Write your **student number** in the boxes above. **Letter**

Mathematical Methods ¾

Examination 2 (Tech-Active)

Question and Answer Book

VCE Examination (Term 1 Mock) – April 2025

- Reading time is **15 minutes**
- Writing time is **2 hours**

Materials Supplied

- Question and Answer Book of 23 pages.
- Multiple-Choice Answer Sheet.

Instructions

- Follow the instructions on your Multiple-Choice Answer Sheet.
- At the end of the examination, place your Multiple-Choice Answer Sheet inside the front cover of this book.

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents

Student's Full Name: ______________________

Student's Email: ______________________

Tutor's Name: ______________________

Marks (Tutor Only): ______________

Section A

Instructions

- Answer **all** questions in pencil on the Multiple-Choice Answer Sheet.
- Choose the response that is **correct** or that **best answers** the question.
- A correct answer scores 1; an incorrect answer scores 0.
- Marks will **not** be deducted for incorrect answers.
- No marks will be given if more than one answer is completed for any question.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1

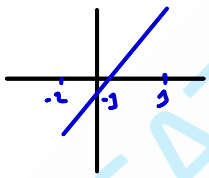
Consider the function $f:[-2,3] \to R, f(x) = ax - 3$. If the range of f is $[-7,3]$, then the value of a is:

A. 1

B. 2

C. 3

D. 4



$f(-2)=-7$

Question 2

Let $p(x) = x^3 - ax + 1$ where $a \in \mathbb{R}$. If the remainder of $p(x)$ when divided by $ax + 8$ is 1, then the value of a is:

$p\left(-\frac{8}{a}\right)=1$

A. 3

B. 4

C. 8

D. -2

Question 3

The maximal domain of the function with rule $f(x) = \frac{1}{\sqrt{\log_e(x)}}$ is:

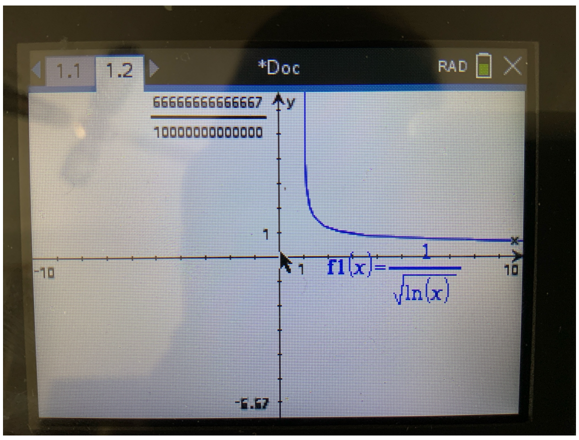
A. $[0, \infty)$

B. $(1, \infty)$

C. $(0, 1)$

D. $(0, \infty)$

① Domain
② Graph
③ Composite Func



Question 4

Which one of the following statements about transformations is false?

A. A horizontal shift of a function $f(x)$ by 3 units to the right is represented by $f(x+3)$.

B. A reflection of a function $f(x)$ across the x-axis is represented by $-f(x)$.

C. A vertical stretch by a factor of 2 is represented by $2f(x)$.

D. A reflection across the y-axis is represented by $f(-x)$.

Question 5

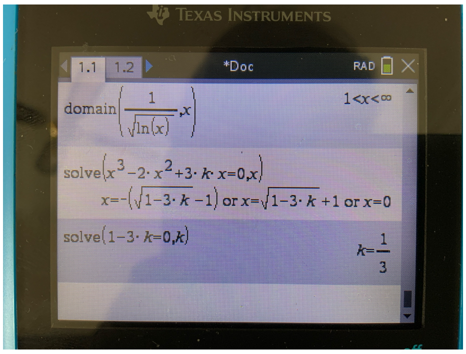
The equation $x^3 - 2x^2 + 3kx = 0$ has two solutions for:

A. $k = \frac{1}{3}$

B. $k > \frac{1}{3}$

C. $k < \frac{1}{3}$

D. $-\frac{1}{3} < k < \frac{1}{3}$



Question 6

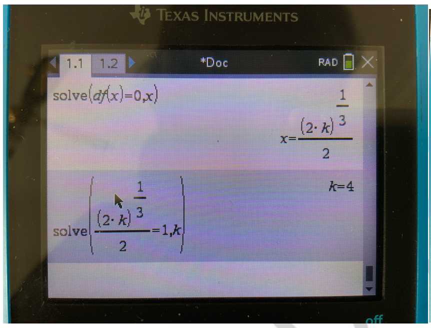
The polynomial $p(x) = x^4 - kx + 1$ where $k > 0$ has an inverse on the interval $(-\infty, 1]$ for:

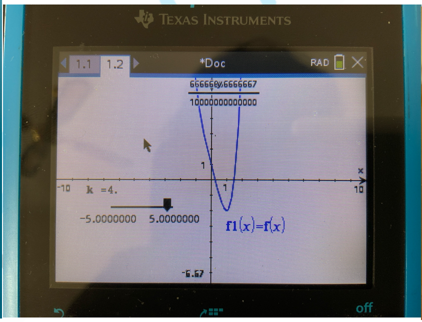
A. $k \geq 1$

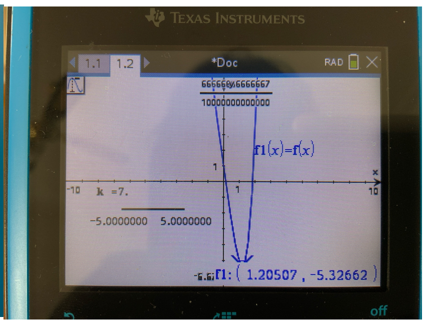
B. $k \geq 4$

C. $k \leq 1$

D. $k \leq 4$







Question 7

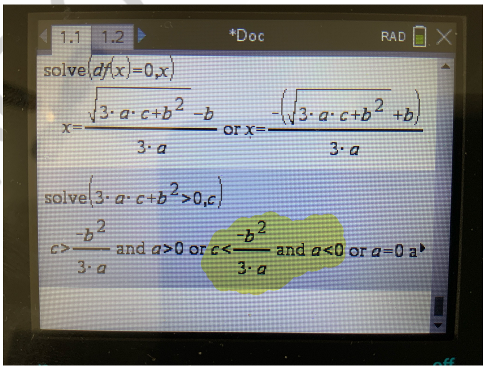
The function $f(x) = ax^3 + bx^2 - cx$ has two stationary points for:

A. $a > 0$ and $c < -\frac{b^2}{3a}$.

B. $a > 0$ and $c = -\frac{b^2}{3a}$.

C. $a < 0$ and $c > -\frac{b^2}{3a}$.

D. $a < 0$ and $c < -\frac{b^2}{3a}$.



Question 8

Let B be the domain of f and C be the domain of g. What is the largest domain B such that $g(f(x))$ exists?

A. $\{\, x \in B \mid f(x) \in C \,\}$

KIDo

B. $B \cap C$

C. B

D. C

Question 9

Let f and g be differentiable functions. The derivative of $f\left(e^{g(x)}\right)$ is:

A. $e^{x}e^{g'(x)}f\left(e^{g(x)}\right)$

B. $e^{g'(x)}f'\left(e^{g(x)}\right)$

C. $e^{g(x)}g'(x)f'\left(e^{g(x)}\right)$

D. $e^{g(x)}f'\left(e^{g(x)}\right)$

Question 10

Which of the following statements is false?

A. The product of two odd functions is an even function.

B. The product of an odd and an even function is an odd function.

C. The composition of two odd functions is an even function.

D. The composition of two even functions is an even function.

$f(x)=x^3$
$g(x)=x^3$
$f(g(x)) = x^9 \to$ ODD

Question 11

The minimum distance between the graph of $y = \sqrt{x}$ and the point $(2, 0)$ is:

A. $\frac{3}{2}$

B. $2\sqrt{2}$

C. $\frac{\sqrt{5}}{2}$

D. $\frac{\sqrt{7}}{2}$

$d(x) = \sqrt{(x-2)^2 + (\sqrt{x}-0)^2}$
$d'(x)=0$ or fMin
$x = 3/2$
$d(\tfrac{3}{2}) =$

Question 12

The angle, in degrees, between the lines $y = 2x - 1$ and $y = 3x - \frac{1}{3}$ is closest to:

A. 11.31

B. 0.20

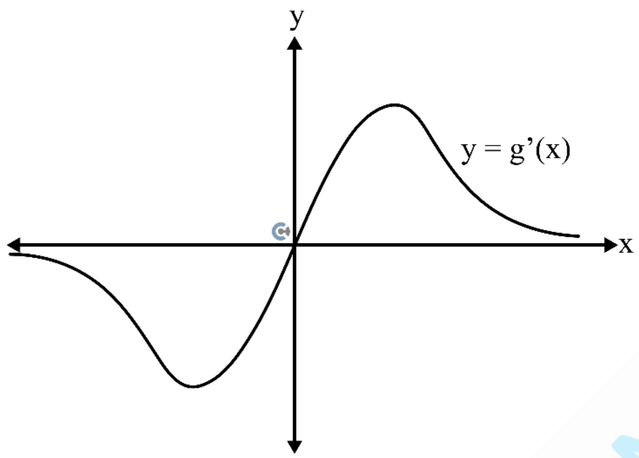
C. 8.13

D. 0.14

$\left|\tan^{-1}(m_1) - \tan^{-1}(m_2)\right|$

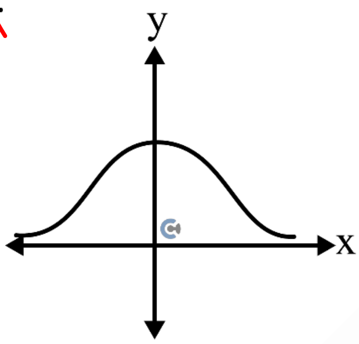
Question 13

Part of the graph of $y = g'(x)$ is shown below.

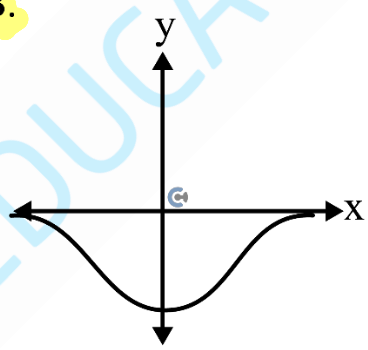


That part of the graph of $y = g(x)$ that corresponds to this graph, could be represented by:

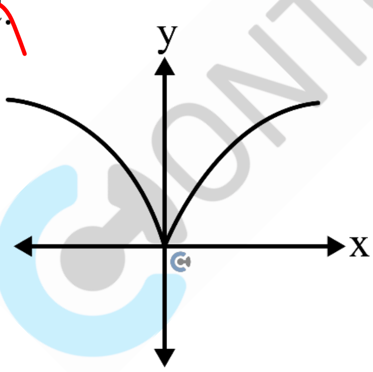
A.



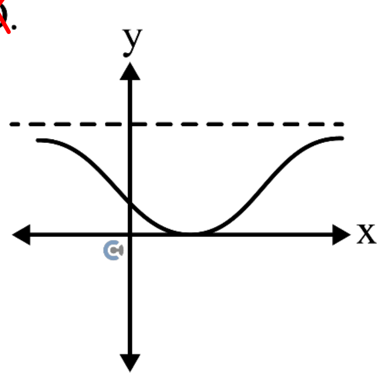
B.



C.



D.

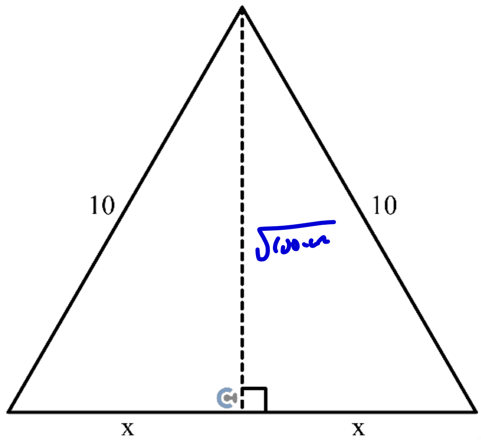


Question 14

The value of x that maximises the area of the triangle below is equal to

$A = \frac{1}{2} \times (2x) \times \sqrt{100-x^2}$

$A'(x) = 0$ or fMax



A. 5

B. $5\sqrt{2}$

C. $\sqrt{20}$

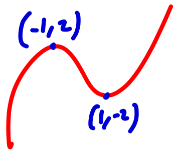
D. 10

Question 15

The value of k such that the polynomial $p(x) = x^3 - 3x + k$ has exactly two x-intercepts is equal to

$p'(x) = 0$

$x = \pm 1$



A. $k = 0$

B. $k = 1$

C. $k = 2$

D. $k = 3$

Question 16

The following algorithm applies Newton's method using a for loop with 4 iterations.

Inputs:

$f(x)$, a function of x

$df(x)$, the derivative of $f(x)$

$x0$, an initial estimate

define newton$(f(x), df(x), x0)$

for i from 1 to 4

if $df(x0) = 0$ Then

return "Error: Division by zero"

else

$x0 \leftarrow x0 - f(x0) \div df(x0)$

end if

end for

return $x0$

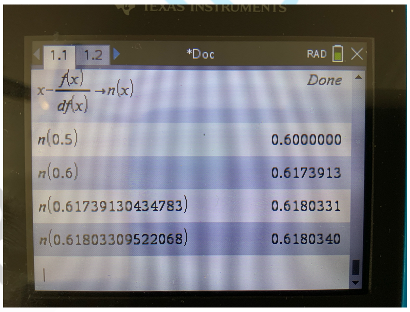
The return value of the function newton $(x^3 - 2x + 1, 3x^2 - 2, 0.5)$ is closest to:

A. 0.500

B. 0.618

C. 0.717

D. 0.698



Question 17

Let f be a one-to-one differentiable function such that $f(1) = 3$ and $f'(1) = 4$. Let g be the inverse of f. If $g(3) = 1$ and $g'(1) = 2$, then $g'(3)$ is:

A. 1

B. 4

C. $\frac{1}{2}$

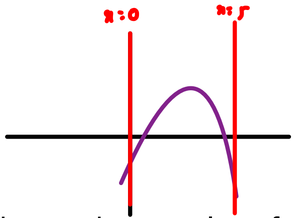
D. $\frac{1}{4}$

Question 18

Consider the family of functions:

$$f_k(x) = kx - x^2, x \in [0,5]$$

$= -(x - \frac{k}{2})^2 + \frac{k^2}{4}$



Where k is a real constant. For which of the following ranges of k does the maximum value of $f_k(x)$ occur at an endpoint?

$f_k'(0)=0$ | $f_k'(5)=0$

$k=0$ | $k=10$

A. $k < 0$ or $k > 10$

B. $k \leq 0$ or $k \geq 10$

C. $0 < k < 10$

D. $k \leq 0$ or $k > 10$

Question 19

A possible value of a for which the tangent to the curve $f(x) = x^2$ at $x = a$ makes an angle of $45°$ with the line $y = 2x + 1$ is:

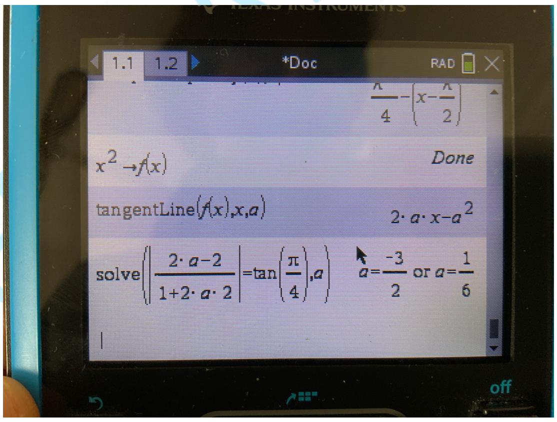
A. $-\frac{1}{6}$

B. $\frac{1}{3}$

C. $\frac{1}{6}$

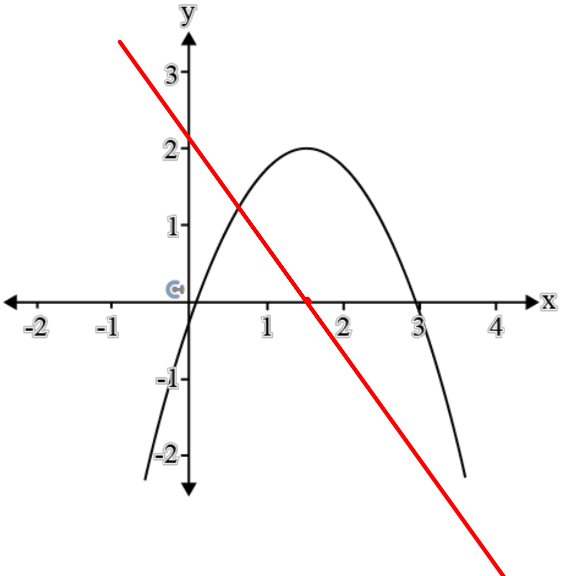
D. $-\frac{1}{3}$

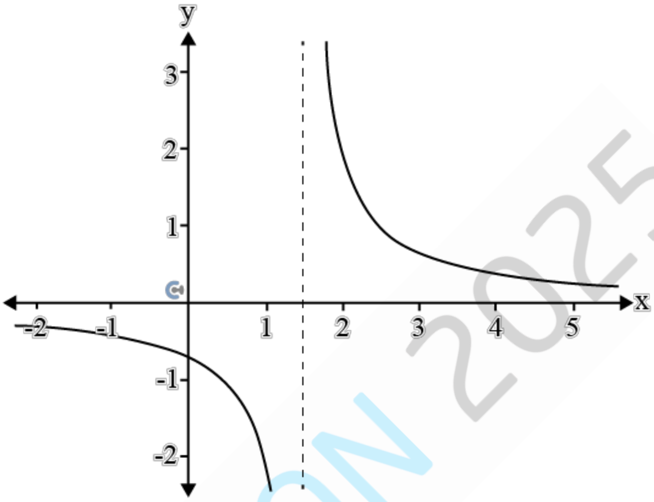
$\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \tan(\theta)$



Question 20

Consider the graphs of two functions f and g, shown on the axes below.





The turning point of f is at $x = 1.5$ and occurs where $y > 0$.

The vertical asymptote of g is $x = 1.5$ and g has no x-intercepts.

On the interval $x \in [0, 4]$, the number of x-intercepts for the graph of the function h where $h = f' \times g$ is:

A. 0

B. 1

C. 2

D. 3

Section B

Instructions

- Answer **all** questions in the spaces provided.
- Write your responses in English.

Question 1 (13 marks)

Consider the function $f(x) = -\frac{x^4}{16} + \frac{x^3}{6} + \frac{15x^2}{8} - 9x + 1$.

a. Find the rule of the derivative of $f(x)$. 1 mark

$f'(x) = -\frac{x^3}{4} + \frac{x^2}{2} + \frac{15x}{4} - 9$

b. Find the x coordinates of the stationary points of f. 1 mark

$f'(x) = 0$

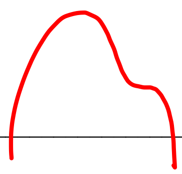
$x = -4, 3$

c. State the values of x where the function $f(x)$ is strictly decreasing. 1 mark

$f'(x) \leq 0$

$x \geq -4$

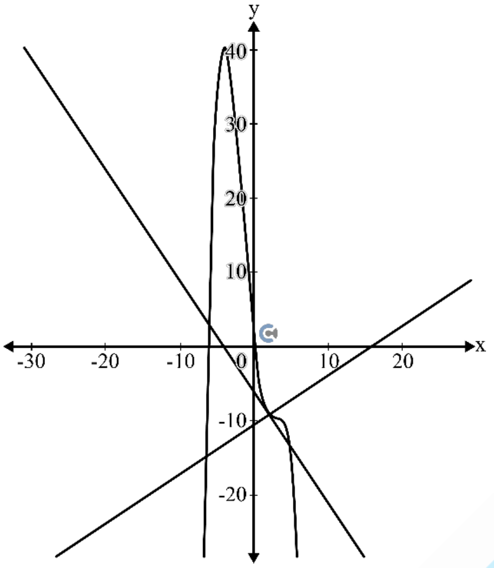
d. State the nature of the stationary points of $f(x)$. 2 marks



$x = -4$: Local Max

$x = 3$: SPI

The diagram below shows part of the graph of $y = f(x)$, the tangent to the graph at $x = 2$, and a straight line drawn perpendicular to that tangent at $x = 2$. The equation of the tangent at $x = 2$ is $y = -\frac{3x}{2} - \frac{37}{6}$.



e. Using Newton's method with $x_0 = 2$, find the next iterate x_1. 1 mark

$-\frac{3x}{2} - \frac{37}{6} = 0$

$x = -\frac{37}{9}$

f. Find the y-axis intercept of the tangent to the graph of f at $x = 2$. 1 mark

$y = -\frac{37}{6}$

g. Find the equation of the line perpendicular to the graph of f at $x = 2$. Hence, find the coordinates of the y-intercept of a line perpendicular to the graph of f at $x = 2$. 2 marks

$y = \frac{2x}{3} - \frac{21}{2}$

$\therefore$ y int: $-\frac{21}{2}$

h. Find the area of the triangle formed by the y-axis intercepts of both the tangent line and the line perpendicular to it at $x = 2$, and the point $(2, f(2))$. 2 marks

$A = \frac{1}{2} b \times h$

$= \frac{1}{2} \times \left(-\frac{17}{6} - \left(-\frac{21}{2}\right)\right) \times 2$

$= \frac{13}{3}$

i. The tangent to the graph of f at another point to the left is parallel to the tangent at $x = 2$. 2 marks

Find the coordinates of this point.

$y = -\frac{3}{2}x - \frac{17}{6}$

$f'(x) = -\frac{3}{2}$

$\therefore x = -\sqrt{15}, 2, \sqrt{15}$

BUT, x is "to the left"

$\therefore x = -\sqrt{15}$

$\therefore \left(-\sqrt{15}, \frac{13\sqrt{15}}{2} + \frac{241}{16}\right)$

Do not write in this area.

Question 2 (18 marks)

A new rollercoaster at Funland theme park is modelled by the height function:

$$h(x) = \begin{cases} -0.5x^2 + 4x + 2, & 0 \le x \le 5, \\ ax + b, & 5 < x \le 10, \end{cases}$$

where $h(x)$ gives height (in metres) above ground when the car is x metres from the station.

a. State the initial height. 1 mark

$h(0) = 2\text{m}$

b. Find the average rate of change in height from $x = 0$ to $x = 5$. 2 marks

$\text{ARoC} = \frac{h(5) - h(0)}{5 - 0}$

$= \frac{1}{2}$

c. Determine the maximum height reached for $0 \le x \le 5$. 2 marks

Let $f(x) = -0.5x^2 + 4x + 2$

$f'(x) = 0$

$x = 4$

$f(4) = 10$

$\therefore$ Max height $= 10\text{m}$

It is known that the two pieces join smoothly at $x = 5$.

d. Find the gradient of the rollercoaster at the joining point $P(5, h(5))$. 2 marks

$f'(x) = 4 - x$

$f'(5) = 4 - 5$

$= -1$

e. Show that the conditions for smooth joining give $a = -1$ and $b = 14.5$. 2 marks

Let $g(x) = ax + b$, $g'(x) = a$

$f'(5) = g'(5)$

$-1 = a$

$\therefore a = -1$

$f(5) = g(5)$

$-0.5 \times 5^2 + 4 \times 5 + 2 = 5a + b$

$\frac{19}{2} = 5 \times -1 + b$

$\therefore b = \frac{19}{2} + 5$

$= \frac{29}{2}$

$= 14.5$

f. Find the concavity of h at $x = 3$. 2 marks

$f''(x) = -1$

$\therefore$ As $f''(x) < 0$

$h(x)$ is concave down at $x = 3$

g. Find the total horizontal distance (in metres) over which the rollercoaster remains at or above $8\,m$ in height. 2 marks

$h(x)=8$

$x = 2, \frac{13}{2}$

$\therefore \frac{13}{2} - 2 = \frac{9}{2}\,m$

After safety testing, ride engineer Sam wants to make the coaster "more thrilling" by vertically stretching every height by a factor $k > 0$. The transformed height function is $h^*(x) = k\,h(x)$.

h. State the single transformation (in terms of k) mapping h to h^*. 1 mark

Dilation by a factor of k from the x axis

i. Find the value of k such that the new maximum height is $15\,m$. 1 mark

OG Max height $= 10$

New height $= 10k$

$\therefore 10k = 15$

$k = 3/2$

Sam now wants to install a horizontal support beam at height of 12 metres which joins two points of the rollercoaster.

j. Determine all values of k, correct to three decimal places, for the support beam to be longer than 5 metres. 3 marks

$h^*(x)=12$

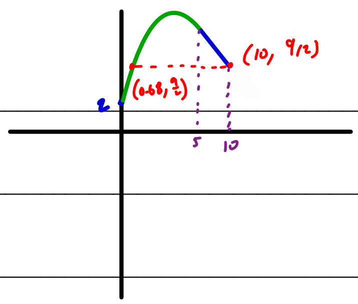
$x = \frac{2(\sqrt{k(5k-6)}+2k)}{k}, \frac{-2(\sqrt{k(5k-6)}-2k)}{k}, \frac{29k-24}{2k}$

$\text{dist} = \frac{29k-24}{2k} - \left(\frac{-2(\sqrt{k(5k-6)}-2k)}{k}\right)$

$5 < \text{dist} \leq (10 - 0.68)$

$5 < \text{dist} \leq 9.3166$

$\therefore 1.566 < k \leq 2.667$



$-0.5x^2+4x+2 = 9/2$

$x = \pm\sqrt{11}+4$

$= 0.68$ or 7.32

But $0 \leq x \leq 5$

$\therefore x = 0.68$

Question 3 (16 marks)

Consider the polynomial $p(x) = x^3 - kx^2$ where $k \in [0, \infty)$.

a. Find the coordinate(s) of the stationary point(s) of $p(x)$ in terms of k. 2 marks

$p'(x) = 0$

$x = 0, \frac{2k}{3}$

$\therefore (0,0), \left(\frac{2k}{3}, -\frac{4k^3}{27}\right)$

b. Find the values of k such that there are two stationary points. 1 mark

$k \in (0, \infty)$

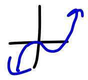
c. State the value of k such that $p(x)$ is an odd function. 1 mark

$-p(x) = p(-x)$

$k = 0$

d. Suppose $k = 1$. Find the exact value(s) of d for which the graph of $y = p(x) + d$ has:

TP: $(0,0), \left(\frac{2}{3}, -\frac{4}{27}\right)$



i. Exactly one x-intercept. 1 mark

$d < 0$ or $d > \frac{4}{27}$

ii. Exactly three x-intercepts. 1 mark

$0 < d < \frac{4}{27}$

e. Find the value of k for which the tangent to the function p at $x = 1$ hits the x-axis at an angle of 60 degrees. 2 marks

tangentLine(p(x), x, 1)

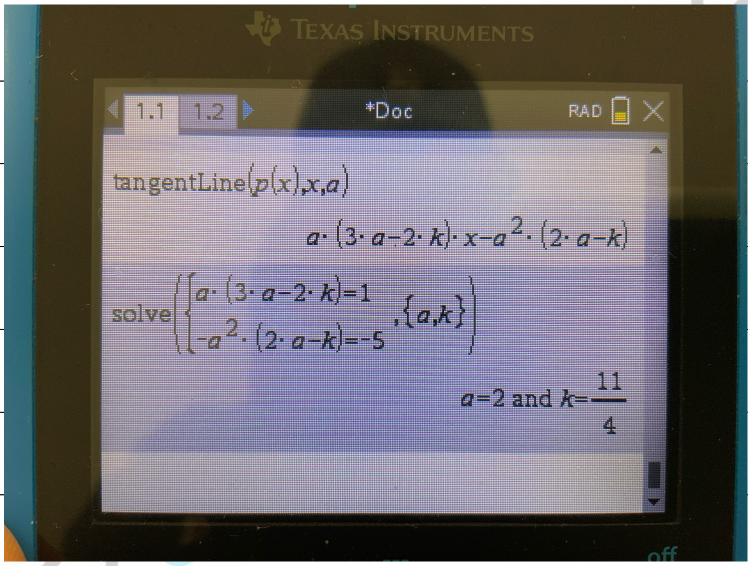
$y = -(2k-1)x + k-2$

$m = \tan(\theta)$

$-(2k-1) = \tan(60)$

$\therefore\ k = \frac{1-\sqrt{3}}{2}$

f. Find the value of k for which the graph of $y = p(x)$ has a tangent of $y = x - 5$. 3 marks



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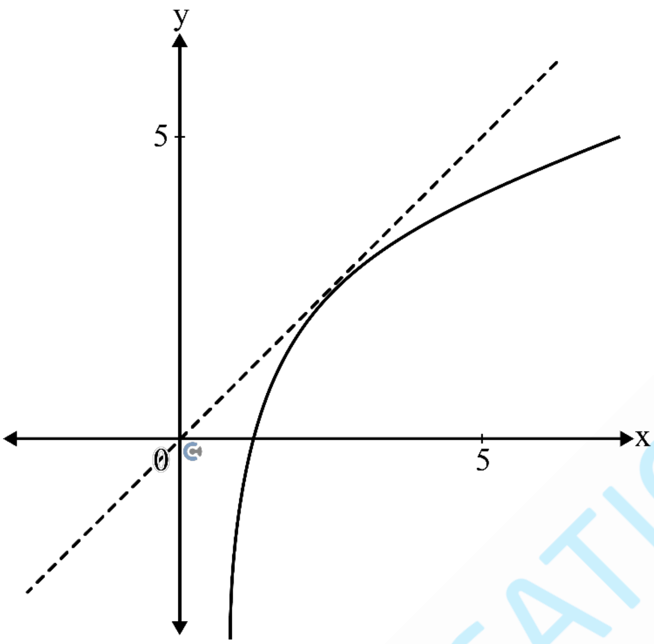
A new function is defined by composing the function $\log_e(x)$ with the derivative of $p(x)$. Let $f:(c,\infty)\to\mathbb{R}, f(x)=\log_e\left(p'(x)\right)$ be a one-to-one function. A section of the function is shown below.



g. Find the smallest possible value of c in terms of k. 2 marks

DOMAIN FUNC doesn't work!

$C = \frac{2k}{3}$

Solve ($p'(x) > 0$, x)

doesn't work!

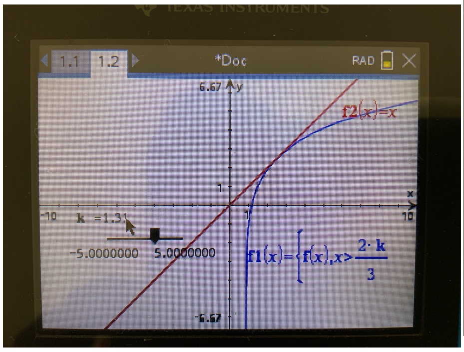
h. Find the value of k correct to three decimal places for which the line $y = x$ is tangent to the graph of $y = f(x)$. 2 marks

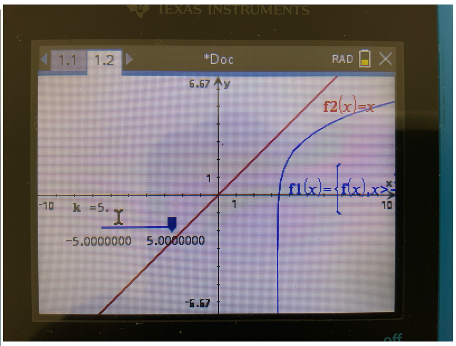
$f(x) = x$, $f'(x) = 1$

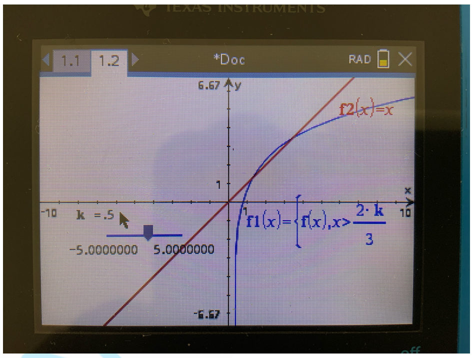
$\therefore x = 2.530$, $k = 1.314$

i. Hence, or otherwise, find the values of k for which the functions f and f^{-1} have two points of intersection. Give your answer correct to three decimal places. 1 mark

$0.000 \le k < 1.314$



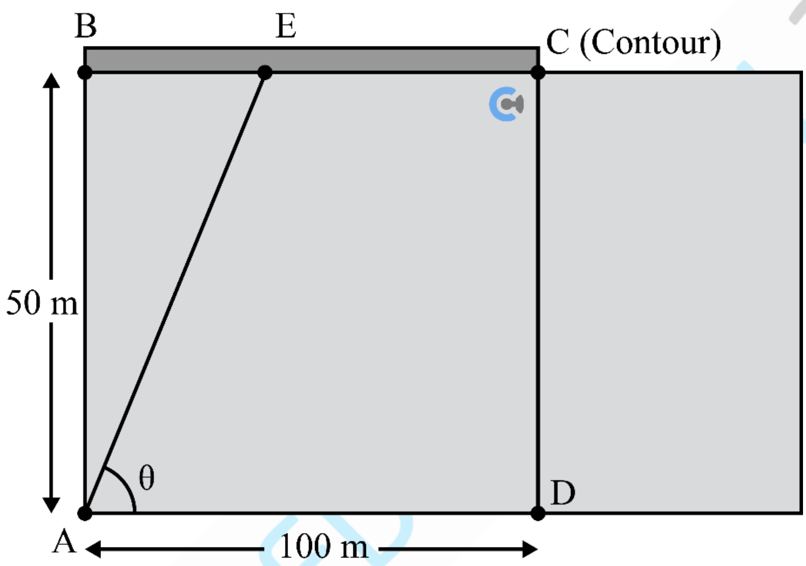




Question 4 (13 marks)

The Late Contour Students, on their way to the Contour Term 1 Exam, find themselves at point A on the Southwest corner of a rectangular car park that is $50\ m$ wide. The students need to get to point C at the Contour Centre, $100\ m$ to the east and on the edge of the car park.

Point B is directly to the north of point A and on the edge of the car park as shown. There is a normal footpath running along the far side BC of the car park. They need to get from A to C. They can travel at a speed of 2 metres per second on the footpath, but only at a speed of 1 metre per second through the car park. Note that the diagram is **NOT** to scale.



a. Calculate the time taken to travel directly from A through the car park to B, and then from B to C using the footpath. 1 mark

$S = \frac{d}{t}$

$t = \frac{d}{s}$

$t_{AB} = \frac{50}{1}$

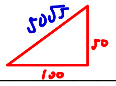
$= 50s$

$t_{BC} = \frac{100}{2}$

$= 50s$

Do not write in this area.

b. Calculate the time taken, to the nearest second, to travel directly from A to C through the car park. 2 marks



$t = \frac{d}{s}$

$= \frac{50\sqrt{5}}{1}$

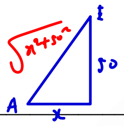
$= 50\sqrt{5}$ s

$= 111.8 \approx 112$ secs

The Late Contour Students realise that they will not make it in time for the exam if they were to travel either of the ways calculated in **part a.** and **part b.** They decide to travel from point A to E, which is x m horizontally away from B, and travel from E to C via the footpath to optimise their time.

c. Show that the total time taken for the Late Contour Students to go from A to C can be modelled by: 2 marks

$$T(x) = \sqrt{x^2 + 2500} + \frac{100 - x}{2}$$



$t_{AE} = \frac{\sqrt{x^2+50^2}}{1}$

$= \sqrt{x^2+2500}$

$t_{EC} = \frac{100-x}{2}$

$\therefore T(x) = t_{AE} + t_{EC}$

$= \sqrt{x^2+2500} + \frac{100-x}{2}$

d. State the appropriate domain of $T(x)$. 1 mark

$x \in (0, 100)$

Cannot travel the same as part a and e, hence we exclude endpoints.

e. Write the equation for $\frac{dT}{dx}$, and hence find the value of x required to minimise the time taken to travel from A to C correct to one decimal place. 2 marks

$\frac{dT}{dx} = \frac{x}{\sqrt{x^2+2500}} - \frac{1}{2}$

$\frac{dT}{dx} = 0$

$x = \frac{50\sqrt{3}}{3}$

≈ 28.9 m

f. Calculate the minimum time taken to travel from A to C, to the nearest second. 1 mark

$T(28.9) = 93.3$

≈ 93 sec

It is known that the Contour Term 1 Exam is starting in 80 seconds. The students decide to increase their speed on the footpath to be $a\ m/s$ instead of $2\ m/s$, while their speed through the car park stays the same.

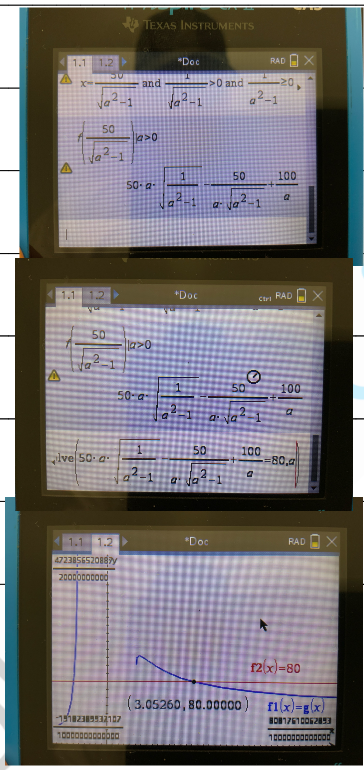
g. Find the minimum value of a, correct to two decimal places, such that they can make it to the Contour Centre before the start of the Contour Term 1 Exam. 4 marks

Let $f(x)=\sqrt{x^2+2500}+\frac{100-x}{a}$

$f'(x)=0 \mid x>0$

$x=\frac{50}{\sqrt{a^2-1}}$

$\boxed{\therefore a=1.05 \text{ m/s}}$



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