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VCE Mathematical Methods ¾
Differentiation Exam Skills [0.11]
Workshop

Error Logbook:

Not knowing how to do the q		Algebraic/Calculator Mistakes	
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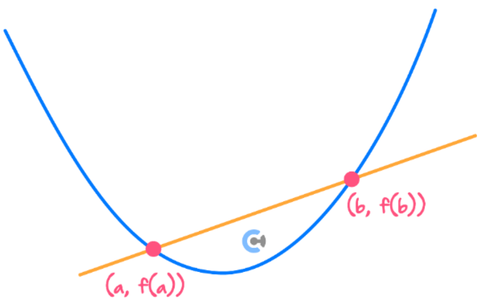
Section A: Recap

Average Rate of Change

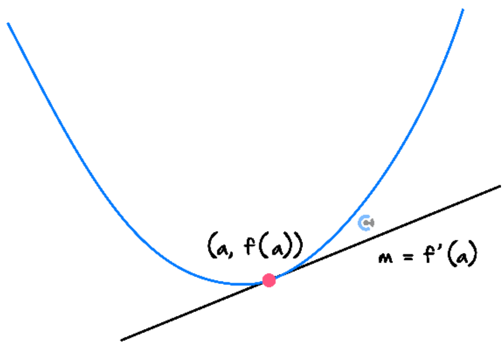


- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:

$$\text{Average rate of change} = \frac{f(b)-f(a)}{b-a}$$

- It is the gradient of the line joining the two points.

Instantaneous Rate of Change



- **Instantaneous rate of change** is a gradient of a graph at a single point/moment.

$$\text{Instantaneous rate of change} = f'(x)$$

- **Differentiation** is the process of finding the derivative of a function.
- **First principles** derivative definition:

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Definition

Alternative Notation for Derivative

$$f'(x) = \frac{dy}{dx}$$

Definition

Derivatives of Functions

- The derivatives of many of the standard functions are in the summary table below:

$f(x)$	$f'(x)$
x^n	$n \times x^{n-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
e^x	e^x
$\log_e(x)$	$\frac{1}{x}$

Definition

The Product Rule

- The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

- Or, in another form:

$$\frac{d}{dx}(u \cdot v) = u'v + v'u$$

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The Quotient Rule

- The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

- Or, written in another form:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

  - Always differentiate the top function first.

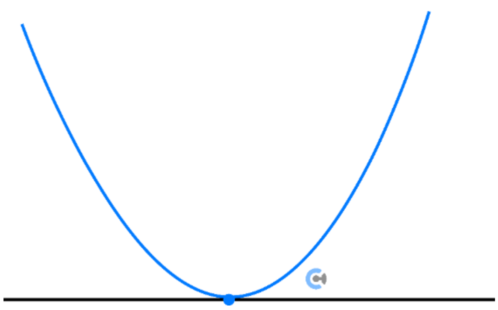
The Chain Rule

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

- The process for finding derivatives of **composite functions.**

Stationary Points



- The point where the gradient of the function is zero.

$$f'(x) = 0, \frac{dy}{dx} = 0$$

Calculator Commands: Finding Derivatives

- **Mathematica**

```
f'[x]
```

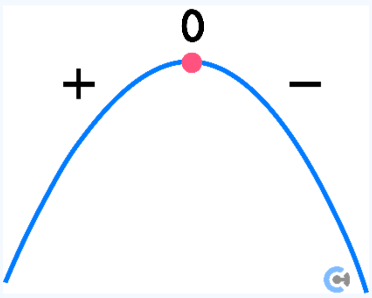
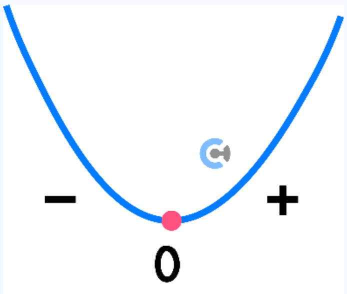
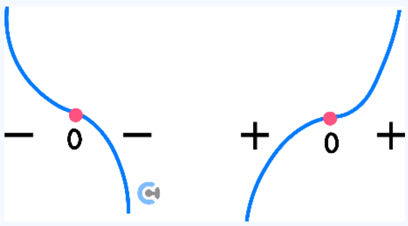
- **TI**
  - Shift Minus

$$\frac{d}{dx}(f(x))$$

- **Casio**
  - Math 2

$$\frac{d}{dx}(f(x))$$

Types of Stationary Points

Local Maximum	Local Minimum	Stationary Point of Inflection
		

- Sign test.

- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing curve	Stationary point	∪ - Increasing curve

- Find the gradient of the neighbouring points.

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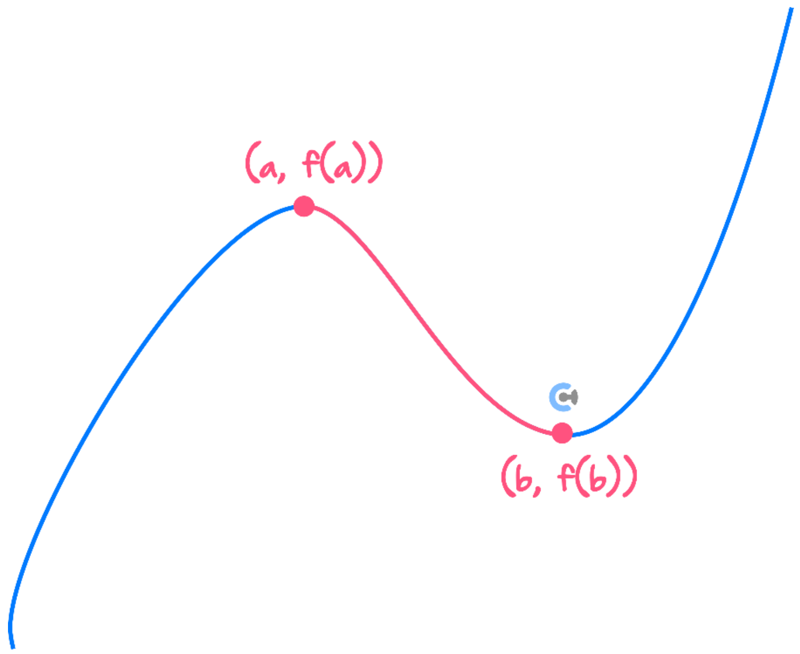
Strictly Increasing and Strictly Decreasing Functions



Strictly Increasing: $x \in (-\infty, a] \cup [b, \infty)$

Strictly Decreasing: $x \in [a, b]$

- Steps:

  1. Find the turning points.

  2. Consider the sign of the derivative between/outside the turning points.

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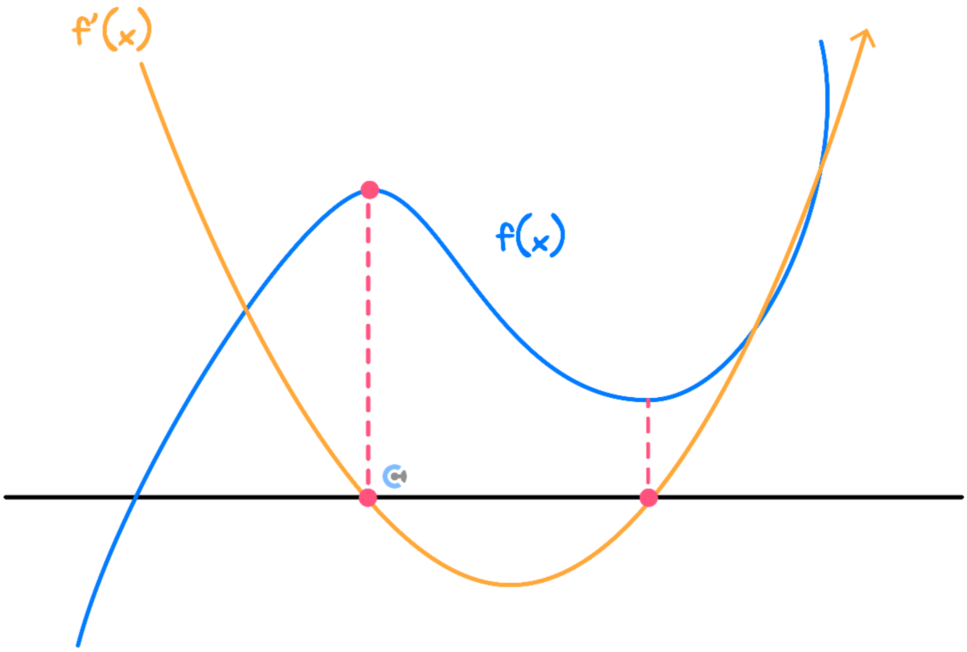
Graphs of the Derivative Function



$f(x)$	$f'(x)$
Stationary Point	x-intercepts
Increasing	Positive
Decreasing	Negative

$$y\text{-value of } f'(x) = \text{Gradient of } f(x)$$

- **Steps**

1. Plot x-intercept at the same x-value as the stationary point of the original.
2. Consider the trend of the original function and sketch the derivative.
   - Original is increasing → Derivative is above the x-axis.
   - Original is decreasing → Derivative is below the x-axis.

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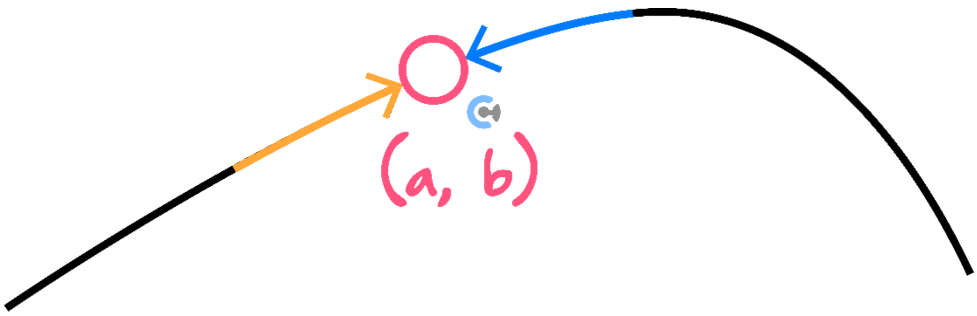
Limits

$$\lim_{x \to a} f(x) = L$$

"The function $f(x)$ approaches L as x approaches a."

- Limit is the value that a function (y-value) approaches as the x-value approaches a value.

Validity of Limits



$$\lim_{x \to a} f(x) \text{ exists if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

- Limit is defined when the left limit equals the right limit.

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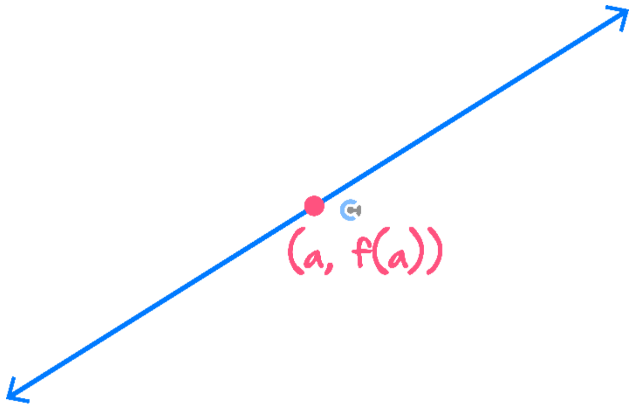
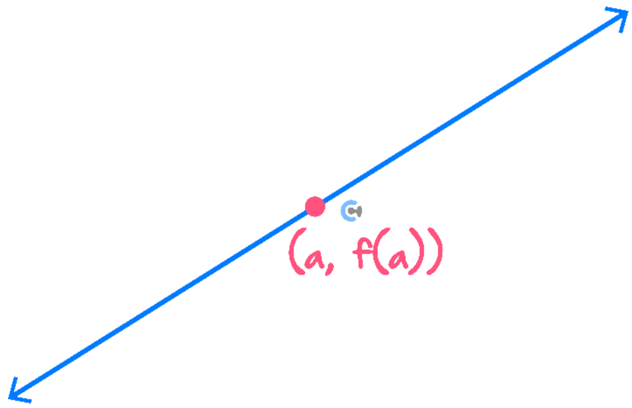
Continuity



- A function f is said to be continuous at a point $x = a$ if:

  1. $f(x)$ is defined at $x = a$.
  2. $\lim\limits_{x \to a} f(x)$ exists.
  3. $\lim\limits_{x \to a} f(x) = f(a)$.

Differentiability



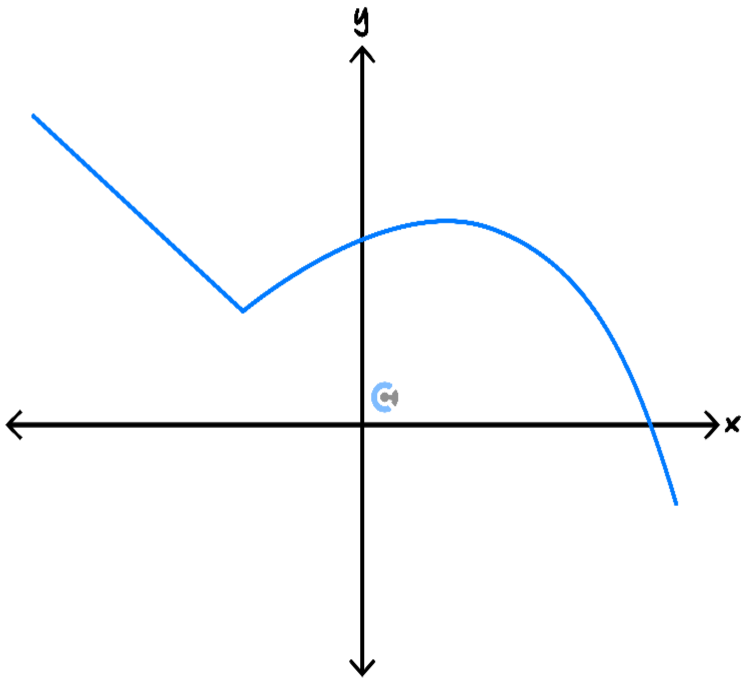
- A function f is said to be differentiable at a point $x = a$ if:

  1. $f(x)$ is continuous at $x = a$.
  2. $\lim\limits_{x \to a} f'(x)$ exists.
     - Limit exists when the left and right limits are the same.
     - Gradient on the LHS and RHS must be the same.

- We **cannot** differentiate:
  1. Discontinuous points.
  2. Sharp points.
  3. Endpoints.

Definition

<u>Finding the Derivative of Hybrid Functions</u>



1. Simply derive each function.
2. Reject the values for x that are not differentiable from the domain.

Definition

<u>Second Derivatives</u>

- The derivative of the derivative.
- To get the second derivative, we can **differentiate the original function twice**.

$$\frac{d^2y}{dx^2} = f''(x)$$

Concavity

- Concave up is when the gradient is increasing.

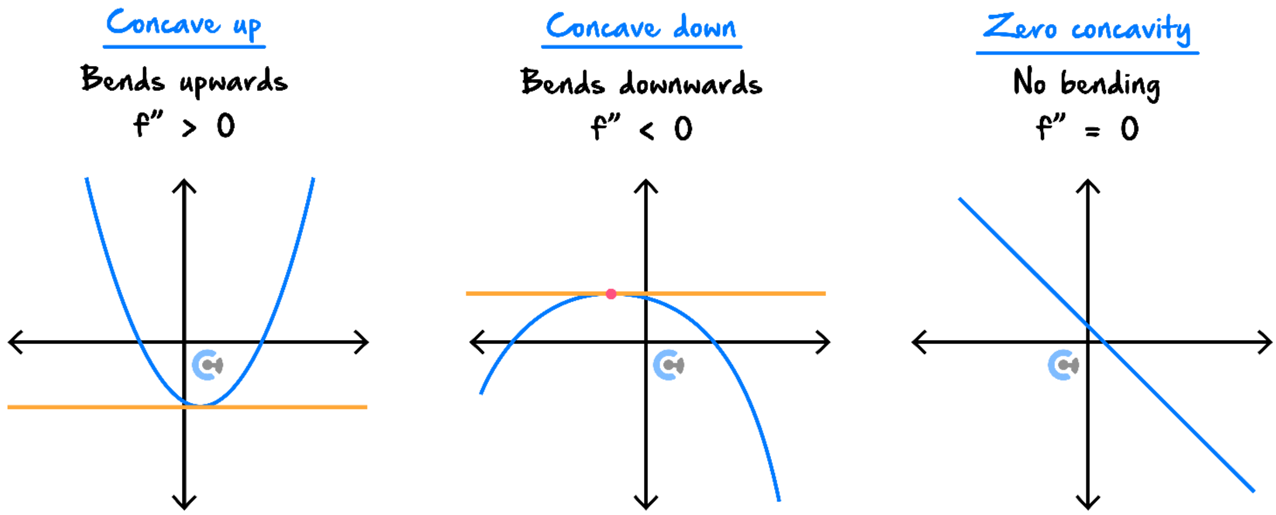
$$f''(x) > 0 \to \text{Concave Up}$$

- Concave down is when the gradient is decreasing.

$$f''(x) < 0 \to \text{Concave Down}$$

- Zero concavity is when the gradient is neither increasing nor decreasing.

$$f''(x) = 0 \to \text{Zero Concavity}$$



Concavity is also linked to how the curve is bent.

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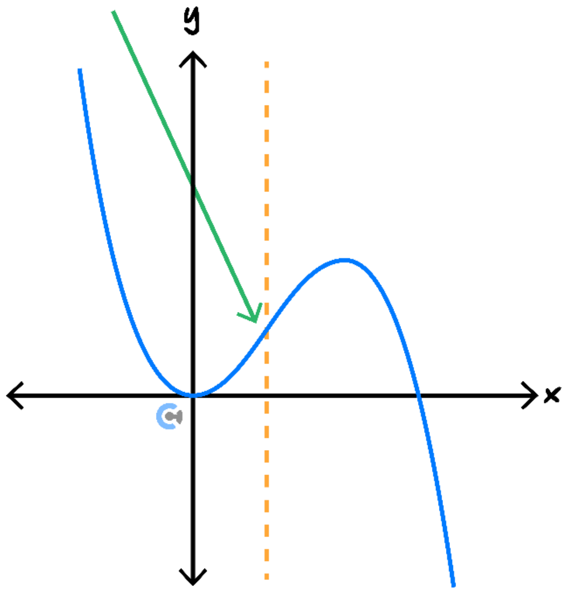
Points of Inflection

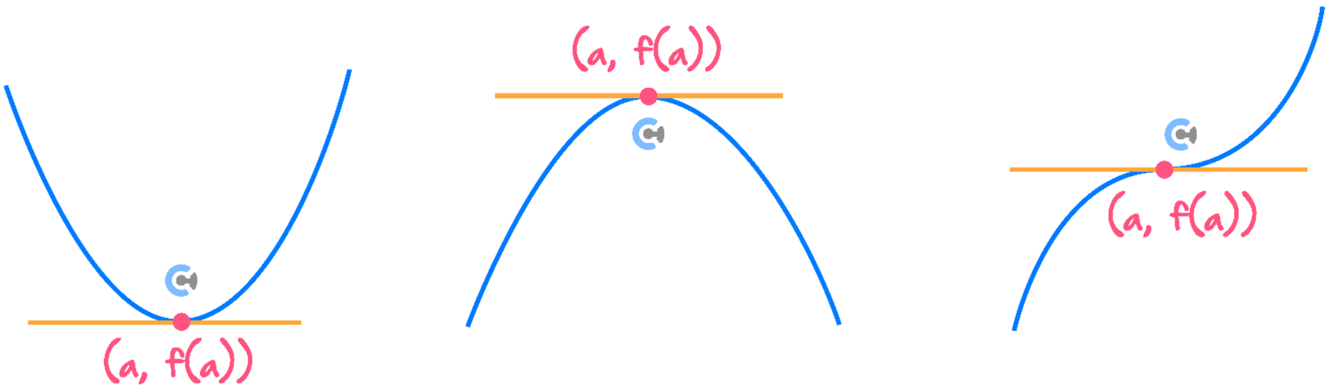
- A point at which a curve **changes concavity** is called a **point of inflection**.



$$f''(x) = 0$$

- Simply, it is when the bending changes.

The Second Derivative Test



- Suppose that $f'(a) = 0$ and hence, f has a stationary point at $x = a$. The second derivative test states:

  - **Concave up gives us a local minimum.**

$$f''(x) > 0 \rightarrow \text{Local Minimum}$$

- **Concave down gives us a local maximum.**

$$f''(x) < 0 \rightarrow \text{Local Maximum}$$

- **Zero concavity gives us a stationary point of inflection.**

$$f''(x) = 0 \rightarrow \text{Stationary Point of Inflection}$$

Definition

Joining Smoothly

- Let two different curves be defined as $f(x)$ and $g(x)$. For these two curves to join smoothly at $x = a$, they have to satisfy:
  - $f(a) = g(a)$
  - $f'(a) = g'(a)$
- In other words, the function must be **continuous** and **differentiable** at that point!

Definition

Steps for Finding Strictly Increasing/Decreasing Regions

1. Plot the graph on CAS.
2. Find stationary points.
3. Use a graph to determine which regions are increasing/decreasing.

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Section B: Warmup

Question 1

a. Let $g(x) = e^{\sin(x)}$. Find $g'(x)$.

$g'(x) = \cos(x)e^{\sin(x)}$

b. Let $h(x)$ be a differentiable function. Find the derivative of $x^2h(x)$, with respect to x.

Use the product rule.

$$\frac{d}{dx}\left(x^2h(x)\right) = 2xh(x) + x^2h'(x).$$

c. Consider the function f given by:

$$f(x) = \begin{cases} 2 - ax & x < 1 \\ ax^2 + bx + 4 & x \geq 1 \end{cases}$$

Find the integer values of a and b such that the graph of f joins smoothly at $x = 1$.

Solution: Let $g(x) = 2 - ax$ then $g'(x) = -a$. Also let $h(x) = ax^2 + bx + 4$, then $h'(x) = 2ax + b$.
Now $g(1) = h(1) \implies 2 - a = a + b + 4 \implies 2a + b = -2 \implies b = -2 - 2a$.
Also $g'(1) = h'(1) \implies -a = 2a + b \implies b = -3a$.
So we solve $-3a = -2 - 2a \implies a = 2$ then $b = -6$.
$a = 2, b = -6$.

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Section C: Exam 1 Questions (19 Marks)

INSTRUCTION: 19 Marks. 28 Minutes Writing.

Question 2 (4 marks)

a. Let $y = \frac{\sin(x)}{x^2+4}$.

Find $\frac{dy}{dx}$. (2 marks)

$$\frac{dy}{dx} = \frac{\cos(x) \cdot (x^2+4) - \sin(x) \cdot 2x}{(x^2+4)^2}$$

$$= \frac{(x^2+4)\cos(x) - 2x\sin(x)}{(x^2+4)^2}$$

b. Let $f(x) = x^2 e^{7x}$. Evaluate $f'(1)$. (2 marks)

$$f'(x) = 2xe^{7x} + x^2 \cdot e^{7x} \cdot 7$$

$$f'(1) = 2e^7 + 7e^7$$

$$= 9e^7$$

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Question 3 (4 marks)

Let $f : [-\pi, \pi] \to \mathbb{R}, f(x) = \cos(2x) + 1$.

a. Calculate the average rate of change of f between $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{4}$. (2 marks)

$$\frac{f(\frac{\pi}{4})-f(-\frac{\pi}{3})}{\frac{\pi}{4}-(-\frac{\pi}{3})} = \frac{1/2}{7\pi/12}$$

$$= \frac{6}{7\pi}$$

$f(\frac{\pi}{4})-f(-\frac{\pi}{3})$

$= [\cos(\frac{\pi}{2})+1]-[\cos(-\frac{2\pi}{3})+1]$

$= 1-(-\frac{1}{2}+1) = 1-\frac{1}{2} = \frac{1}{2}$

b. Find the angle that a tangent to f makes with the positive x-axis when $x = \frac{\pi}{3}$. (2 marks)

$f'(x) = -2\sin(2x)$

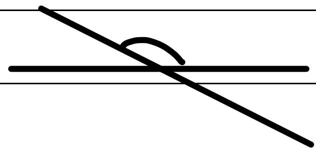
$f'(\frac{\pi}{3}) = -2\sin(\frac{2\pi}{3})$

$= -2 \times \frac{\sqrt{3}}{2}$

$= -\sqrt{3}$.

$\tan(\theta) = -\sqrt{3}$

$\theta = -\frac{\pi}{3}, \frac{2\pi}{3}$.



$\therefore \frac{2\pi}{3}^c$

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Question 4 (8 marks)

Let $f : [-2, 1] \to \mathbb{R}, f(x) = (x + 1)^2(x - 1)$.

a. Show that $f(x) = x^3 + x^2 - x - 1$. (1 mark)

$f(x) = (x+1)^2(x-1)$
$= (x+1)(x+1)(x-1)$
$= (x+1)(x^2-1)$
$= x^3 - x + x^2 - 1 = x^3 + x^2 - x - 1$

b. Find the x-values for which the graph of $y = f(x)$ has stationary points. (2 marks)

$f'(x) = 3x^2 + 2x - 1 = 0$

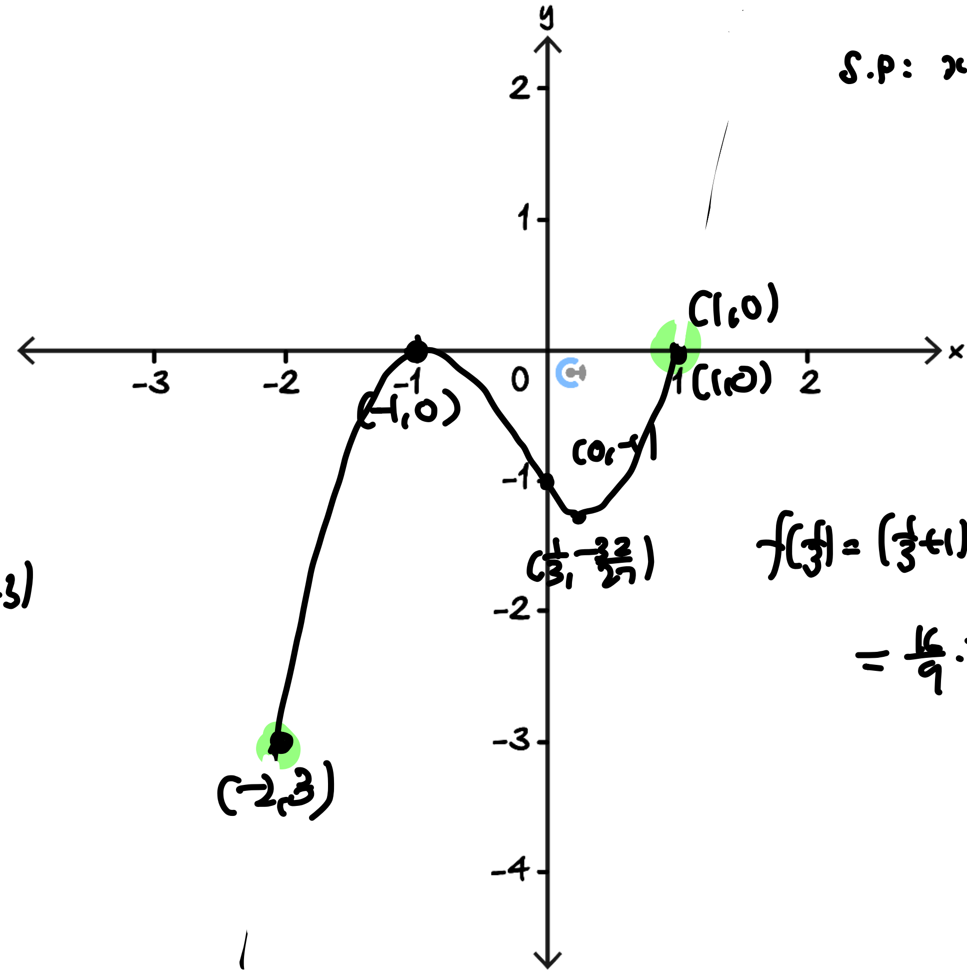
$(3x - 1)(x + 1) = 0$

$x = \frac{1}{3}, -1$

c. Hence, sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts, stationary points, and endpoints with coordinates. (2 marks)

$(x+1)^2(x-1) = y$

S.P: $x = \frac{1}{3}, -1$



$f(\frac{1}{3}) = (\frac{1}{3}+1)^2 \cdot (\frac{1}{3}-1)$

$= \frac{16}{9} \cdot \frac{-2}{3} = \frac{-32}{27}$

$f(-2) = (-1)^2 \cdot (-3)$

$= -3$

d. The gradient of f at $x = a$ is equal to the average rate of change of f on the interval $x \in [-2, 0]$. Determine the possible value(s) of a. (2 marks)

$f'(a) = \frac{f(0) - f(-2)}{0 - (-2)}$

$3a^2 + 2a - 1 = 1$

$3a^2 + 2a - 2 = 0$

$a = \frac{-2 \pm \sqrt{4 - 4 \cdot 3 \cdot (-2)}}{6}$

$= \frac{-1 \pm \sqrt{7}}{3}$

$f'(x) = 3x^2 + 2x - 1$

$f(0) - f(-2) = -1 - [-3]$

$= 2$

$\therefore \frac{f(0) - f(-2)}{0 - (-2)} = \frac{2}{2} = 1$

e. The function f is mapped to the function g according to $g(x) = 2f\left(\frac{1}{3}(x+2)\right)$. State the x-values for the stationary points of g. (1 mark)

f: $(-1,0)$ & $(-\frac{1}{3}, -\frac{32}{27})$

$(-1,0)$ & $(\frac{1}{3}, \frac{-64}{27})$

$(-3,0)$ & $(1, \frac{-64}{27})$

$(-5,0)$ & $(-1, -\frac{64}{27})$

Transform: Dil 2 from x

Dil 3 from y

2 left

$\boxed{x = -5, -1}$

Question 5 (3 marks)

A differentiable function $f : \mathbb{R} \to \mathbb{R}$ has the following two properties:

- $f'(x) = f(x)(6 - f(x))$
- ??? The range of f is $(0, 6)$.

a. Find $f'(0)$ if $f(0) = 2$. (1 mark)

$f'(0) = f(0) \cdot (6 - f(0))$

$= 2 \times (6-2) = 2 \times 4 = 8.$

b. Determine the number of stationary points of the graph of f and justify your answer. (1 mark)

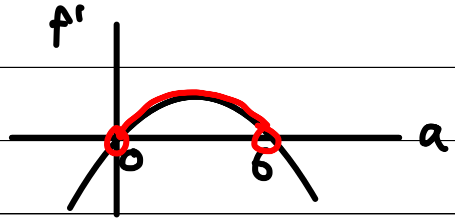
$f' = 0.$

let $f(x) = a$.

$f'(x) = a(6-a)$

$a \in (0, 6)$

Range of f':



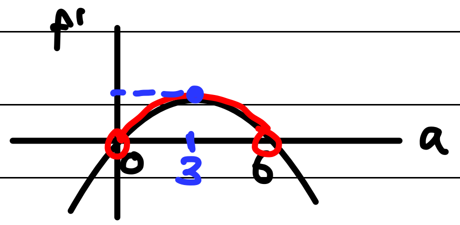
$f' = 0.$

No soln

$\therefore$ Zero Stationary Points.

c. Find the range of f'. (1 mark)

$f' \in (0, 9]$



$a(6-a)$

$f'(x) = 3 \times (6-3)$

$= 3 \times 3 = 9$

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Section D: Tech Active Exam Skills

Calculator Commands: Finding Derivatives

Mathematica	TI	Casio
`f'[x]` `D[f[x], x]`	Shift Minus $\frac{d}{dx}(f(x))$	Math 2 $\frac{d}{dx}(f(x))$

Calculator Commands: Finding Second Derivatives

Mathematica	TI	Casio
`f''[x]` `D[f[x], {x, 2}]`	Shift Minus $\frac{d^2}{dx^2}(f(x))$	Math 2 $\frac{d^2}{dx^2}(f(x))$

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Calculator Commands: Stationary Point

- ALWAYS sketch the graph first to get an idea of the nature of the stationary point.
- The turning points for a function $f(x)$ can be found by solving $f'(x) = 0$ and subbing the result into f.
- **Example:** Find the turning point for $f(x) = e^{-x^2+2x}$.
- **TI:**

Define $f(x)=e^{-x^2+2\cdot x}$	*Done*
solve$\left(\frac{d}{dx}(f(x))=0,x\right)$	$x=1$
$f(1)$	e

- **Casio:**

define $f(x) = e^{-x^2+2x}$	
	done
solve$(\frac{d}{dx}(f(x))=0, x)$	
	$\{x=1\}$
$f(1)$	
	e

- **Mathematica:**

```
In[4]:= f[x_] := Exp[-x^2 + 2 x]
In[5]:= Solve[f'[x] == 0 && y == f[x], Reals]
Out[5]= {{x → 1, y → e}}
```

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Calculator Commands: Using Sliders/Manipulate on CAS

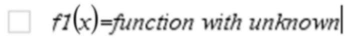
Mathematica

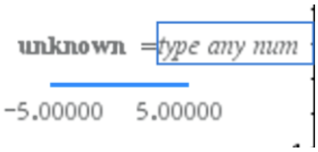
```
Manipulate[Plot[function, {x, xmin, xmax}],

   {unknown, lowerbound, upperbound}]
```

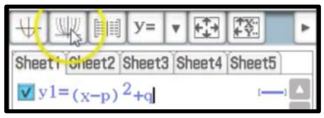
- **NOTE**: The function **must** be typed out instead of using its saved name.

TI-Nspire





Casio Classpad



Calculator Commands: Joining Smoothly

Mathematica

```
f[x_] := One Function

g[x_] := Another Function

Solve[f[x value] == g[x value] && f'[x value] == g'[x value]]
```

TI and Casio

- Define each branch as $f(x)$ and $g(x)$.
- TI: Define its derivative as $df(x)$ and $dg(x)$.

  Casio: Define them as different names.
- Solve $f(a) = g(a)$ and $df(a) = dg(a)$ simultaneously.

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Section E: Exam 2 Questions (22 Marks)

INSTRUCTION: 22 Marks. 32 Minutes Writing.

Question 6 (1 mark)

If $y = \frac{\tan x}{h(x)}$, then $\frac{dy}{dx}$ is:

A. $\frac{h(x)\sec^2(x) + \tan(x)h'(x)}{h(x)^2}$

B. $\frac{h(x)\sec^2(x) + \tan(x)h'(x)}{h(x)^4}$

C. $\frac{h(x)\sec^2(x) - \tan(x)h'(x)}{h(x)^2}$

D. $\frac{h'(x)\sec^2(x) - \tan(x)h(x)}{h(x)^2}$

Question 7 (1 mark)

A cubic function has the rule $y = f(x)$. The graph of the derivative function $f'(x)$ crosses the x-axis at $(2, 0)$ and $(-3, 0)$. The maximum value of the derivative function is 10. The value of x for which the graph of $y = f(x)$ has a local maximum is:

Quad.

A. -2

B. 2

C. -3

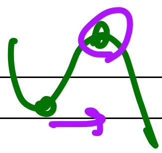
D. 3

$f' = 0$

$x = 2, -3$

(−) Quad

(−) Cubic



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Question 8 (1 mark)

Let $h(x) = g(x)e^{f(x^2)}$ be a differentiable function. Then $h'(x)$ is equal to:

A. $2e^{f(x^2)}g(x)f'(x^2) + e^{f(x^2)}g(x)$

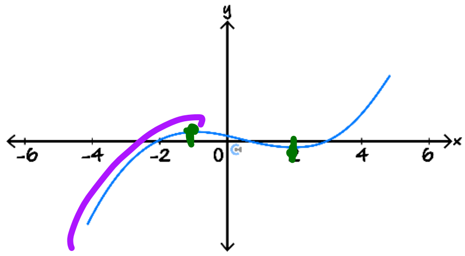
B. $2xe^{f(x^2)}g(x)f'(x^2) + e^{f(x^2)}g'(x)$

C. $2x^2e^{f(x^2)}g'(x)f'(x^2) + e^{f(x^2)}g(x)$

D. $x^2e^{f(x^2)}g(x)f'(x^2) + e^{f(x^2)}g'(x)$

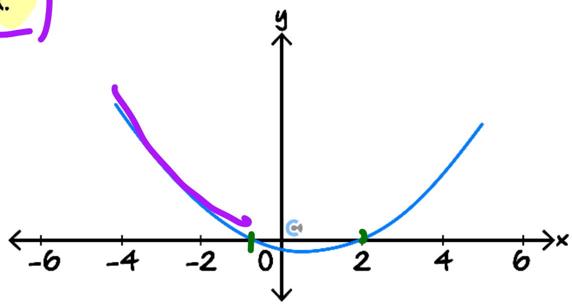
Question 9 (1 mark)

The graph of the function with the equation $y = f(x)$ is shown below.

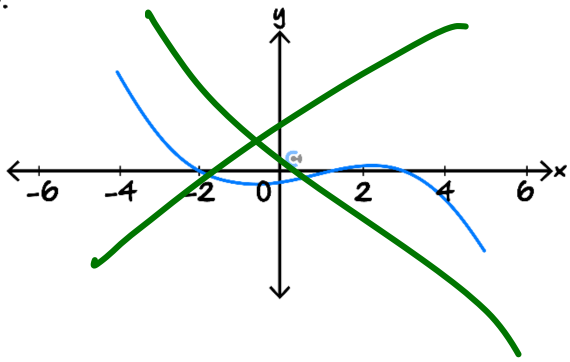


Which one of the following is most likely to be the graph of the derivative function with equation $y = f'(x)$?

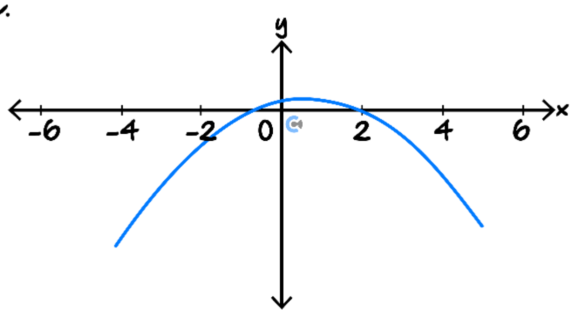
A.



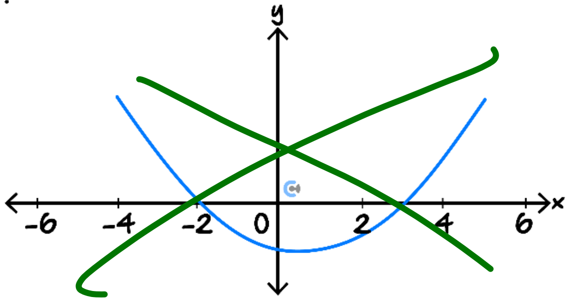
B.



C.



D.



Question 10 (1 mark)

Consider the function h, where

$$h(x) = \begin{cases} -x^2 + 2ax + 1 & x < 1 \\ a(x-b)^2 + a & x \geq 1 \end{cases}$$

The values of a and b such that the graph of $y = f(x)$ joins smoothly at $x = 1$ are:

A. $a = 2, b = 1$

B. $a = 1, b = 2$

C. $a = \frac{1}{2}, b = 2$

D. $a = 1, b = \frac{1}{2}$

$f(1) = g(1)$

$df(1) = dg(1)$

Question 11 (1 mark)

Let f be a one-to-one differentiable function. The points $(3, 5)$ and $(5, 8)$ lie on the graph of f. It is known that f has; a gradient of 2 when $x = 3$, a gradient of 3, when $x = 5$ and a gradient of 7, when $x = 8$.

The function g is differentiable and $g(x) = f^{-1}(x)$ for all x.

$g'(5)$ is equal to:

A. $\frac{1}{2}$

B. 2

C. $\frac{1}{8}$

D. $\frac{1}{3}$

f: $(3,5)$ $\cancel{(5,8)}$

$m = 2$

$g = f^{-1}$: $(5,3)$ $\cancel{(8,5)}$

$m = \frac{1}{2}$

$\frac{dy}{dx} \rightarrow \frac{dx}{dy}$

Space for Personal Notes

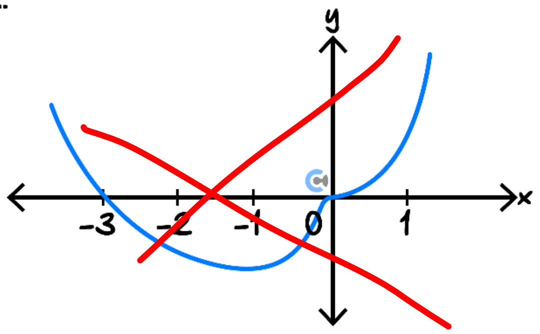
Question 12 (1 mark)

A continuous function f has the following properties:

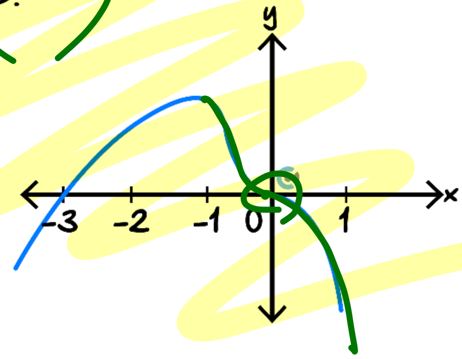
$$f(0) = 0 \qquad f'(0) = 0$$
$$f(-3) = 0 \qquad f'(-1) = 0$$
$$f'(x) > 0 \text{ for } x \in (-\infty, -1)$$
$$f'(x) < 0 \text{ for } x \in (-1, 0) \cup (0, \infty)$$

Which one of the following is most likely to represent the graph of f?

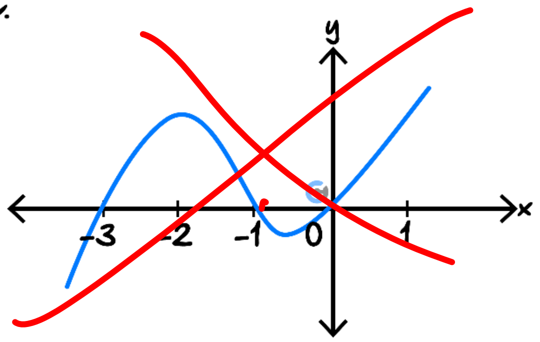
A.



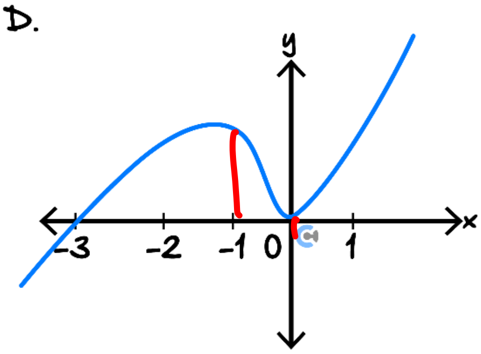
B.



C.



D.



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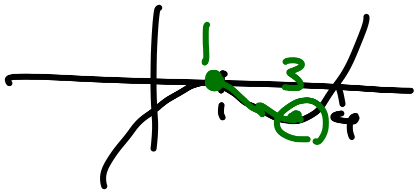
Question 13 (15 marks)



Let $f : \mathbb{R} \to \mathbb{R}, f(x) = (x - 1)^2(x - 4)$.

a. Find $f'(x)$. (1 mark)

$f'(x) = 3x^2 - 12x + 9$

= Squen

b. For what values of x is $f(x)$ strictly decreasing? (1 mark)

$f'(x) = 0$

$x = 1, 3$

$x \in [1, 3]$

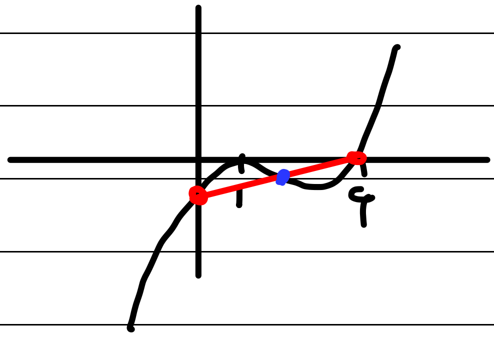
c.

i. Find the gradient of the line segment joining the points on the graph of $y = f(x)$ where $x = 0$ and $x = 4$. (1 mark)

$\frac{f(4) - f(0)}{4 - 0} = 1$

ii. Show that the midpoint of the line segment in **part c.i.** also lies on the graph of $y = f(x)$. (2 marks)

$(0,-4)$ & $(4,c)$



Midpoint: $(2,-2)$

$f(2) = (2-1)^2 \cdot (2-4)$

$= 1^2 \cdot (-2)$

$= -2 \quad \therefore$ lies.

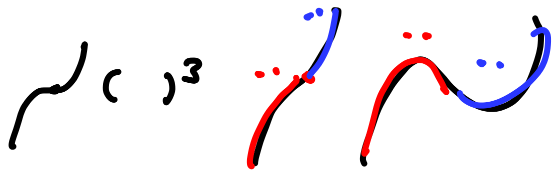
iii. Find the values of x for which the tangent to the graph of $y = f(x)$ is equal to the gradient of the line segment joining the points on the graph where $x = 0$ and $x = 4$. (2 marks)

$f'(x) = \dfrac{f(4)-f(0)}{4-0}$

$= 1$

$x = \dfrac{6 \pm 2\sqrt{3}}{3}$

Let $g : \mathbb{R} \to \mathbb{R}, g(x) = (x - a)^2(x - 4)$, where $a \in \mathbb{R}$.



d.

Cubic

i. State the value of a for which $g(x)$ has a stationary point of inflection. (1 mark)

$(x-a) = (x-4)$

$a = 4$

ii. Find the coordinates for the stationary points of g, in terms of a. (2 marks)

$g'(x) = 0.$

$x = a, \ \frac{a+8}{3}.$

$(a, 0). \ \left(\frac{a+8}{3}, \frac{4(a-4)^3}{27}\right)$

e. Find the values of a for which the gradient of $g(x)$ when $x = \frac{10+a}{3}$ is negative. (1 mark)

$g'\left(\frac{10+a}{3}\right) < 0$

$a > 5$

f. Suppose the tangent to the graph of $y = g(x)$ at $x = \frac{10+a}{3}$ has a positive gradient. $a < 5.$

i. Find the coordinates of another point where the tangent to the graph of $y = g(x)$ is parallel to the tangent at $x = \frac{10+a}{3}$. (2 marks)

$g'(x) = g'\left(\frac{10+a}{3}\right)$

$x = \frac{3a-2}{3}, \cancel{\frac{10+a}{3}}.$

$\therefore \left(\frac{3a-2}{3}, \frac{4}{27}(3a-14)\right)$

ii. Find the value(s) of a for which the points that these parallel tangents are drawn at have the same y-value. (2 marks)

$f\left(\frac{10+a}{3}\right) = f\left(\frac{3a-2}{3}\right)$

$a = 6, 3 \pm \sqrt{3}.$

$a \neq 6, \quad a < 5$

$\therefore \boxed{a = 3 \pm \sqrt{3}}$

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Section F: Extension Exam 1 (12 Marks)

INSTRUCTION: 12 Marks. 18 Minutes Writing.

Question 14 (9 marks)

Consider the function $f : [0, 4] \to \mathbb{R}, f(x) = \frac{\sqrt{x}(4-x)}{2}$.

a.

i. For $x \in (0, 4)$, show that the gradient of the tangent to the graph of f is $\frac{4-3x}{4\sqrt{x}}$. (1 mark)

You get this

ii. Hence, find the coordinates of any stationary points of f. (2 marks)

$\frac{4-3x}{4\sqrt{x}} = 0$

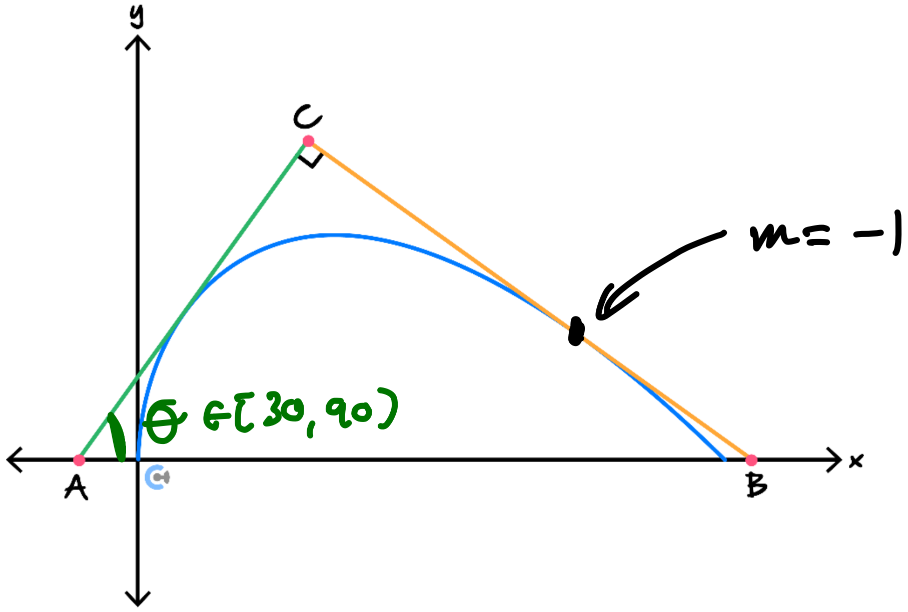
$4-3x=0$

$x = \frac{4}{3}$.

$\left(\frac{4}{3}, \frac{8\sqrt{3}}{9}\right)$

The edges of the **right-angled** triangle ABC are the line segments AC and BC which are tangent to the graph of f and the line segment AB, which is part of the horizontal axis, as shown below.

Let θ be the angle that AC makes with the positive direction of the horizontal axis, where $30° \leq \theta < 90°$.



b. Find the equation of the line through B and C, in the form $y = mx + c$, for $\theta = 45°$. (2 marks)

$m_{AC} = \tan(45°) = 1$

$m_{BC} = -1$

$\therefore y = -x + c$

Sub $(4, 0)$

$0 = -4 + c$

$c = 4$

$\boxed{\therefore y = -x + 4}$

$f'(x) = -1$

$\frac{4-3x}{4\sqrt{x}} = -1$

Check the sign

$4 - 3x = -4\sqrt{x}$

Square "tidy up the sign"

$\boxed{(4-3x)^2 = 16x}$

$9x^2 - 24x + 16 = 16x$

$9x^2 - 40x + 16 = 0$

$(9x - 4)(x - 4) = 0$

$x = \frac{4}{9}, \quad x = 4.$

$4 - 3x < 0$

$\frac{4}{3} < x$

$\therefore x = 4$

$(4, 0)$

c. Find the coordinates of C when $\theta = 45°$. (4 marks)

Solution: We now need to find the equation of line segment AC. We know it has gradient 1, so solve $f'(x) = 1$.
We solve $4 - 3x = 4\sqrt{x}$. Let $a = \sqrt{x}$

$$\begin{aligned} 4 - 3a^2 &= 4a \\ 3a^2 + 4a - 4 &= 0 \\ (3a-2)(a+2) &= 0 \\ a &= -2, \frac{2}{3} \end{aligned}$$

Only $a = \frac{2}{3}$ valid $\implies x = \frac{4}{9}$.
Now $f\left(\frac{4}{9}\right) = \frac{2\left(\frac{36}{9} - \frac{4}{9}\right)}{3 \times 2} = \frac{64}{54} = \frac{32}{27}$.
Then

$$\begin{aligned} y - \frac{32}{27} &= x - \frac{12}{27} \\ y &= x + \frac{20}{27}. \end{aligned}$$

Now we find C at the intersection of $y = x + \frac{20}{27}$ and $y = -x + 4$.
$\implies 2x = \frac{108}{27} - \frac{20}{27} = \frac{88}{27} \implies x = \frac{44}{27} \implies y = \frac{64}{27}$.
Thus $C\left(\frac{44}{27}, \frac{64}{27}\right)$

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Question 15 (3 marks)

A triangle is formed in the first quadrant of the unit circle with vertices at $O(0,0)$, C and B. The vertex C lies on the unit circle, and the vertex B lies directly under C, on the x-axis. Let θ be the angle that the line segment OC makes with the positive x-axis where $0 < \theta < \frac{\pi}{2}$.

Find the maximum area of the triangle and show that it is a maximum.

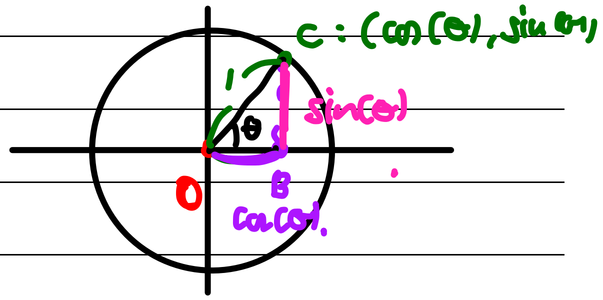
1) "Area of the triangle"

$A(\theta) = \frac{1}{2} \cdot \cos(\theta) \sin(\theta)$

$= \frac{1}{4} \sin(2\theta)$

2) "Domain": $\theta \in (0, \frac{\pi}{2})$

3) Find tp & endpoint



$A'(\theta) = \frac{1}{2} \cdot -\sin(\theta) \cdot \sin(\theta) + \frac{1}{2} \cos(\theta) \cdot \cos(\theta)$

$= \frac{1}{2}(\cos^2(\theta) - \sin^2(\theta)) = 0$

$\cos^2(\theta) = \sin^2(\theta)$

$1 = \tan^2(\theta)$

$\tan(\theta) = \pm 1$

$\theta = \frac{\pi}{4}, ~~\frac{\pi}{4}, \frac{3\pi}{4}~~$

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4) $A(\frac{\pi}{4}) = \frac{1}{2} \cdot \cos(\frac{\pi}{4}) \cdot \sin(\frac{\pi}{4})$

$= \frac{1}{2} \times \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4}.$

Show: $A'(\frac{\pi}{8})$

$= \frac{1}{2}(\cos^2(\frac{\pi}{8}) - \sin^2(\frac{\pi}{8}))$

$= \frac{1}{2} \cdot \left(\frac{3}{4} - \frac{1}{4}\right) = \frac{1}{4}.$

; $A'(\frac{\pi}{3})$

$= \frac{1}{2}(\cos^2(\frac{\pi}{3}) - \sin^2(\frac{\pi}{3}))$

$= \frac{1}{2} \cdot \left(\frac{1}{4} - \frac{3}{4}\right) = (-)$

$\therefore$ local max

Question 15 (3 marks) Alternative Method.

A triangle is formed in the first quadrant of the unit circle with vertices at $O(0,0)$, C and B. The vertex C lies on the unit circle, and the vertex B lies directly under C, on the x-axis. Let θ be the angle that the line segment OC makes with the positive x-axis where $0 < \theta < \frac{\pi}{2}$.

Find the maximum area of the triangle and show that it is a maximum.

$\therefore \text{Area} = x\sqrt{1-x^2}$

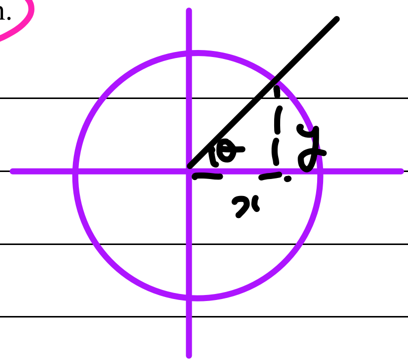
$\frac{dA}{dx} = \sqrt{1-x^2} + x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot -2x = 0$

$\sqrt{1-x^2} = \frac{x^2}{\sqrt{1-x^2}}$

$1-x^2 = x^2$

$1 = 2x^2$

$x = \frac{1}{\sqrt{2}}$ as $x > 0$ for $\theta \in (0, \frac{\pi}{2})$



$x^2 + y^2 = 1$

$y = \sqrt{1-x^2}$ as $y > 0$ for $\theta \in (0, \frac{\pi}{2})$

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$\therefore f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \cdot \sqrt{1-\frac{1}{2}} = \frac{1}{2}$

Show it is max

Something bigger than $\frac{1}{\sqrt{2}}$

$A'\left(\frac{1}{2}\right) = \sqrt{1-\frac{1}{4}} - \frac{1/4}{\sqrt{1-\frac{1}{4}}}$

$= \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

$A'(\quad) = (-)$

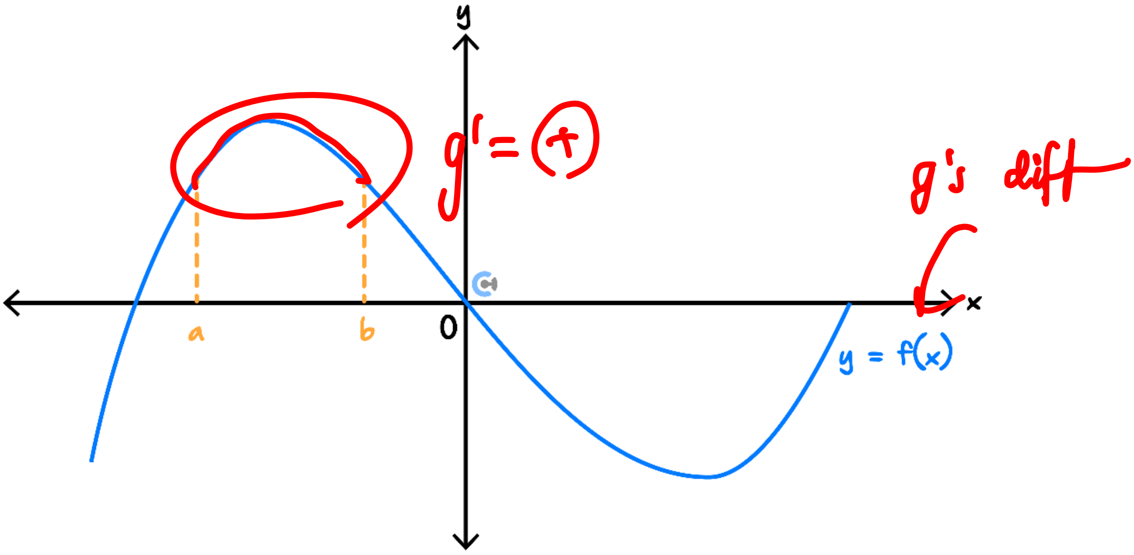
$\therefore$ local max

Section G: Extension Exam 2 (17 Marks)

INSTRUCTION: 17 Marks. 24 Minutes Writing.

Question 16 (1 mark)

The graph of the function with equation $y = f(x)$ is shown below.



Let g be the function such that $g'(x) = f(x)$.

On the interval (a, b), the graph of g will:

A. Have a negative gradient.

B. Have a positive gradient.

C. Have a local minimum value.

D. Have a local maximum value.

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Question 17 (1 mark)

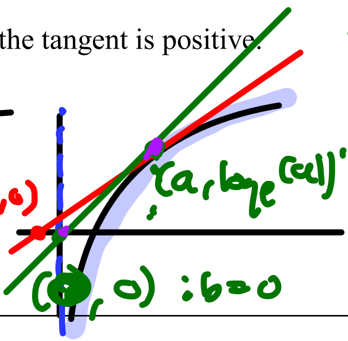
The tangent to the graph of $y = \log_e(x)$ at the point $(a, \log_e(a))$ crosses the x-axis at the point $(b, 0)$ where $b < 0$. Which of the following is **false**?

A. The gradient of the tangent is positive.

B. $a > e$ True

C. $1 < a < e$

D. $a > 0$



$(b,0)$

$(a, \log_e(a))$

$(0,0)$: $b=0$

tangentLine$(\log_e(x), x, a)$

$y =$ ______

sub $(0,0)$

$a = e$.

Gradient!

$f'(a) =$

$\frac{1}{a} = \frac{\log_e(a) - 0}{a - 0}$

$1 = \log_e(a)$

$a = e$.

Question 18 (1 mark)

A function $f(x)$ satisfies $f'(x) = xf(x)$ for all $x \in \mathbb{R}$ with $f(0) = 2$. Which one of the following statements is false?

A. $f(x)$ is increasing for $x > 0$.

B. $f(x)$ has a local maximum at $x = 0$.

C. $f(x)$ satisfies $f(x) = Ce^{x^2/2}$ for some constant C.

D. $f(x)$ grows faster than any polynomial function as $x \to \infty$.

$\underbrace{f'(0) = 0 \cdot f(0) = 0 \times 2 = 0}_{\text{Stationary Pt}}$

$f''(x) = 1 \cdot f(x) + x \cdot f'(x)$

$f''(0) = f(0) + 0 = 2 =$ ☺

Question 19 (1 mark)

Let $p(x)$ be a degree seven polynomial with seven real roots. What is the minimum amount of real roots that $q(x)$ can have if $q'(x) = p(x)$?

A. 0

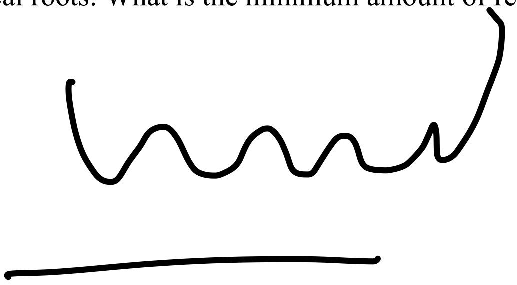
B. 1

C. 7

D. 8

$q' =$ deg 7

$q =$ deg 8 :



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Question 20 (13 marks)

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = 3^{x+2} - 4$.

No translate

a. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x, ay + d)$ maps the graph of $y = 3^x$ onto the graph of f.

State the values of a and d. (2 marks)

$3^x \cdot 3^2 - 4$

$= 9 \cdot 3^x - 4$ y

$a = 9$

$b = -4$

b. Find the rule and domain for f^{-1}, the inverse function of f. (2 marks)

Let $y = f^{-1}(x)$.

$f^{-1}(x) = \log_3(x+4) - 2$

Dom f^{-1} = Rge $f = (-4, \infty)$

Exam tip

"When a q seems redundant easy

⇒ Hint for next question"

c. Find the gradient of f and the gradient of f^{-1} at $x = -1$. (2 marks)

$f'(-1) = 3\log_e(3)$

$(f^{-1})'(-1) = \frac{1}{3\log_e(3)}$

Gradients are reciprocal.

 $\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

d. The graphs of f and f^{-1} intersect each other at two points. Let θ_1 be the angle that f and f^{-1} make at their first point of intersection and let θ_2 be the angle that f and f^{-1} make at their second point of intersection.

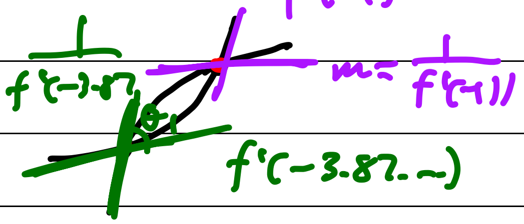
Find $\theta_1 - \theta_2$. Give your answer in degrees correct to two decimal places. (3 marks)

1) Find intersection

$f(x) = x$ (lucky)

$x = -3.87, -1$

$(-3.87, -3.87)$ & $(-1, -1)$



$\theta_2 = 56.243$

2) $\tan(\theta_1) = \left|\dfrac{f'(-3.87...) - \frac{1}{f'(-3.87...)}}{1 + 1}\right|$

$17.76°$

$\theta_1 = 74.007°$

The function f is mapped to the function g when it undergoes a dilation by factor k from the x-axis, where $k > 0$.

⇒ 2 week :)

e. Find the value(s) of k such that g and g^{-1} intersect each other exactly once. Give your answer(s) correct to three decimal places. (2 marks)

$f(x)$ intersect $y = x$ "once"

→ Intersect w same gradient "tangential"

→ $f(x) = x$: intersect

$f'(x) = 1$: Same

$x = 0.729, -1.696$

$k = 0.045, 0.051$

* Important "

↳ Sketch & "see"

↳ $k \times f(x)$. "Sliders"

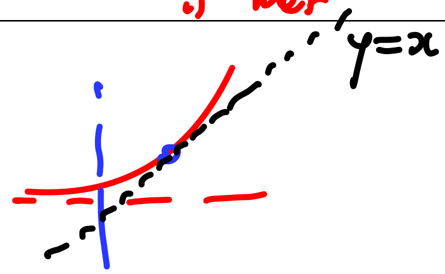
Casio: [slider]

Math: Manipulate[...]

↳ Play around w k value

= Notice when the condition

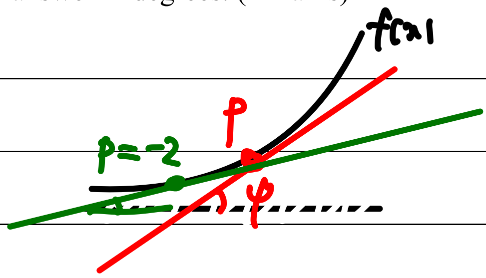
is met



f. Let $p \in (-2, \infty)$. The tangent drawn to the graph of f at $x = p$, makes an angle of $\varphi°$ with the positive x-axis. State the range of values that φ can take. Give your answer in degrees. (2 marks)

$\tan^{-1}(f'(-2)) = \tan^{-1}(\ln(3))$

$\varphi \in (\tan^{-1}(\ln(3)), 90)$



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