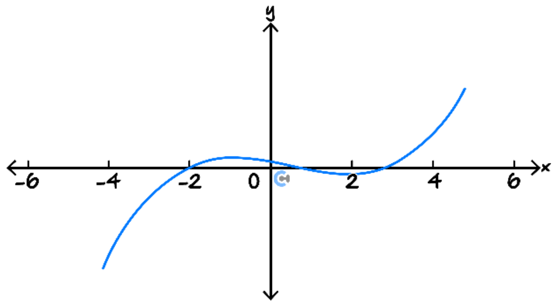
B. $2xe^{f(x^2)}g(x)f'(x^2) + e^{f(x^2)}g'(x)$

C. $2x^{2e^{f(x^2)}}g'(x)f'(x^2) + e^{f(x^2)}g(x)$

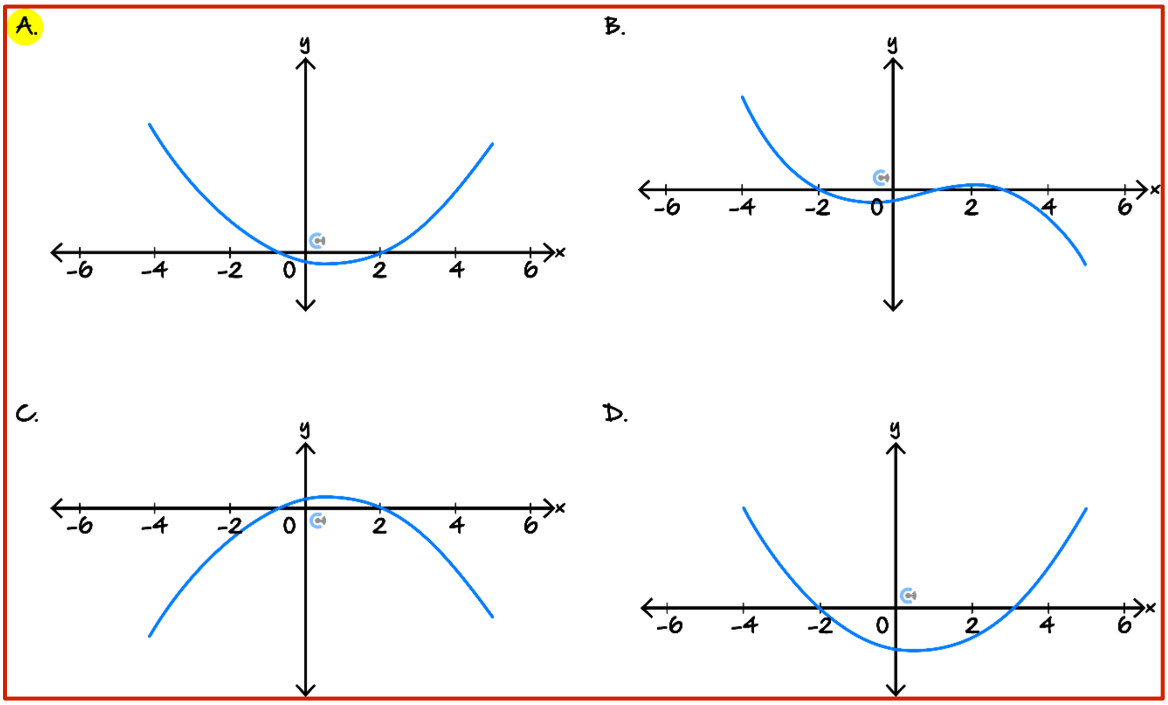
D. $x^2e^{f(x^2)}g(x)f'(x^2) + e^{f(x^2)}g'(x)$

Question 9 (1 mark)

The graph of the function with equation $y = f(x)$ is shown below:



Which one of the following is most likely to be the graph of the derivative function with equation $y = f'(x)$?



Question 10 (1 mark)

If $y = e^{-x} - 1$, then the rate of change of y with respect to x, when $x = 0$ is:

A. $-e$

B. -2

C. -1

D. 0

Question 11 (1 mark)

Let f be a one-to-one differentiable function such that $f(3) = 5, f(5) = 8, f'(3) = 2$, and $f'(5) = 3$.

The function g is differentiable and $g(x) = f^{-1}(x)$ for all x.

$g'(5)$ is equal to:

A. $\frac{1}{2}$

B. 2

C. $\frac{1}{8}$

D. $\frac{1}{3}$

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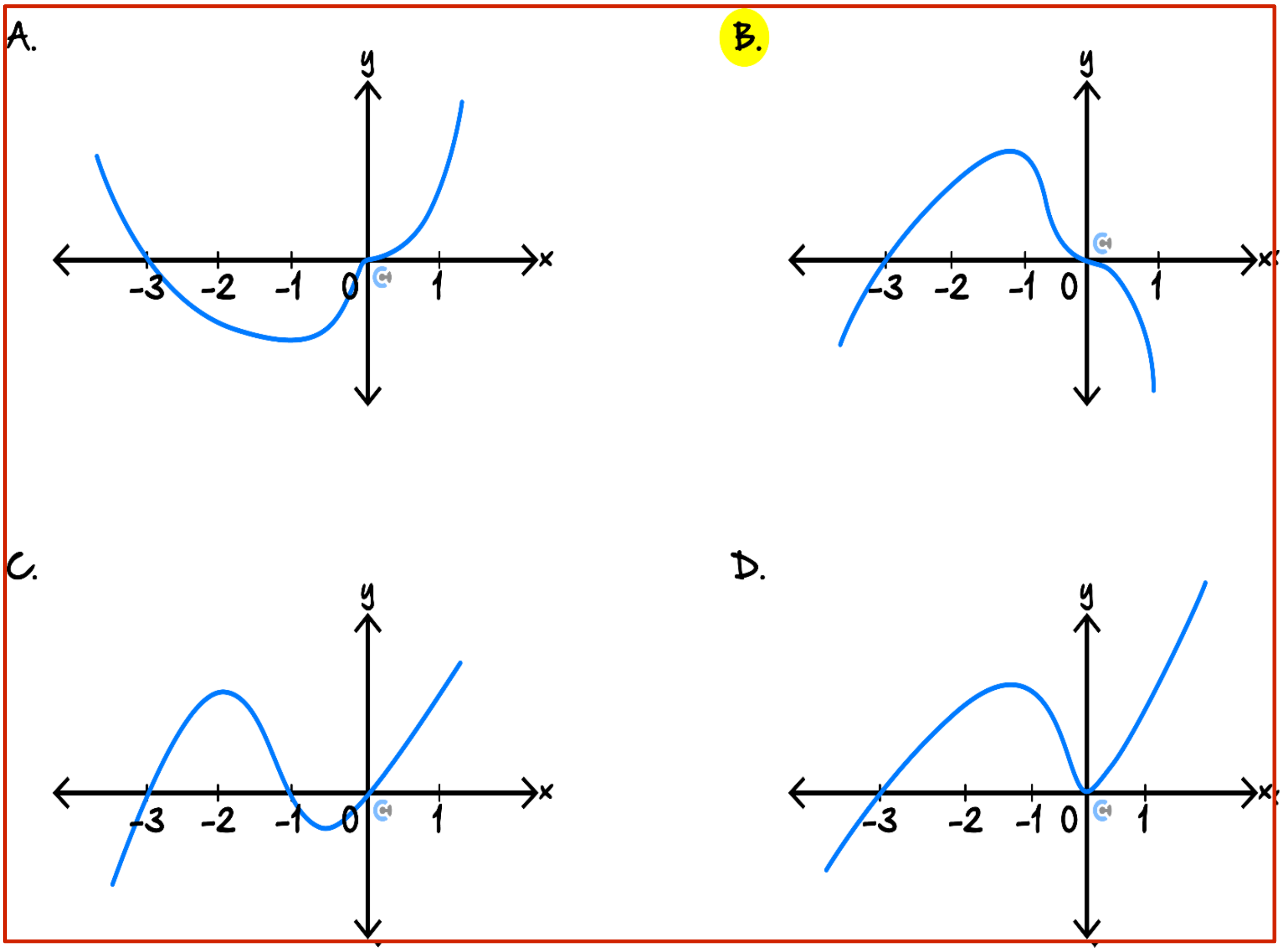
Question 12 (1 mark)

A continuous function f has the following properties:

$$f(0) = 0 \qquad f'(0) = 0$$
$$f(-3) = 0 \qquad f'(-1) = 0$$
$$f'(x) > 0 \text{ for } x \in (-\infty, -1)$$
$$f'(x) < 0 \text{ for } x \in (-1,0) \cup (0, \infty)$$

Which one of the following is most likely to represent the graph of f?



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Question 13 (15 marks)

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = (x-1)^2(x-4)$.

a. Find $f'(x)$. (1 mark)

Solution: $f'(x) = 3(x-3)(x-1) = 3x^2 - 12x + 9$

b. For what values of x is $f(x)$ strictly decreasing? (1 mark)

Solution: $x \in [1, 3]$

c.

i. Find the gradient of the line segment joining the points on the graph of $y = f(x)$ where $x = 0$ and $x = 4$. (1 mark)

Solution: $f(0) = -4$ and $f(4) = 0$. Gradient $= \dfrac{4+0}{4} = 1$.

ii. Show that the midpoint of the line segment in **part c.i.** also lies on the graph of $y = f(x)$. (2 marks)

Solution: Line has equation $y = x - 4$. Midpoint when $x = 2 \implies y = -2$.
Midpoint $(2, -2)$.
Then $f(2) = -2$ so the midpoint lies on the graph of $y = f(x)$

iii. Find the values of x for which the tangent to the graph of $y = f(x)$ is equal to the gradient of the line segment joining the points on the graph where $x = 0$ and $x = 4$. (2 marks)

Solution: We require $f'(x) = 1 \implies x = \frac{2}{3}(3 - \sqrt{3})$ or $x = \frac{2}{3}(3 + \sqrt{3})$

Let $g : \mathbb{R} \to \mathbb{R}, g(x) = (x-a)^2(x-4)$, where $a \in R$.

d.

i. State the value of a for which $g(x)$ has a stationary point of inflection. (1 mark)

Solution: $a = 4$.

ii. Find the coordinates for the stationary points of g, in terms of a. (2 marks)

Solution: We solve $g'(x) = 0 \implies x = a, \frac{a+8}{3}$.

Then $g(a) = 0$ and $g\left(\frac{a+8}{3}\right) = \frac{4}{27}(a-4)^3$.

So stationary points $(a, 0)$ and $\left(\frac{a+8}{3}, \frac{4}{27}(a-4)^3\right)$

e. Find the values of a for which the gradient of $g(x)$ when $x = \frac{10+a}{3}$ is negative. (1 mark)

Solution: We solve $g'\left(\frac{10+a}{3}\right) = -\frac{4}{3}(a-5) < 0 \implies a > 5$

f. Suppose the tangent to the graph of $y = g(x)$ at $x = \frac{10+a}{3}$ has a positive gradient.

i. Find the coordinates of another point where the tangent to the graph of $y = g(x)$ is parallel to the tangent at $x = \frac{10+a}{3}$. (2 marks)

Solution: We solve $g'\left(\frac{10+a}{3}\right) = g'(x) \implies x = \frac{3a-2}{3}$.
So coordinate $\left(\frac{3a-2}{3}, \frac{4}{27}(3a-14)\right)$

ii. Find the value(s) of a for which the points that these parallel tangents are drawn at have the same y value. (2 marks)

Solution: We solve $g\left(\frac{10+a}{3}\right) = \frac{4}{27}(3a-14)$.
This yields $a = 6, 3 \pm \sqrt{3}$. We reject $a = 6$ because $a < 5$ for tangent to be positive.
Thus $a = 3 \pm \sqrt{3}$

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Section F: Extension Exam 1 (9 Marks)

INSTRUCTION:

- Regular: Skip
- Extension: 9 Marks. 12 Minutes Writing.

Question 14 (9 marks)

Consider the function $f : [0,4] \to \mathbb{R}, f(x) = \frac{\sqrt{x}(4-x)}{2}$.

a.

i. For $x \in (0, 4)$, show that the gradient of the tangent to the graph of f is $\frac{4-3x}{4\sqrt{x}}$. (1 mark)

Solution: $f(x) = 2\sqrt{x} - \frac{1}{2}x^{3/2}$

$\implies f'(x) = \frac{1}{\sqrt{x}} - \frac{3}{4}x^{1/2} = \frac{4}{4\sqrt{x}} - \frac{3\sqrt{x} \times \sqrt{x}}{4\sqrt{x}} = \frac{4-3x}{4\sqrt{x}}$

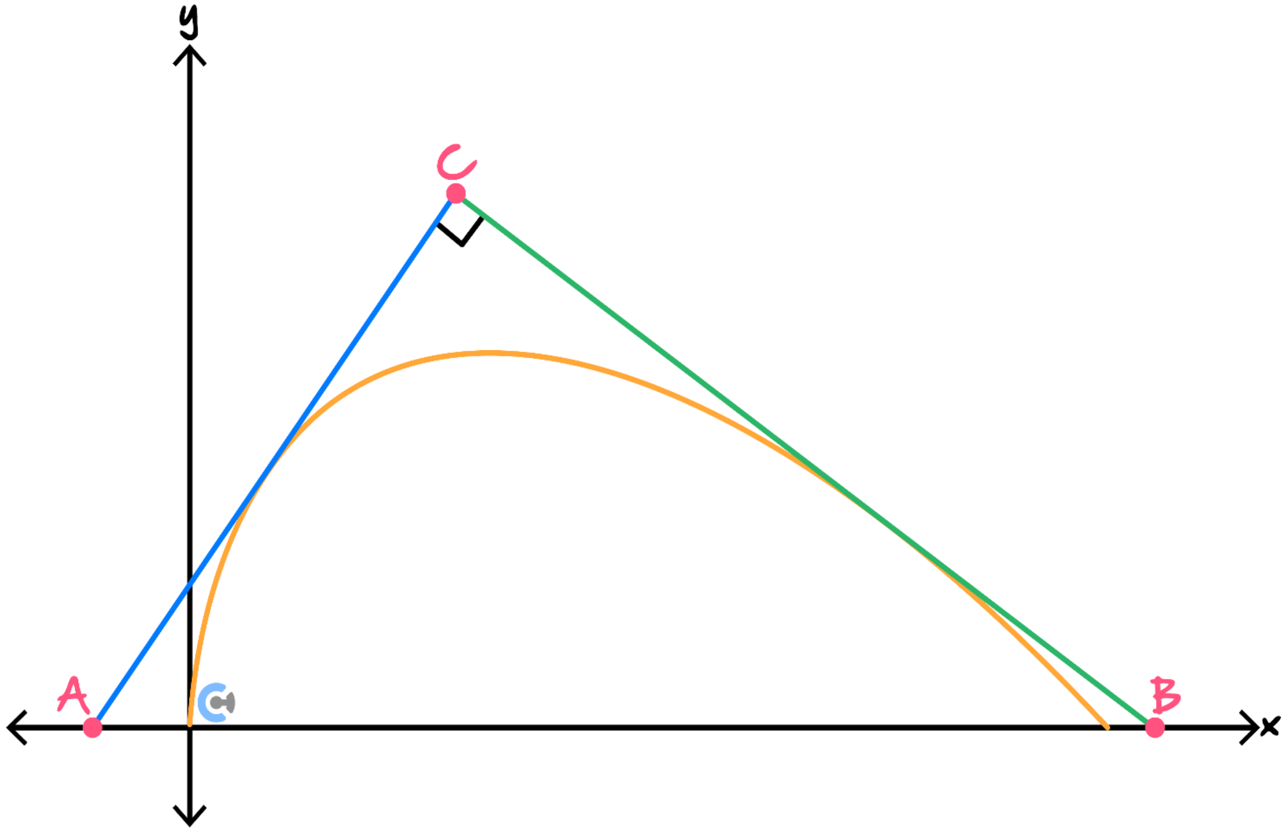
ii. Hence, find the coordinates of any stationary points of f. (2 marks)

Solution: Solve $f'(x) = 0 \implies 4 - 3x = 0 \implies x = \frac{4}{3}$.

Then $f\left(\frac{4}{3}\right) = \frac{1}{2} \times \frac{2}{\sqrt{3}}\left(\frac{12}{3} - \frac{4}{3}\right) = \frac{1}{\sqrt{3}} \times \frac{8}{3} = \frac{8}{3\sqrt{3}} = \frac{8\sqrt{3}}{9}$.

Stationary point at $\left(\frac{4}{3}, \frac{8\sqrt{3}}{9}\right)$

The edges of the **right-angled** triangle ABC are the line segments AC and BC which are tangent to the graph of f and the line segment AB, which is part of the horizontal axis as shown below. Let θ be the angle that AC makes with the positive direction of the horizontal axis, where $30° \leq \theta < 90°$.



b. Find the equation of the line through B and C, in the form $y = mx + c$, for $\theta = 45°$. (2 marks)

Solution: $\theta = 45° \implies AC$ has gradient 1 and therefore BC must have gradient -1.
We solve $f'(x) = \dfrac{4-3x}{4\sqrt{x}} = -1 \implies 4 - 3x = -4\sqrt{x}$. Let $a = \sqrt{x}$. Then

$$4 - 3a^2 = -4a$$
$$3a^2 - 4a - 4 = 0$$
$$(3a+2)(a-2) = 0$$
$$a = -\frac{2}{3}, 2$$

Only $a = 2$ is valid. Therefore $x = 4$.
And so, $y = -x + 4$

c. Find the coordinates of C when $\theta = 45°$. (4 marks)

Solution: We now need to find the equation of line segment AC. We know it has gradient 1, so solve $f'(x) = 1$.
We solve $4 - 3x = 4\sqrt{x}$. Let $a = \sqrt{x}$

$$\begin{aligned} 4 - 3a^2 &= 4a \\ 3a^2 + 4a - 4 &= 0 \\ (3a-2)(a+2) &= 0 \\ a &= -2, \frac{2}{3} \end{aligned}$$

Only $a = \frac{2}{3}$ valid $\implies x = \frac{4}{9}$.
Now $f\left(\frac{4}{9}\right) = \frac{2\left(\frac{36}{9} - \frac{4}{9}\right)}{3 \times 2} = \frac{64}{54} = \frac{32}{27}$.
Then

$$\begin{aligned} y - \frac{32}{27} &= x - \frac{12}{27} \\ y &= x + \frac{20}{27}. \end{aligned}$$

Now we find C at the intersection of $y = x + \frac{20}{27}$ and $y = -x + 4$.
$\implies 2x = \frac{108}{27} - \frac{20}{27} = \frac{88}{27} \implies x = \frac{44}{27} \implies y = \frac{64}{27}$.
Thus $C\left(\frac{44}{27}, \frac{64}{27}\right)$

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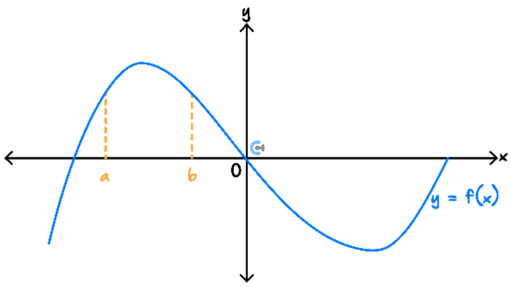
Section G: Extension Exam 2 (14 Marks)

INSTRUCTION:

- Regular: Skip
- Extension: 14 Marks. 17 Minutes Writing.

Question 15 (1 mark)

The graph of the function with equation $y = f(x)$ is shown below:



Let g be the function such that $g'(x) = f(x)$.
On the interval (a, b), the graph of g will:

A. Have a negative gradient.

B. Have a positive gradient.

C. Have a local minimum value.

D. Have a local maximum value.

Question 16 (1 mark)

Consider the function $f : [1, \infty) \to \mathbb{R}, f(x) = x^4 + 4x^3 + 2(a-1)x^2 - 4(a+3)x - 6a - 6$, where $a \in \mathbb{R}$.
The maximal set of values of a for which the inverse function f^{-1} exists is:

A. $(-9, \infty)$

B. $(-\infty, 1)$

C. $(-8, \infty)$

D. $(-9,1)$

Question 17 (1 mark)

Let $p(x)$ be a degree seven polynomial with seven real roots. What is the minimum amount of real roots that $q(x)$ can have if $q'(x) = p(x)$?

A. 0

B. 1

C. 7

D. 8

Question 18 (11 marks)

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = 3^{x+2} - 4$.

a. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x, ay + d)$ maps the graph of $y = 3^x$ onto the graph of f.

State the values of a and d. (2 marks)

Solution: $9 \times 3^x = 3^{x+2}$.
So dilation by factor 9 from the x-axis and translation 4 units down.
Therefore $a = 9$ and $d = -4$

b. Find the rule and domain for f^{-1}, the inverse function of f. (2 marks)

Solution: $x = 3^{y+2} - 4 \implies \log_3(x+4) = y + 2 \implies y = \log_3(x+4) - 2$.
Therefore $f^{-1} : (-4, \infty) \to \mathbb{R},\ f^{-1}(x) = \log_3(x+4) - 2$.

c. Find the point(s) of intersection between f and f^{-1}. Give your answer correct to two decimal places where appropriate. (1 mark)

Solution: We solve $f(x) = f^{-1}(x) \implies x = -3.87, -1$. Thus points of intersection $(-1, -1)$ and $(-3.87, -3.87)$.

d. Find the gradient of f and the gradient of f^{-1} at $x = -1$. (2 marks)

Solution: $f'(-1) = 3\log_e(3)$ and $(f^{-1})'(-1) = \dfrac{1}{3\log_e(3)}$.

e. The function f is mapped to the function g when it undergoes a dilation by a factor k from the x-axis, where $k > 0$. Find the value(s) of k such that g and g^{-1} intersect each other exactly once. Give your answer(s) correct to three decimal places. (2 marks)

Solution: $g(x) = kf(x)$. We solve $g(x) = x$ and $g'(x) = 1$ simultaneously. This yields $k = 0.0454272, x = 0.728531$ or $k = 0.651434, x = -1.6955$. Thus $k = 0.045, 0.651$.

Let $h(x) = 3^{x+2} - 4 - 3^{3x}$.

f. Find the exact values of a for which $h(x) = a$ has two solutions, where $a \in \mathbb{R}$. (2 marks)

Solution: Find that $h(x)$ has a local maximum at $\left(\frac{1}{2}, 6\sqrt{3} - 4\right)$, also $\lim_{x \to -\infty} h(x) = -4$.
Looking at the shape of the graph we see that $a \in (-4, 6\sqrt{3} - 4)$

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