CONTOUREDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Functions & Relations [0.1]
Workshop Solutions

Section A: Recap

Sub-Section: Maximal Domains

Starting with a domain!

Maximal Domain

- **Definition**: The largest possible set of input values (elements of the domain) for which the function is well-defined.
- **Three Important Rules:**

Functions	Maximal Domain
$\sqrt{z}$	$z \geq 0$
$\log(z)$	$z > 0$
$\frac{1}{z}$	$z \neq 0$

Steps

1. Find the restriction of the inside.
2. Sketch the graph if needed.
3. Solve for domain.

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Sub-Section: Domain of Sum, Difference and Product of Functions

What about a domain of the sum of two functions?

Sums, Differences and Products of Functions

- **Rules:**

$$(f+g)(x) = \boxed{f(x)+g(x)}$$

$$(f-g)(x) = \boxed{f(x)-g(x)}$$

$$(f\times g)(x) = \boxed{f(x)\times g(x)}$$

- **Idea:**

$$\textit{Domain of sum or product of two functions} = \boxed{\textit{Intersection}} \textit{ of the two domains}$$

- **Steps:**

1. Find the domain of each function.
2. Find the intersection (draw a number line if needed).

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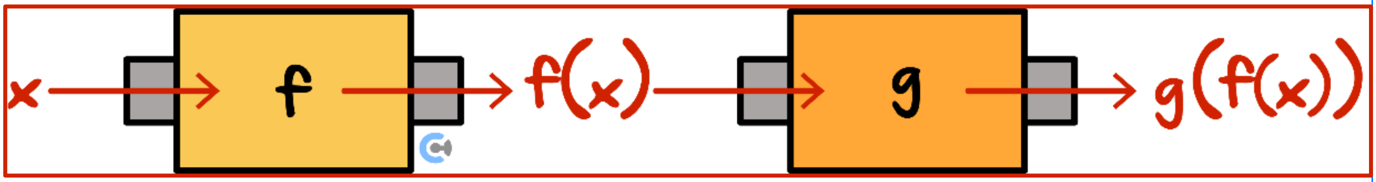
Sub-Section: Basics of Composition

What was the "composition" of functions?

Composite Functions



- **Definition**: A series of functions.
- **Representation of the above:**

$$y = g(f(x)) = g \circ f(x)$$

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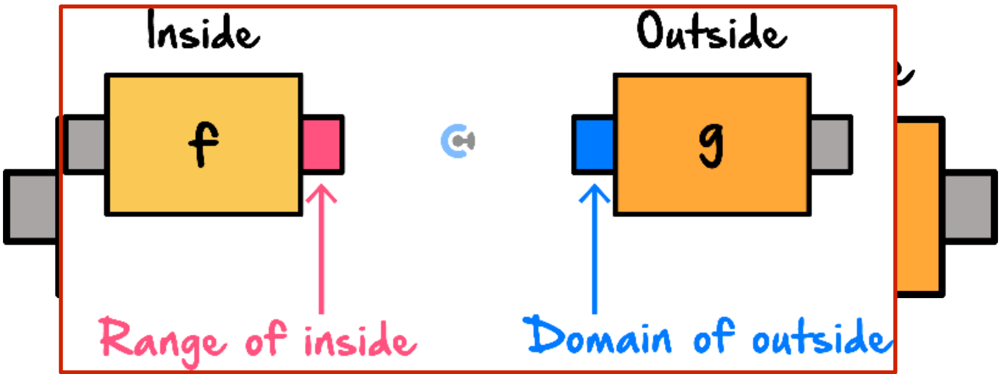
Sub-Section: Validity of Composite Functions

Did composite functions work all the time?

Validity of Composite Functions



- Output of $f(x)$: Range of Inside *(Label Above)*
- Input of $g(x)$: Domain of outside *(Label Above)*
- Composite Function is only valid if:

$$\textit{Range of Inside} \subseteq \textit{Domain of Outside}$$

- Acronym:

RIDO = Range of inside, domain of outside

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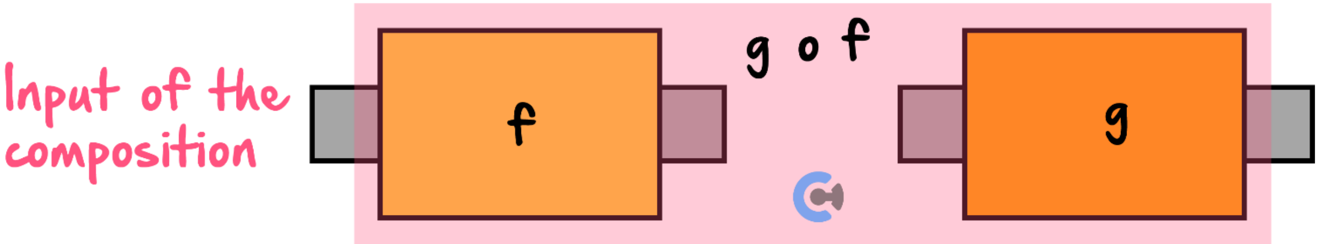
Sub-Section: Domain of Composite Functions

How did we find the domain of a composite function?

Domain of Composite Functions



$$\textbf{Domain of Composite} = \textbf{Domain of Inside}$$

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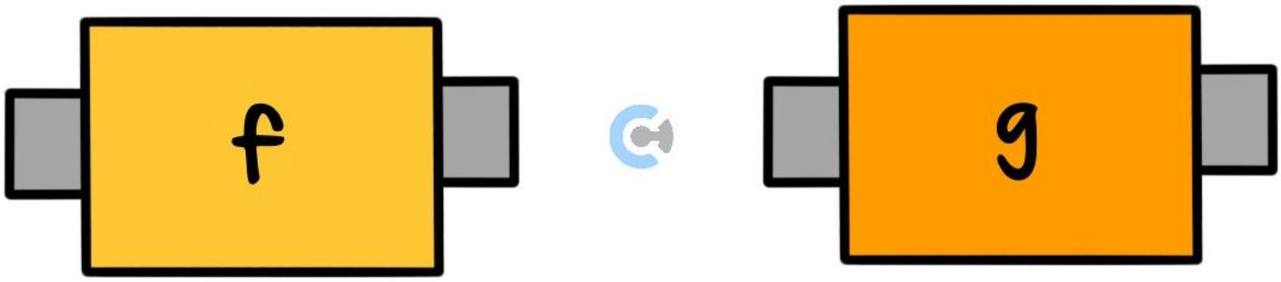
Sub-Section: Range of Composite Functions

Range of the Composite Functions



Range of Composite ⊆ Range of the Outside

- **Finding the range of composition function**: Use the domain and the rule, just like another function.

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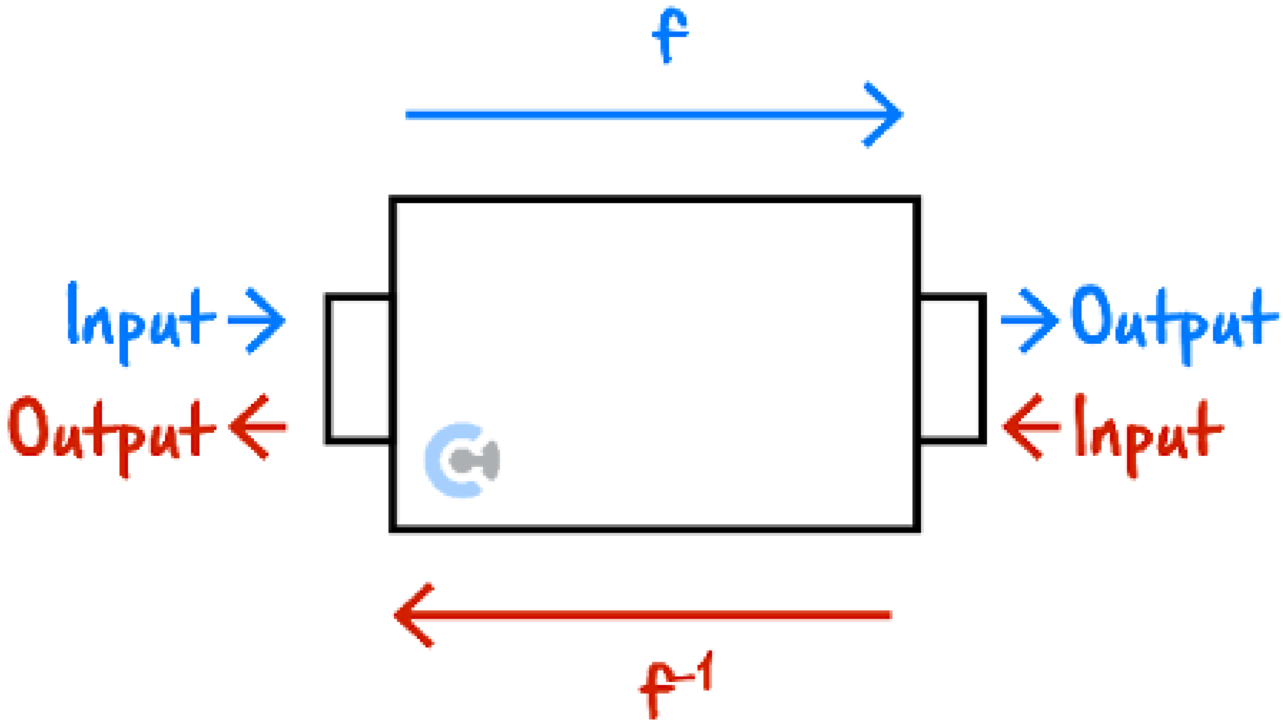
Sub-Section: Basics of Inverses

What did "Inverse" mean?

Inverse Relation

- **Definition**: Inverse is a relation which does the opposite.



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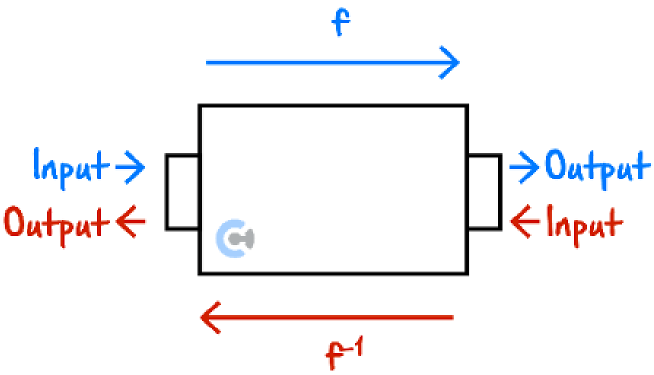
Sub-Section: Swapping x and y

Is there a better way of solving for an inverse relation?

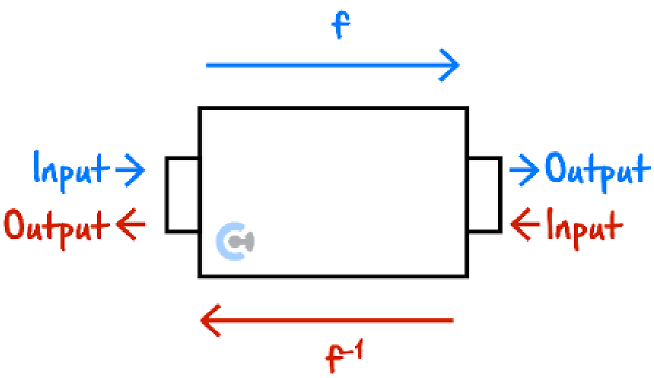
Solving for an Inverse Relation

- Swap x and y.



NOTE: $f(x) = y$.

Domain and Range of Inverse Functions



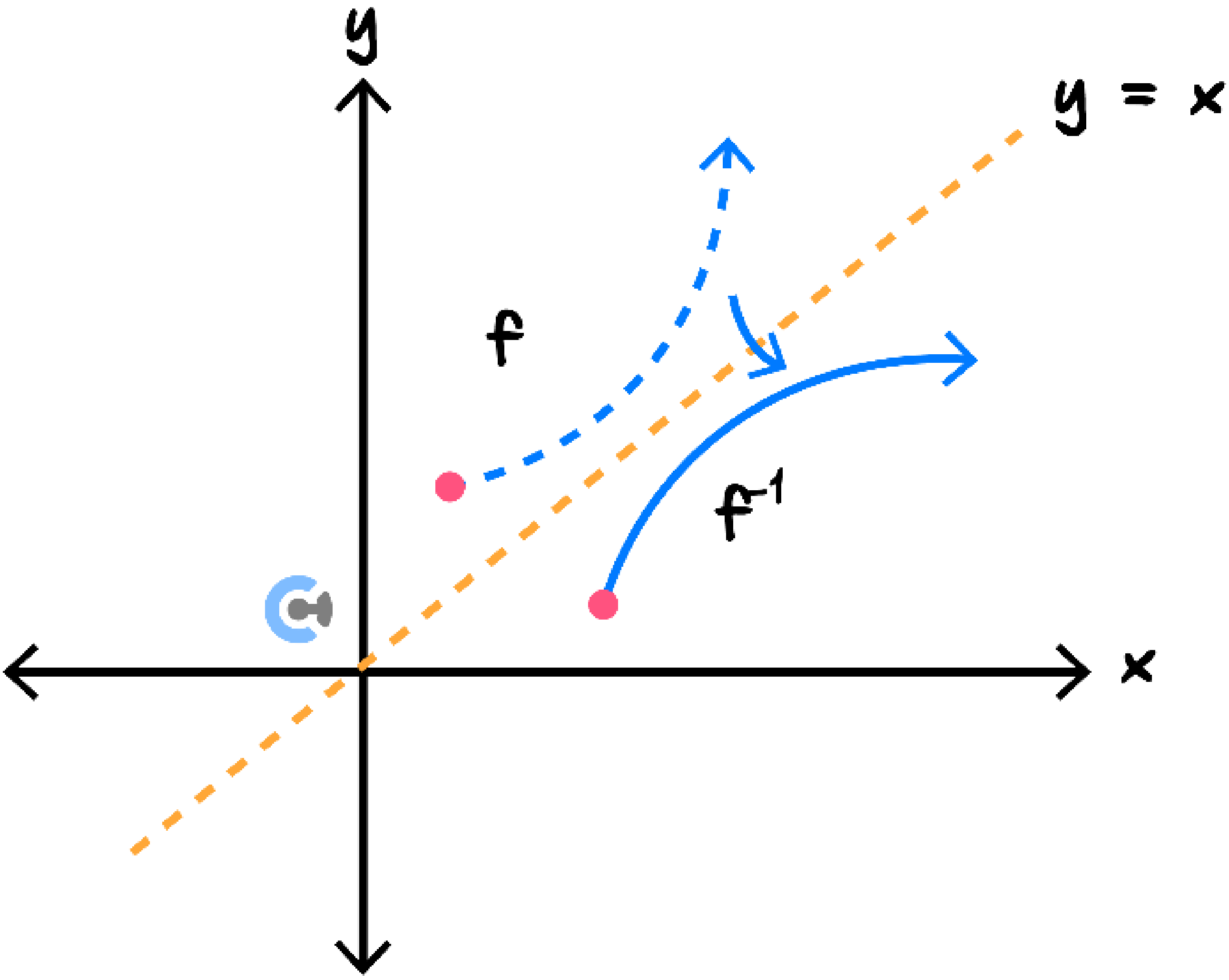
$Dom\ f^{-1} =$ $Ran\ f$

$Ran\ f^{-1} =$ $Dom\ f$

Sub-Section: Symmetry Around $y = x$

Why does this happen?

Symmetry of Inverse Functions



- Inverse functions are always symmetrical around $y = x$.

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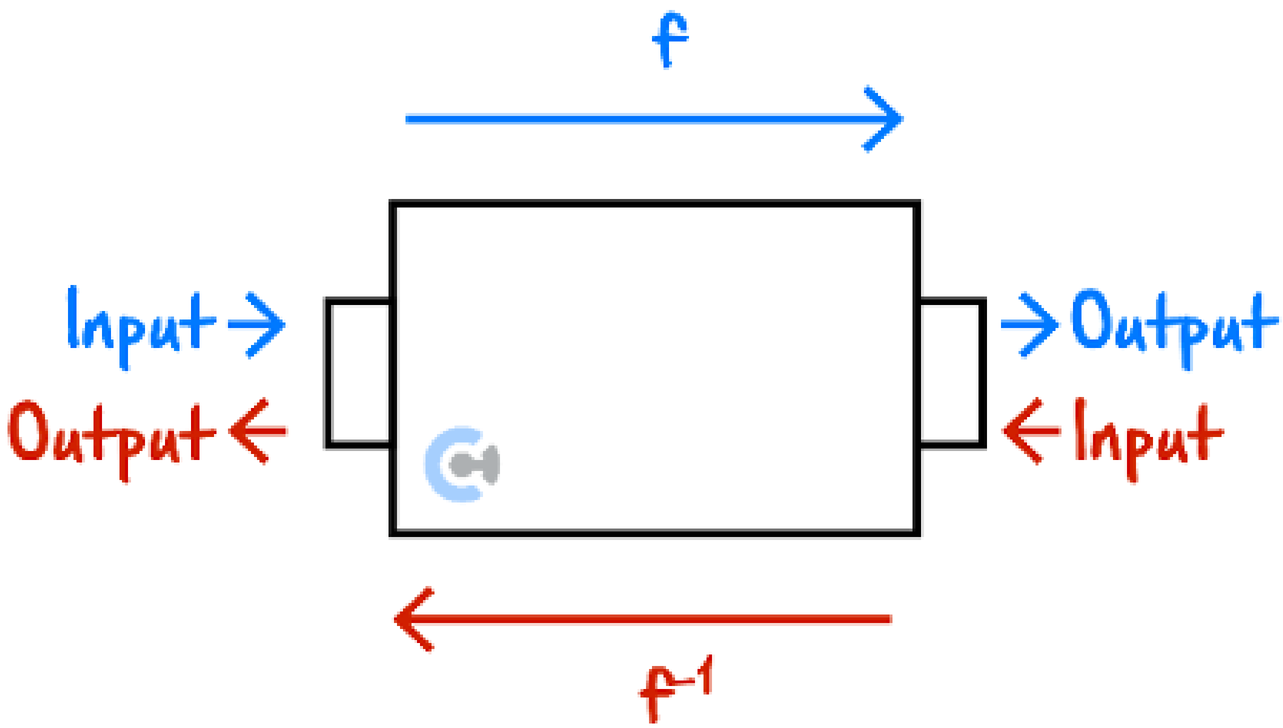
Sub-Section: Validity of Inverse Function

Does an inverse function always exist?

Validity of Inverse Functions



- Requirement for Inverse Function:

f needs to be **1:1**.

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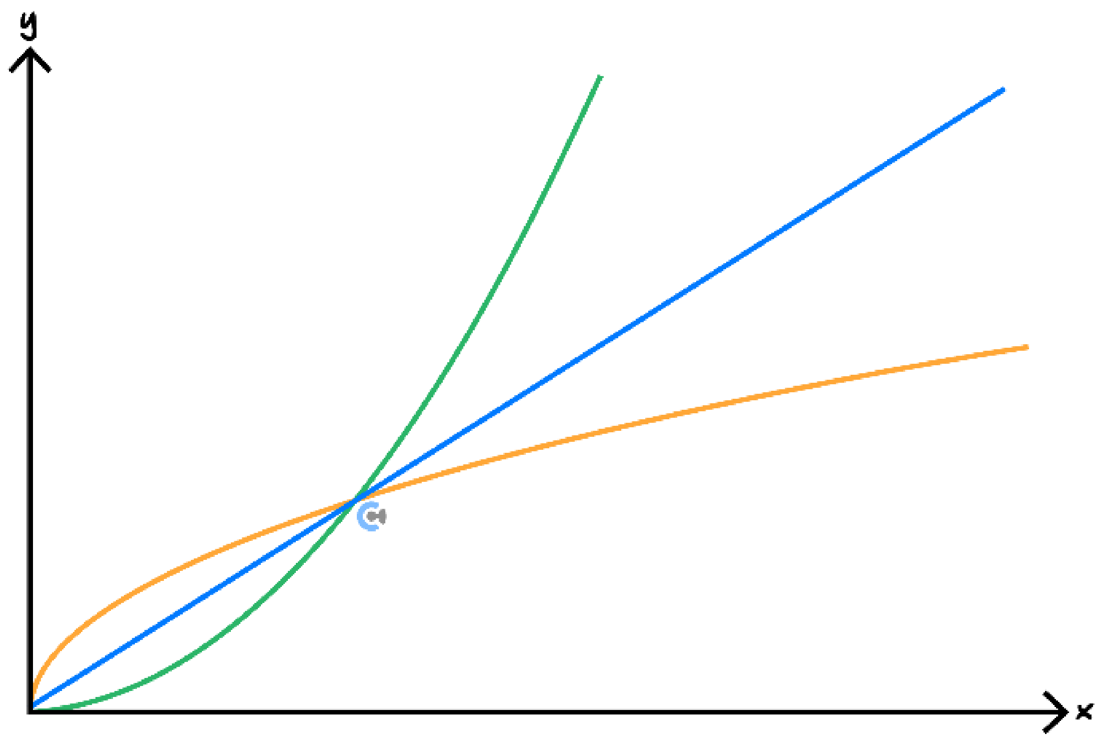
Sub-Section: Intersection Between Inverses

Where do inverses meet?

Intersection Between a Function and its Inverse



- Equate with $y = x$ instead.

$$f(x) = x \text{ OR } f^{-1}(x) = x$$

- We cannot do this when the function is decreasing function.

NOTE: This only works for an increasing function.

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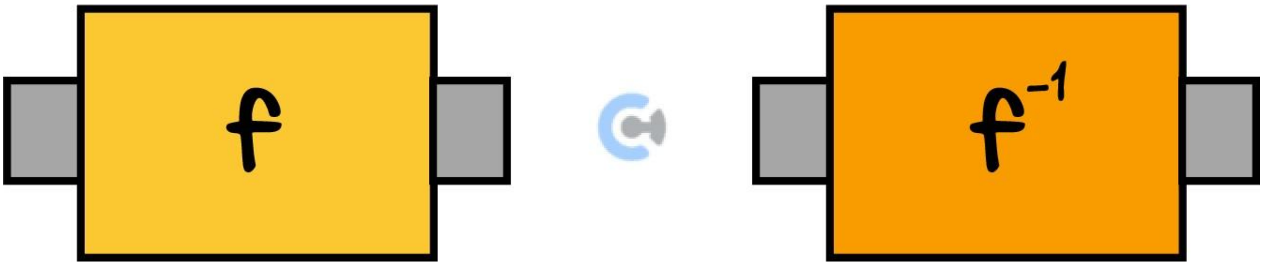
Sub-Section: Composition of Inverses

Composition of Inverse Functions



$f \circ f^{-1}(x) =$ x $for\ all\ x \in$ $dom\ f^{-1}$

$f^{-1} \circ f(x) =$ x $for\ all\ x \in$ $dom\ f$

NOTE: Domain = Domain of Inside.

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Section B: Warm Up (5 Marks)

INSTRUCTION: 5 Marks. 8 Minutes Writing.

Question 1 (5 marks)

Consider the function $f(x) = \sqrt{x+2}$, where f is defined over its maximal domain.

a. State the maximal domain of $h(x) = f(x) + \frac{1}{f(x)}$. (1 mark)

$$(-2, \infty)$$

b. Define the inverse function f^{-1}. (2 marks)

$$f^{-1}: [0, \infty) \to R, f^{-1}(x) = x^2 - 2$$

c. Find the point of intersection between $f(x)$ and $f^{-1}(x)$. (2 marks)

$$(2,2)$$

d. Find the rule and domain for $f^{-1}(f(x))$.

$f^{-1}(f(x)) = x$ for $x \in [-2, \infty)$

e. Let $h(x) = x^2 - 11$, explain why the composition $f(h(x))$ is not valid.

Range of $g = [-11, \infty)$ is not a subset of domain of $f = [-2, \infty)$.

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Section C: Exam 1 (20 Marks)

INSTRUCTION: 20 Marks. 26 Minutes Writing.

Question 2 (5 marks)

Let $f(x) = \sqrt{2x+6} + 4$, where f is defined over its maximal domain.

a. State the maximal domain of f. (1 mark)

$[-3, \infty)$

b. Define the inverse function f^{-1}. (2 marks)

Solution: Rearrange to isolate y

$$x = \sqrt{2y+6} + 4$$
$$(x-4)^2 = 2y + 6$$
$$y = \frac{1}{2}(x-4)^2 - 3$$

Now dom f^{-1} = ran $f = [4, \infty)$. Therefore,

$$f^{-1} : [4, \infty) \to \mathbb{R},\ f^{-1}(x) = \frac{1}{2}(x-4)^2 - 3$$

c. Find the point of intersection between $f(x)$ and $f^{-1}(x)$. (2 marks)

Solution: Intersection will occur along the line $y = x$, therefore we may solve

$$\sqrt{2x+6} + 4 = x$$
$$2x + 6 = (x-4)^2$$
$$x^2 - 10x + 10 = 0$$
$$(x-5)^2 = 15$$
$$x - 5 = \pm\sqrt{15}$$
$$x = 5 \pm \sqrt{15}$$

$x = 5 - \sqrt{15} < 4$ so f^{-1} is not defined. Therefore the only point of intersection is

$$(5+\sqrt{15}, 5+\sqrt{15})$$

Question 3 (8 marks)

Consider the functions, $f: (0, \infty) \to R, f(x) = \log_3(x+1)$ and $g: [-3, \infty) \to R, g(x) = x^2 + 26$.

a. Find the rule for h, where $h(x) = f\big(g(x)\big)$. (1 mark)

$h(x) = \log_3(g(x)+1) = \log_3(x^2+27)$

b. State the domain of h. (1 mark)

$\text{dom } h = \text{dom } g = [-3, \infty)$

c. State the range of h. (2 marks)

Solution: $x^2 + 26$ is minimial when $x = 0$. $h(0) = \log_3(27) = 3$. Therefore, ran $h = [3, \infty)$

Let $k: (-\infty, 0] \to R, k(x) = \log_2(x^2 + 16)$.

d. Define the function k^{-1}. (3 marks)

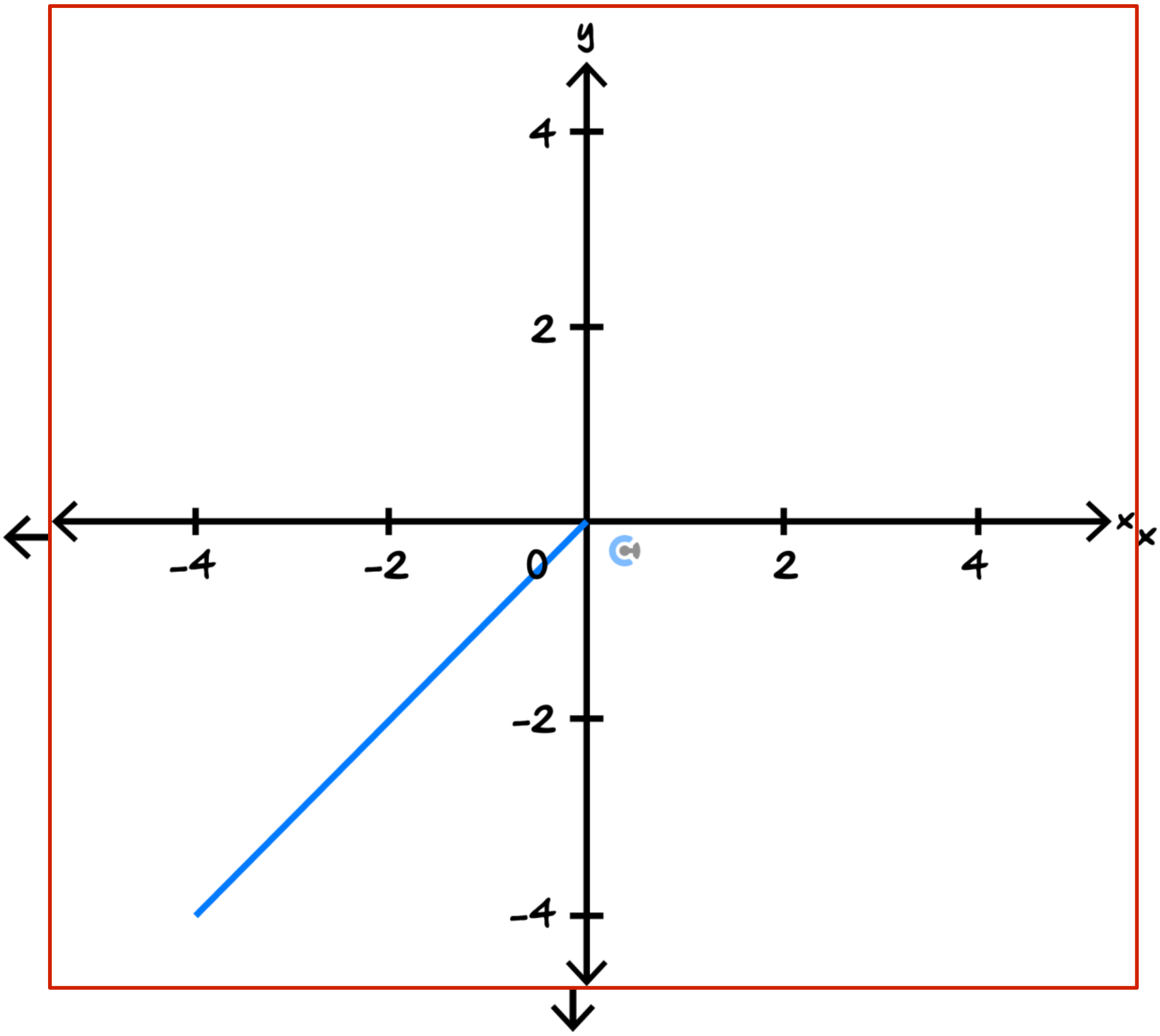
Solution: Rearrange the equation below to find y

$$x = \log_2(y^2+16)$$
$$2^x = y^2 + 16$$
$$y = \pm\sqrt{2^x - 16}$$

Now ran $k^{-1} = \text{dom } k = (-\infty, 0]$, therefore

$$k^{-1} : [4, \infty) \to \mathbb{R},\ k^{-1}(x) = -\sqrt{2^x - 16}$$

e. On the axes below, sketch the graph of $y = k^{-1}(k(x))$. (1 mark)



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Question 4 (7 marks)

Let $f(x) = 2^{-x}$ and $g(x) = x^2 - 2x + 2$.

a.

i. Write down the rule for $f\big(g(x)\big)$. (1 mark)

Solution: $f(g(x)) = 2^{-x^2+2x-2}$

ii. Find the range of $f\big(g(x)\big)$. (1 mark)

Solution: Note that $g(x) = (x-1)^2 + 1$. Larger x values make $f(x)$ smaller. ran $g = [1, \infty)$. Therefore,

$$\text{ran } f(g(x)) = \left(0, \frac{1}{2}\right]$$

b. Consider the function $h: (-\infty, a] \to \mathbb{R}, h(x) = f\big(g(x)\big)$.
Find the largest value of a such that h is a one-to-one function. (1 mark)

Solution: $f(x)$ is one-to-one and $g(x)$ is symmetric about $x = 1$. Therefore $a = 1$

c. Define the inverse function, h^{-1}. (2 marks)

Solution: dom h^{-1} = ran $h = \left(0, \frac{1}{2}\right]$ and ran $h = (-\infty, 1]$.

$$x = 2^{-(y-1)^2-1}$$
$$-\log_2(x) = (y-1)^2 + 1$$
$$(y-1)^2 = -1 - \log_2(x)$$
$$y = 1 \pm \sqrt{-1 - \log_2(x)}$$

Therefore,

$$h^{-1}: \left(0, \frac{1}{2}\right] \to \mathbb{R},\ h^{-1}(x) = 1 - \sqrt{-1 - \log_2(x)}$$

d. Let $k: [b, \infty) \to \mathbb{R}, k(x) = g(f(x))$.
Find the smallest value of b such that k^{-1} exists. (2 marks)

Solution: k needs to be one-to-one. g is one-to-one for $(-\infty, 1]$ or $[1, \infty)$ and f is one-to-one for all $\mathbb{R}$. So find the smallest value of b such that the range of f is a subset of one of these intervals.
The range of f when restricted to $[b, \infty)$ is $(0, 2^{-b}]$
So fitting inside the interval $[1, \infty)$ is not possible.
Want smallest value of b such that $(0, 2^{-b}] \subseteq (-\infty, 1]$
$b = 0$ is the smallest value.

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Section D: Technology Warmup

INSTRUCTION: 5 Minutes Writing.

Calculator Commands: Finding the domain and range

TI

- domain $(f(x), x)$, $f\ Min$ and $Fmax$

Define $f(x)=\sqrt{9-x^2}$	*Done*
domain$(f(x),x)$	$-3\le x\le 3$
fMin$(f(x),x)$	$x=-3$ or $x=3$
fMax$(f(x),x)$	$x=0$
$f(3)$	0
$f(0)$	3

TI-UDF

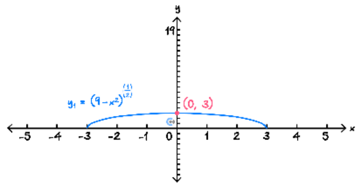
Analyse a Function: Find intercepts, critical points and their nature, maximal domain, asymptote.	analyse(<function>, <variable>) analysed(<function>, <variable>, <lower bound>, <upper bound>)

$$analysed\left(\frac{x^4-2\cdot x^3-3\cdot x^2+3\cdot x+1}{-3\cdot x^3-6\cdot x^2-x+1},x,-5,5\right)$$

- Start Point: $\left[-5 \quad \frac{262}{77}\right]$
- End Point: $\left[5 \quad \frac{-316}{529}\right]$
- Maximal Domain:
  $x\neq-1.68469$ and
  $x\neq-0.629579$ and
  $x\neq0.314273$ and
  $-5\le x\le 5$

Casio Classpad

- Graph the function and use G-Solve to find min and max values for the range.



Mathematica

```
In[127]:= f[x_] := √(9 - x^2)
In[128]:= FunctionDomain[f[x], x]
Out[128]= -3 ≤ x ≤ 3

In[129]:= FunctionRange[f[x], x, y]
Out[129]= 0 ≤ y ≤ 3
```

Mathematica UDF :

- FInfo [f [x], $\{x, x\ min, x\ max\}, y$]

Returns useful information about a function, including derivative, domain, range, period, horizontal intercepts, vertical intercepts, stationary points, inflexion points, left and sided asymptotes, oblique asymptotes and vertical asymptotes.

$$\text{FInfo}\left[\frac{x^2-1}{x\left(x^2-3\right)}, \{x, -\text{Infinity}, \text{Infinity}\}, y\right]$$

The function is $\frac{x^2-1}{x\left(x^2-3\right)}$

The derivative is $-\frac{x^4+3}{x^2\left(x^2-3\right)^2}$

Domain: $x<-\sqrt{3} \vee -\sqrt{3}<x<0 \vee 0<x<\sqrt{3} \vee x>\sqrt{3}$

Range: $y\in\mathbb{R}$

Period: 0

Horizontal Intercepts: $\{-1, 1\}$

Vertical Intercepts: None

Stationary points: $\{\}$

Inflexion points: {{-0.871…, -0.123…}, {0.871…, 0.123…}}

Left sided asymtote: $y=0$

Right sided asymtote: $y=0$

Oblique asymtote: $\{0\}$

Vertical asymtote: $\{x=0, x=-\sqrt{3}, x=\sqrt{3}\}$

Calculator Commands: Finding the composite function

TI

Define $f(x)=\ln(x)$ — *Done*

Define $g(x)=x^2+3$ — *Done*

$f(g(x))$ — $\ln(x^2+3)$

CASIO:

```
define f(x) = ln(x)
                                done
define g(x) = x^2+3
                                done
f(g(x))
                                ln(x^2+3)
```

Mathematica

```
In[141]:= f[x_] := Log[x]
In[142]:= g[x_] := x^2 + 3
In[143]:= f[g[x]]
Out[143]= Log[3 + x^2]
```

Calculator Commands: Finding the inverse function

TI

Define $f(x)=x^2+4\cdot x+9$ — *Done*

$\text{solve}(f(y)=x,y)$ — $y=-(\sqrt{x-5}+2)$ or $y=\sqrt{x-5}-2$

CASIO:

```
define f(x) = x^2+4x+9
                                done
solve(f(y)=x, y)
                                {y=−√(x−5)−2, y=√(x−5)−2}
```

Mathematica

```
In[154]:= f[x_] := x^2 + 4 x + 9
In[155]:= Solve[f[y] == x, y]
Out[155]= {{y → -2 - √(-5 + x)}, {y → -2 + √(-5 + x)}}
```

NOTE: It doesn't tell us which branch is correct.

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Question 5 Tech-Active.

Let $f(x) = \sqrt{x-2}$ and $g(x) = 3x + 4$ be defined on their maximal domains.

Consider the function $h(x) = f\big(g(x)\big)$.

a. Find the rule for $h(x)$.

Define $f(x)=\sqrt{x-2}$	*Done*
Define $g(x)=3\cdot x+4$	*Done*
Define $h(x)=f(g(x))$	*Done*
$h(x)$	$\sqrt{3\cdot x+2}$

b. Find the domain of $h(x)$.

domain$(h(x),x)$	$\frac{-2}{3}\leq x<\infty$

c. Define h^{-1}, the inverse function of h.

solve$(x=h(y),y)$	$y=\frac{x^2-2}{3}$ and $x\geq 0$

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Section E: Exam 2 (24 Marks)

INSTRUCTION: 24 Marks. 30 Minutes Writing.

Question 6 (1 mark)

The graph of $y = x^2 - 2ax$ has a range of $[-16, \infty)$, where a is a positive constant. The value of a is:

A. 2

B. 4

C. 8

D. 16

Question 7 (1 mark)

The domain of the inverse of $\{(1, -4), (2, -3), (3, -2), (4, -1)\}$ is D. Which of the following statements is true?

A. D is $\{x: -1 < x < 4\}$

B. D is $\{x: 1 < x < 4\}$

C. D is $\{-4, -3, -2, -1\}$

D. D is $\{1, 2, 3, 4\}$

Question 8 (1 mark)

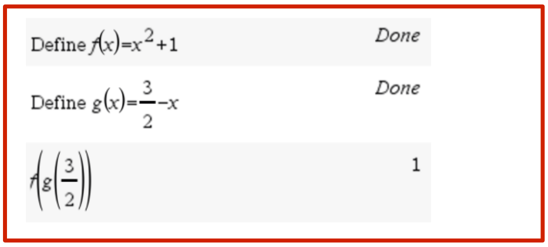
The functions f and g are such that $f(x) = x^2 + 1$ and $g(x) = \frac{3}{2} - x$.The value of $f\left(g\left(\frac{3}{2}\right)\right)$ is:

A. $\frac{1}{4}$

B. 2

C. 1

D. $-\frac{1}{4}$



Question 9 (1 mark)

The domain of the composite function $(f \ o \ g)$ where $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{6}{5-x}$ is:

A. R

B. $R\backslash\{-1\}$

C. $R\backslash\{5\}$

D. $R\backslash\{-1, \ 5\}$

Question 10 (1 mark)

Which of the following functions does not have an inverse function?

A. $f: R \to R, f(x) = 2x - 7$

B. $f: [0, \infty) \to R, f(x) = x^2 + 3$

C. $h: R \to R, h(x) = x^3$

D. $g: [0, \infty) \to R, g(x) = (x-1)^2 + 4$

Question 11 (1 mark)

The function f and its inverse f^{-1} are one-to-one for all values of x. If $f(a) = b$, $f(b) = c$, $f(c) = d$, then $f^{-1}(c)$ is equal to:

A. a

B. b

C. c

D. d

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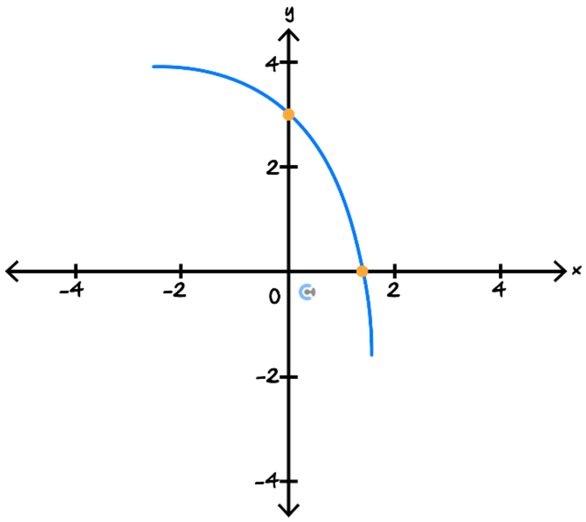
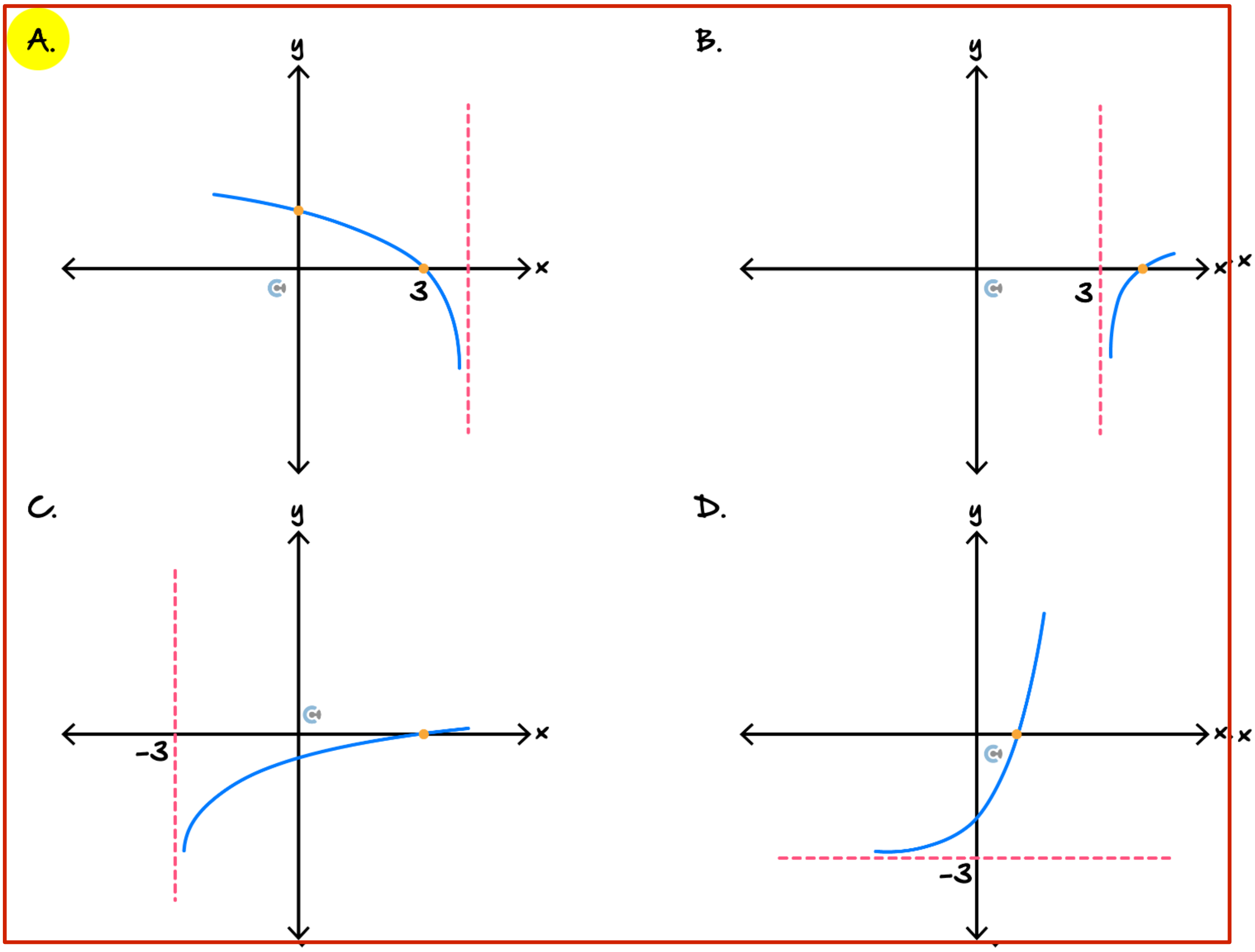
Question 12 (1 mark)

The graph of the function $f(x) = 4 - e^x$ is given below.



Which of the following will represent the inverse function f^{-1}?



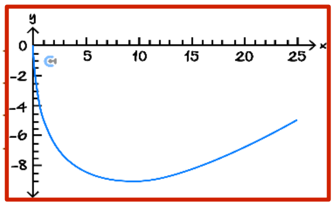
Question 13 (8 marks)

Let $f: [0, \infty) \to R, f(x) = \sqrt{x}$ and $g(x) = f(x) \times (f(x) - 6)$.

Let $h: [s, \infty) \to R, h(x) = g(x)$.

a. Find the minimum value of s for which the inverse function $h^{-1}(x)$ to exist.

```
f[x_] := √x
g[x_] := f[x] * (f[x] - 6)
Plot[g[x], {x, 0, 25}]
```




```
Minimize[g[x], x]
{-9, {x → 9}}
```

$s = 9$

b. Show that the rule of the inverse function can be written as $h^{-1}(x) = x + 18 + 6\sqrt{9 + x}$. (2 marks)

```
Solve[g[y] == x, y]
Solve: There may be values of the parameters for which some or all solutions are not valid.
{{y → 18 + x - 6 √(9 + x)}, {y → 18 + x + 6 √(9 + x)}}
```

c. State the domain and range of the inverse function $h^{-1}(x)$. (1 mark)

dom h^{-1} = ran h = $[-9, \infty)$

ran h^{-1} = dom h = $[9, \infty)$

d. Let $d(x) = h^{-1}(x) - h(x)$.

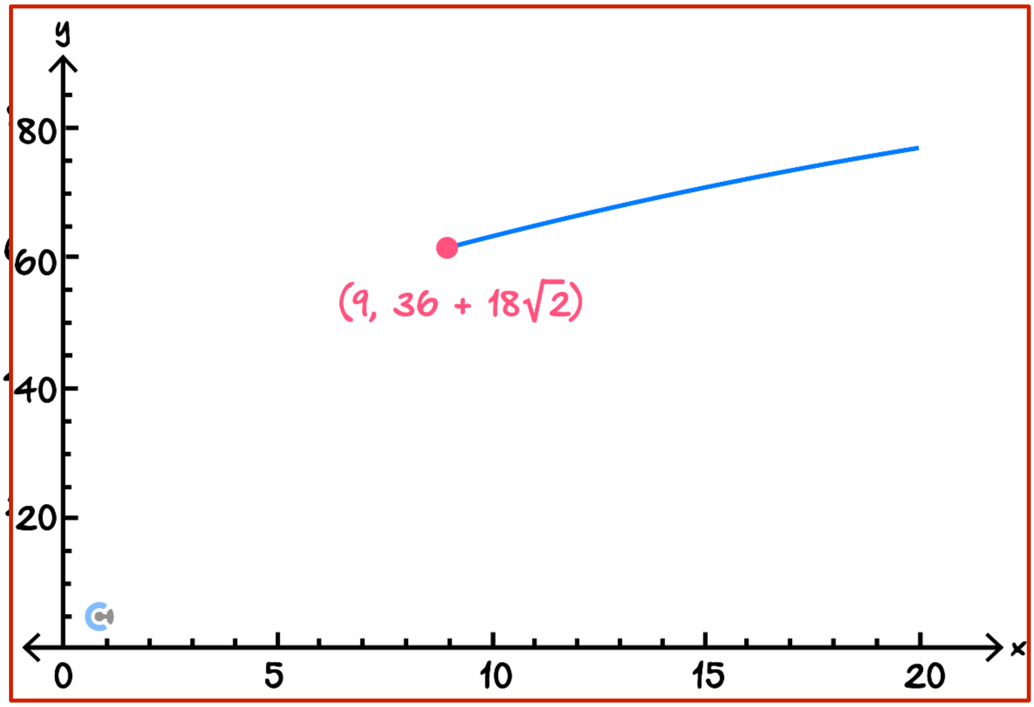
i. Find the maximal domain of d. (1 mark)

$[9, \infty)$

ii. Find the rule of the function $d(x)$. (1 mark)

```
In[89]:= d[x_] := h1[x] - h[x]
In[90]:= d[x] // Simplify
Out[90]= 6 (3 + √x + √(9 + x))
```

e. Sketch the graph of the function $y = d(x)$ on the axes below. Label the endpoint with coordinates. (2 marks)



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Question 14 (9 marks)

The price of a certain rare mineral is modelled by the function:

$$P(t) = at^2 + bt + c, t \geq 0.$$

Where P represents the cost of the mineral in thousands of dollars and t represents the time elapsed since the start of the year in months.

Jenny expects the price to drop from \$35000 to \$ $\frac{115000}{4}$ in 5 months, however, in the long term, she expects the stock price to be \$30000 in 10 months.

a. Solve for values a, b and c which satisfy Jenny's expectations. (2 marks)

Solution: From the information given (remember P is in thousands of dollars) we have that

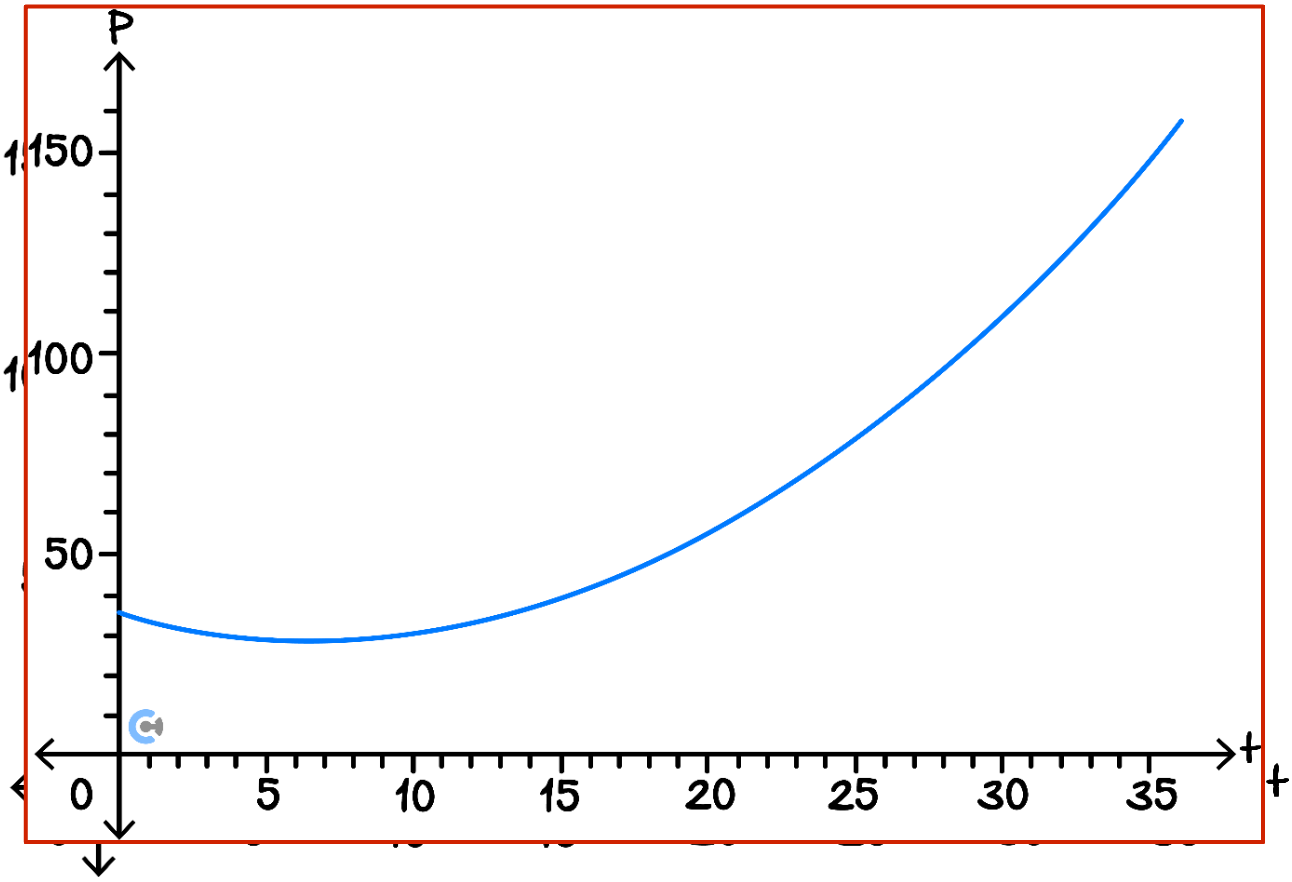
$$p(0) = 35$$
$$p(5) = \frac{115}{4}$$
$$p(10) = 30$$

Solve using CAS to obtain

$$a = \frac{3}{20} \quad \text{and} \quad b = -2 \quad \text{and} \quad c = 35$$

```
In[31]:= p[t_] := a*t^2 + b*t + c
In[32]:= Solve[{p[0] == 35, p[5] == 115/4, p[10] == 30}, {a, b, c}]
Out[32]= {{a -> 3/20, b -> -2, c -> 35}}
```

b. Graph the mineral stock price (in thousands of dollars) for the first 3 years. (2 marks)



As the mineral is very rare, its market price changes often and can be sold for a profit. However, due to fees associated with selling the mineral, the profit earned on the mineral follows the following model.

$$M(t) = \log_e\left(70 - \frac{P(t)}{2}\right)$$

Where M is the profit earned in thousands of dollars and P is the corresponding market price for the mineral at that moment in time. (2 marks)

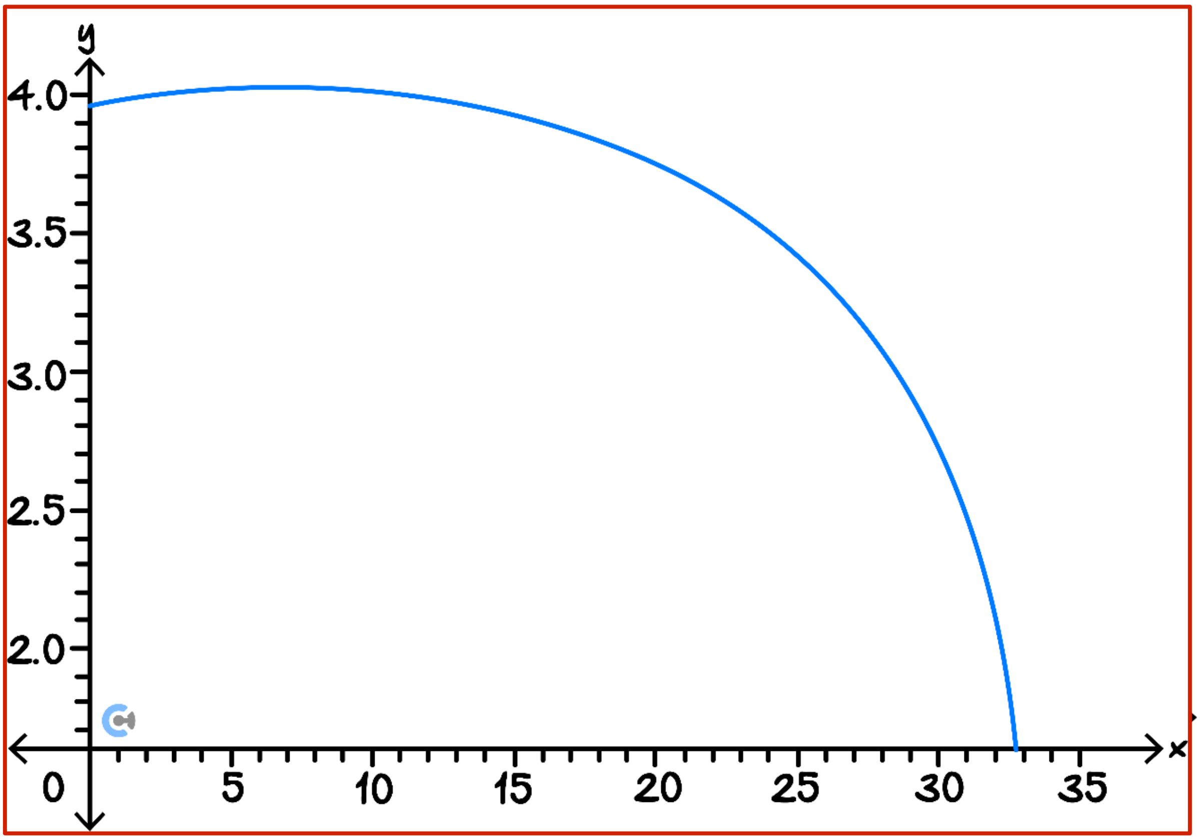
c. When can't Jenny sell her stock for profit? Give your answer in terms of both stock price and months, correct to three decimal places. (2 marks)

Solution: For $M(t)$ to be defined we require that $70 - \frac{P(t)}{2} > 0$.
Jenny can't sell her stock for profit when $M(t)$ is not defined.
$M(t)$ is not defined when $P > 140$ which happens when $t > 33.951$

```
In[38]:= Reduce[p[t] > 140, t] // N
Out[38]= t < -20.6178 || t > 33.9512

In[39]:= FunctionDomain[Log[70 - p[t]/2], t] // N
Out[39]= -20.6178 < t < 33.9512
```

d. On the axes below sketch the graph of $y = M(t)$. (2 marks)



e. Find the maximum profit that Jenny can make correct to the nearest dollar. (1 mark)

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Section F: Extension Exam 1 (15 Marks)

Let's take a BREAK (Extension Stream)!

INSTRUCTION: 15 Marks. 20 Minutes Writing.

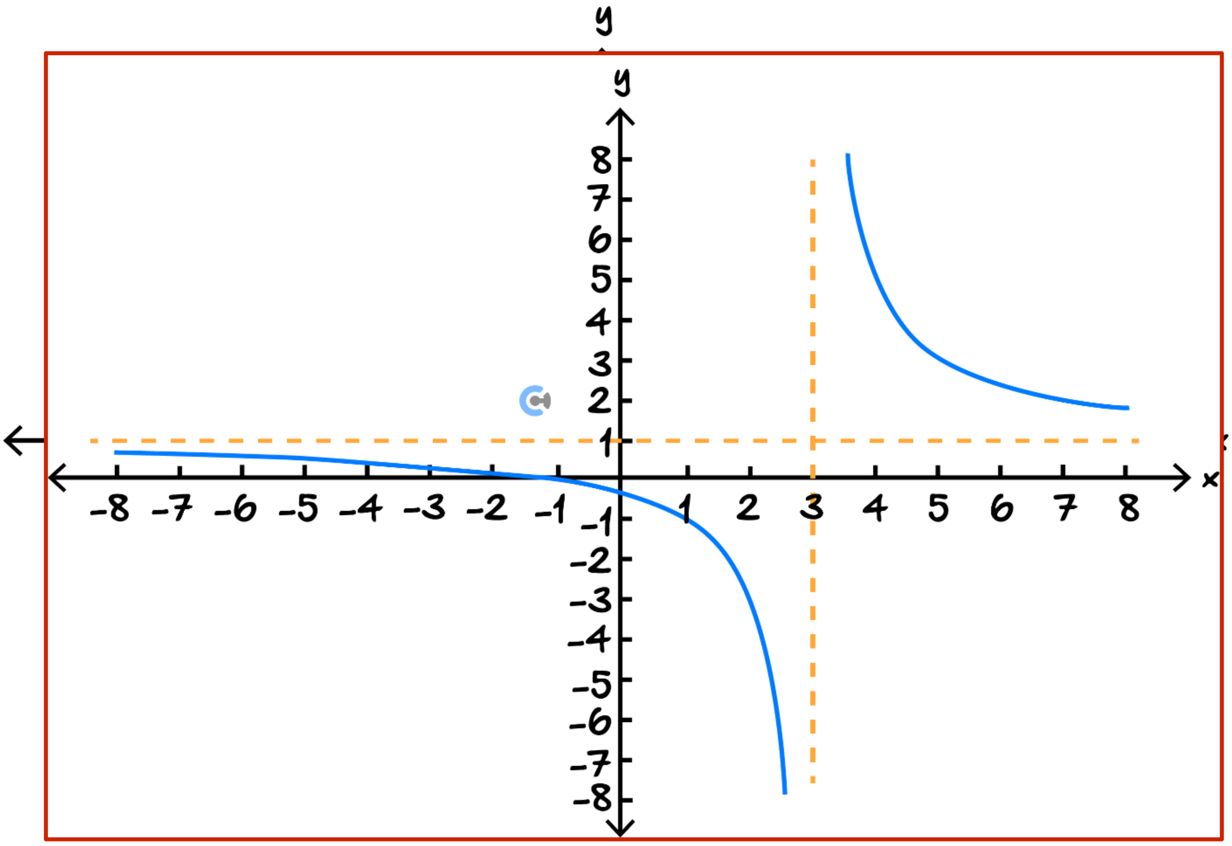
Question 15 (9 marks)

Let $f(x) = \frac{x+1}{x-3}$ be defined on its maximal domain.

a. Write $f(x)$ in the form $A + \frac{B}{x-3}$ for integers A and B. (1 mark)

Solution: $f(x) = 1 + \frac{4}{x-3}$

b. Sketch the graph of $y = f(x)$ on the axes below. Label the coordinates of all axes intercepts and the equations of any asymptotes. (2 marks)



Solution: y intercept: $\left(0, -\frac{1}{3}\right)$ and x intercept: $(-1, 0)$
Asymptotes: $x = 3$ and $y = 1$

c. Find the maximal domain of $g(x) = \sqrt{\frac{x+1}{x-3}} + \log_2(-x^2 + x + 12)$. (2 marks)

Solution: $\{(-\infty, -1] \cup (3, \infty)\} \cap (-3, 4) = (-3, -1] \cup (3, 4)$

Let $h: (a, \infty) \to \mathbb{R}, h(x) = f(x)$, where $a > 3$, be a function.

d. Define h^{-1}, the inverse function of h. (2 marks)

Solution:

$$x = 1 + \frac{4}{y-3}$$
$$x - 1 = \frac{4}{y-3}$$
$$y = \frac{4}{x-1} + 3$$

dom h^{-1} = ran $h = \left(1, \frac{a+1}{a-3}\right)$ The inverse function is

$$h^{-1}: \left(1, \frac{a+1}{a-3}\right) \to \mathbb{R},\ h^{-1}(x) = \frac{4}{x-1} + 3$$

e. Find the smallest value of a such that h and h^{-1} never intersect. (2 marks)

Solution:

$$\frac{x+1}{x-3} = x$$
$$x + 1 = x^2 - 3x$$
$$x^2 - 4x - 1 = 0$$
$$(x-2)^2 = 5$$
$$x = 2 \pm \sqrt{5}$$

Therefore $a = 2 + \sqrt{5}$

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Question 16 (6 marks)

Consider the function $f: \mathbb{R} \to \mathbb{R}, f(x) = 3 - 2^x$.

a. State the range of f. (1 mark)

Solution: $\text{ran } f = (-\infty, 3)$

b. Define f^{-1}, the inverse function of f. (2 marks)

Solution:

$$x = 3 - 2^y$$
$$3 - x = 2^y$$
$$y = \log_2(3 - x)$$

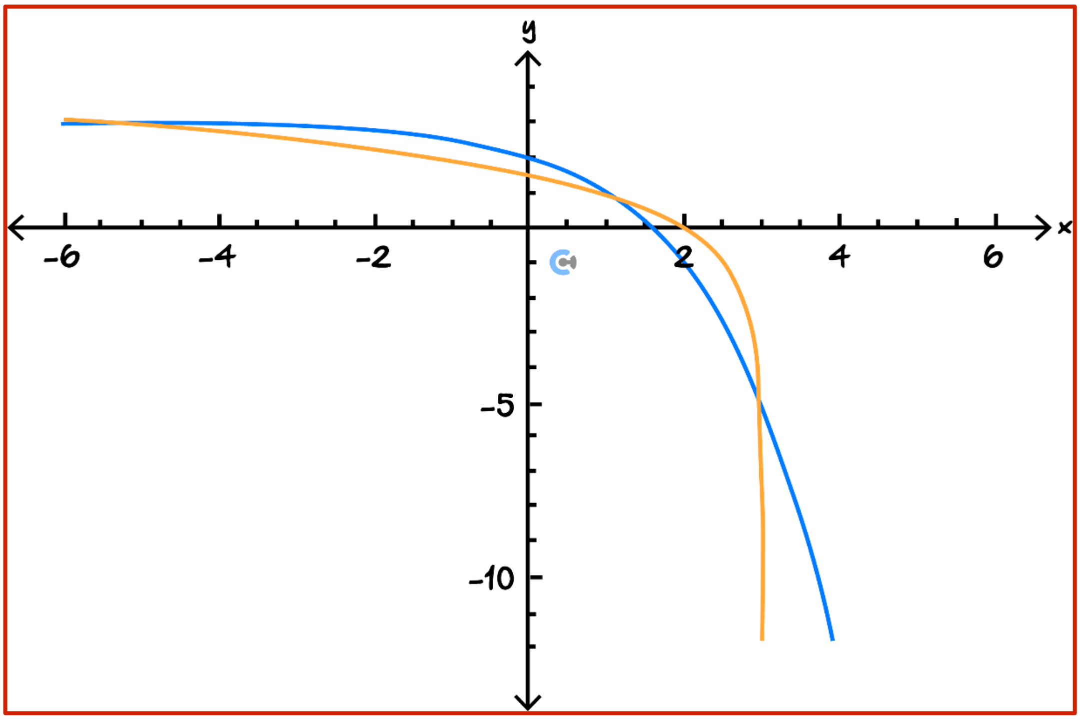
$\text{dom } f^{-1} = \text{ran } f = (-\infty, 3)$

$$f^{-1}: (-\infty, 3) \to \mathbb{R},\ f^{-1}(x) = \log_2(3 - x)$$

c. Find a point of intersection of f and f^{-1} with integer coordinates. (1 mark)

Solution: $(1, 1)$

d. Determine the total number of points of intersection of f and f^{-1}. Justify your answer. (2 marks)



Solution: We have a POI at $(1, 1)$.

Now $f(2) = -1$ and $f^{-1}(2) = 0$ and since $f^{-1}(x)$ has an asymptote at $x = 3$ f must intersect f^{-1} somewhere in $2 < x < 3$.

Similarly $f(0) = 2$ and $f^{-1}(0) = \log_2(3) < 2$ and since $f(x)$ has an asymptote at $y = 3$ f must intersect f^{-1} somewhere in $x < 0$.

So a total of 3 points of intersection. A rough graph should make this clear.

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Section G: Extension Exam 2 (16 Marks)

INSTRUCTION: 16 Marks. 20 Minutes Writing.

Question 17 (1 mark)

If $h: (1,3] \to \mathbb{R}$, where $h(x) = (x-1)^2(x+3)$ and $f: [-1,3) \to \mathbb{R}$, where $f(x) = 1 - x$, then $g = h \times f$ is defined by:

A. $g: (1,3) \to \mathbb{R}$, where $h(x) = -(x-1)^3(x+3)$

B. $g: (1,3] \to \mathbb{R}$, where $h(x) = -(x-1)^3(x+3)$

C. $g: (1,3) \to \mathbb{R}$, where $h(x) = -(x-1)(x+3)^2$

D. $g: [-1,3] \to \mathbb{R}$, where $h(x) = -(x-1)^3(x+3)$

Question 18 (1 mark)

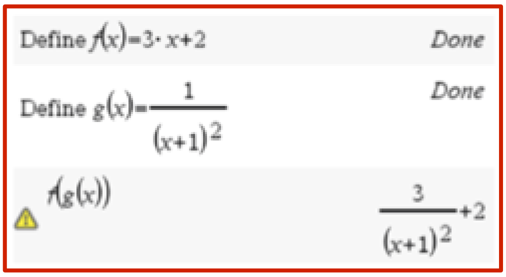
Consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 2$ and $g: \mathbb{R}\backslash\{-1\} \to \mathbb{R}$, $g(x) = \frac{1}{(x+1)^2}$, then the range of $f(g(x))$ is:

A. $\mathbb{R}\backslash\{-1\}$

B. $(1, \infty)$

C. $(2, \infty)$

D. $\mathbb{R}\backslash\{0\}$



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Question 19 (1 mark)

The functions f and g are such that $f(x) = 2x - 1$ and $g(x-1) = \sqrt{2x}$.

Then $f\big(g(x)\big)$ is given by:

A. $2\sqrt{2x} - 1$

B. $2\sqrt{2x} + 1$

C. $\sqrt{2\sqrt{2}x + 1}$

D. $2\sqrt{2}(\sqrt{x+1}) - 1$

Define $f(x)=2\cdot x-1$	*Done*
Define $g(x)=\sqrt{2\cdot x+2}$	*Done*
$g(x-1)$	$\sqrt{2\cdot x}$
$f(g(x))$	$2\cdot\sqrt{2\cdot(x+1)}-1$

Question 20 (1 mark)

Consider the function $f: [-2, \infty) - \mathbb{R}, f(x) = x^2 + 4x + 1$. The function $h = f \text{ o } f^{-1}$ is defined by:

A. $h: [0, \infty) - \mathbb{R}, h(x) = x$

B. $h: [-2, \infty) - \mathbb{R}, h(x) = x$

C. $h: [-3, \infty) - \mathbb{R}, h(x) = x$

D. $h: [-\infty, 2) - \mathbb{R}, h(x) = x$

completeSquare$(x^2+4\cdot x+1,x)$	$(x+2)^2-3$

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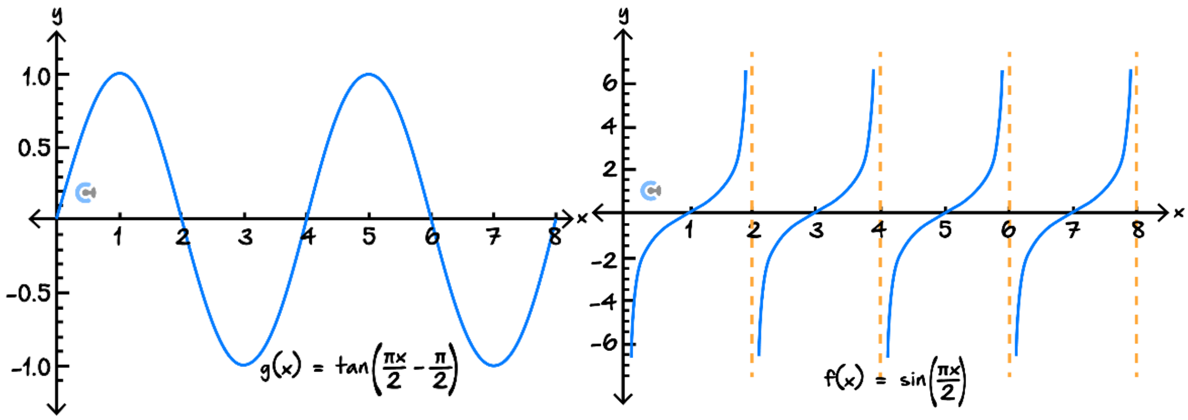
Question 21 (1 mark)

Consider the graphs of two circular functions f and g, shown on the axes below.



For the interval $x \in [0,4]$, the number of x-intercepts on the graph of $h(x) = f(x) \times g(x)$ is:

A. 4

B. 6

C. 8

D. 9

Question 22 (11 mark)

Let $f(x) = x^2 + \frac{1}{x^2} + 2$ and $g(x) = x^2$ be functions defined on their maximal domains.

a. Define the function h such that $f = g \ o \ h$. (2 marks)

Solution: $h : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ h(x) = x + \frac{1}{x}$

b. Find the maximum value of h and the x-value where this occurs when $x > 0$. (2 marks)

Solution: Minimum of 2 when $x = 1$

c. Hence, state the range of f. (1 mark)

Solution: ran $f = [4, \infty)$

d. Consider the functions $k: [1, \infty) \to \mathbb{R}, k(x) = f(x)$ and $p: [a, 0) \to \mathbb{R}, p(x) = f(x)$, where a is a real number.

i. Find the smallest value of a such that p^{-1} exists. (1 mark)

Solution: $a = -1$

ii. Show that the inverse function, $k^{-1}(x)$, satisfies the equation. (2 marks)

$$[k^{-1}(x)]^2 = \frac{\left(\sqrt{x} \pm \sqrt{x-4}\right)^2}{4}$$

Solution: We start from $k(k^{-1}(x)) = x$. Let $k^{-1}(x) = a$. Then,

$$x = \frac{(1+a^2)^2}{a^2}$$

$$a^2 x - (1+a^2)^2 = 0$$

Now let $[k^{-1}(x)]^2 = a^2 = b$

$$bx - (1+b)^2 = 0$$

$$b = \frac{x - 2 \pm \sqrt{x(x-4)}}{2}$$

$$= \frac{2x - 4 \pm 2\sqrt{x}\sqrt{x-4}}{4}$$

$$= \frac{(\sqrt{x} \pm \sqrt{x-4})^2}{4}$$

iii. Hence, define k^{-1}. (1 mark)

Solution: $k^{-1} : [4, \infty) \to \mathbb{R}, \; k^{-1}(x) = \dfrac{\sqrt{x} + \sqrt{x-4}}{2}$

Suppose now that $f(x) = x^2 + \frac{1}{x^2} + 2$ is defined on some arbitrary domain $D \subseteq \mathbb{R}\backslash\{0\}$ where it is one-to-one.

e. Write down a piecewise definition for the rule of $f^{-1}(x)$ that depends on the domain D. (2 marks)

$$f^{-1}(x) = \begin{cases} \frac{\sqrt{x} + \sqrt{x-4}}{2}, & D \subseteq \{x : x \geq 1\} \\ \frac{\sqrt{x} - \sqrt{x-4}}{2}, & D \subseteq \{x : 0 < x < 1\} \\ \frac{-\sqrt{x} - \sqrt{x-4}}{2}, & D \subseteq \{x : x < -1\} \\ \frac{\sqrt{x-4} - \sqrt{x}}{2}, & D \subseteq \{x : -1 < x < 0\} \end{cases}$$

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