Write your **student number** in the boxes above. **Letter**

Mathematical Methods ¾

Examination 1 (Tech-Free)

Question and Answer Book - SOLUTIONS

VCE Examination (Term 1 Mock) – April 2025

- Reading time is **15 minutes.**
- Writing time is **1 hour.**

Materials Supplied

- Question and Answer Book of 12 pages.

Instructions

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents

Student's Full Name: ______________________________

Student's Email: ______________________________

Tutor's Name: ______________________________

Marks (Tutor Only): ____________________

Section A

Instructions

- Answer **all** questions in the spaces provided.
- Write your responses in English.

Question 1 (4 marks)

Let $y = \frac{\tan(2x)}{x^3}$.

a. Find $\frac{dy}{dx}$. Give your answer in the simplest form. **D2** 2 marks

Learning Objective [2.3.1] Find general derivatives with functional notation.

$$\frac{dy}{dx} = \frac{(2\sec^2(2x)\cdot x^3) - (\tan(2x)\cdot 3x^2)}{x^6} \ \mathbf{1M}$$

$$\frac{dy}{dx} = \frac{2x\sec^2(2x) - 3\tan(2x)}{x^4} \ \mathbf{1A}$$

Note: Other equivalent forms exist. Must cancel the x^2 for simplest form

Let $f(x) = x^4 e^{4x-1}$.

D2

b. Evaluate $f'(1)$. 2 marks

Learning Objective [2.3.1] Find general derivatives with functional notation.

$$f'(x) = 4x^3e^{4x-1} + x^4(4e^{4x-1})$$

$$= 4e^{4x-1}(x^3 + x^4) \ \mathbf{1M\ for\ this\ or\ above\ line}$$

$$f'(1) = 4e^{4(1)-1}(1^3 + 1^4)$$

$$= 8e^3 \ \mathbf{1A}$$

Do not write in this area.

Question 2 (3 marks)

Learning Objective [1.5.4] Find the unknown value for systems of linear equations.

Find the value of $k \in R$ for which the system of linear equations:

$$3x - (k+1)\,y = 2k + 4,$$
$$(k+2)\,x - 10\,y = 4k + 8$$

Has infinitely many solutions.

D3

$$\frac{3}{k+2} = \frac{-(k+1)}{-10} = \frac{2k+4}{4k+8}\ \mathbf{1M}\ (\textit{other methods exist})$$

$$\frac{3}{k+2} = \frac{k+1}{10}$$

$$3 \times 10 = (k+2)(k+1) \Longrightarrow 30 = k^2 + 3k + 2 \Longrightarrow k^2 + 3k + 2 - 30 = 0 \Longrightarrow k^2 + 3k - 28 = 0$$

$$k^2 + 3k - 28 = 0 \Longrightarrow (k+7)(k-4) = 0 \Longrightarrow k = -7 \text{ or } k = 4\ \mathbf{1M}$$

$$\frac{3}{k+2} = \frac{2k+4}{4k+8}$$

Check $k = 4$: $\frac{3}{4+2} = \frac{3}{6} = \frac{1}{2}, \frac{2(4)+4}{4(4)+8} = \frac{8+4}{16+8} = \frac{12}{24} = \frac{1}{2}$ so both ratios match

Check $k = -7$: $\frac{3}{-7+2} = \frac{3}{-5} = -\frac{3}{5}, \frac{2(-7)+4}{4(-7)+8} = \frac{-14+4}{-28+8} = \frac{-10}{-20} = \frac{1}{2}$ ratios don't match

$\boxed{k = 4}$ *for infinitely many solutions* **1A**

Question 3 (5 marks)

Consider the quartic polynomial $p(x) = x^4 - 5x^2 + 4$.

D2

a. Express $p(x)$ as a product of linear factors. 2 marks

2M Learning Objective [1.7.2] Find factored form of polynomials.

3M Learning Objective [1.7.3] Graph factored and unfactored polynomials.

$p(x) = x^4 - 5x^2 + 4$

Let $A = x^2$: $p(x) = A^2 - 5A + 4$

$p(x) = (A - 4)(A - 1)$

$p(x) = (x^2 - 4)(x^2 - 1)$ **1M**

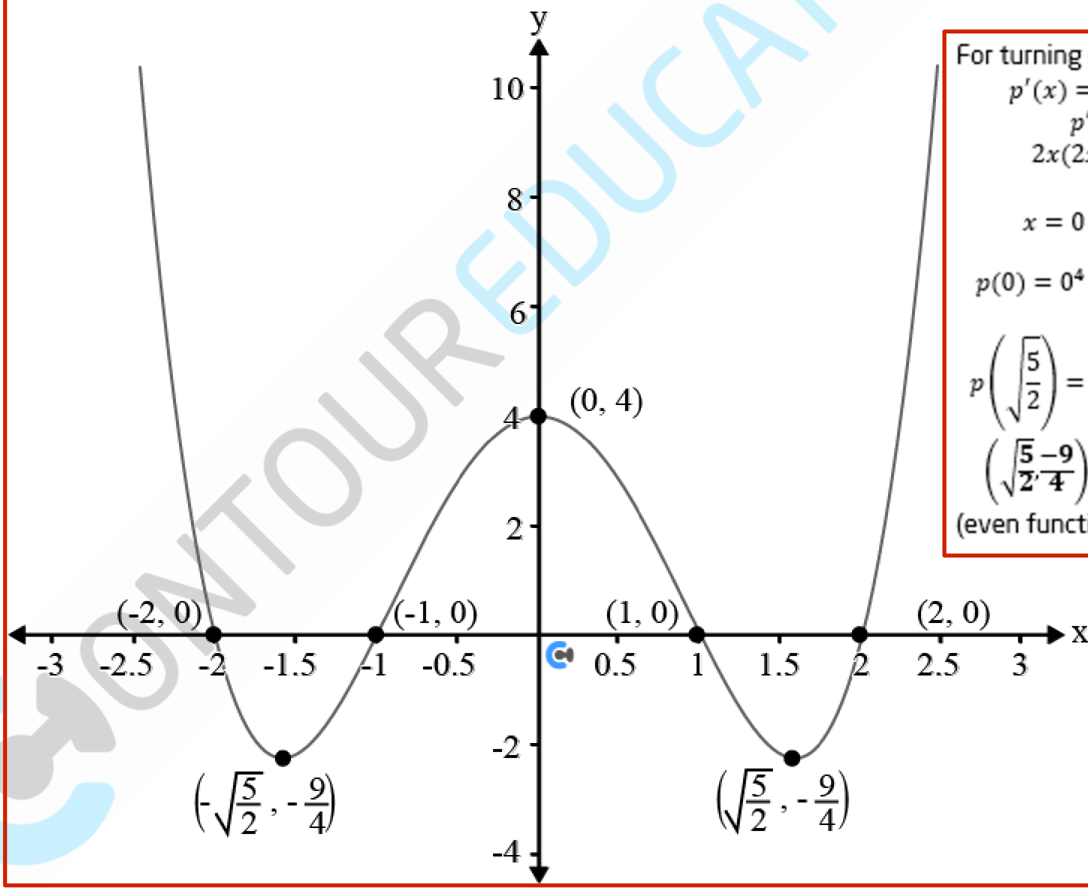
$p(x) = (x - 2)(x + 2)(x - 1)(x + 1)$ **1A**

D2

b. Sketch the graph of $y = p(x)$, labelling all axial intercepts and turning points. Use the fact that $\frac{\sqrt{10}}{2} \approx 1.6$. 3 marks

2M Learning Objective [1.7.2] Find factored form of polynomials.

3M Learning Objective [1.7.3] Graph factored and unfactored polynomials.



For turning points:

$p'(x) = 2x(2x^2 - 5)$

$p'(x) = 0$

$2x(2x^2 - 5) = 0$

$x = 0 \text{ or } x = \pm\sqrt{\frac{5}{2}}$

$p(0) = 0^4 - 5(0^2) + 4 = 4$

$\mathbf{(0, 4)}$

$p\left(\sqrt{\frac{5}{2}}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 4$

$\left(\sqrt{\frac{5}{2}}, \frac{-9}{4}\right)$ and $\left(-\sqrt{\frac{5}{2}}, \frac{-9}{4}\right)$

(even function)

Accept minimum turning points with x-value in the range $(-1.8, -1.2)$ *and* $(1.2, 1.8)$

Question 4 (5 marks)

Let $f: \mathbb{R}\backslash\{0\} \to \mathbb{R}, f(x) = \frac{1}{x}$ and let $g(x) = f(x+1)$.

a. State the rule of the function $f(f(x))$ and state its domain and range. **D1** 2 marks

2M Learning Objective [1.1.2] - Find the rule, domain and range of a composite function (Range doesn't not require splitting to find as the function is easy to draw).

3M Learning Objective [1.2.1] Find a new domain to fix composite functions.

$$f(f(x)) = f\left(\frac{1}{x}\right)$$

$$f(f(x)) = \frac{1}{\left(\frac{1}{x}\right)}$$

$$f(f(x)) = x \ \mathbf{1A}$$

$$dom\ f = \mathbb{R}\setminus\{0\}, ran\ f = \mathbb{R}\setminus\{0\}\ \mathbf{1A}$$

b. Does the function $g(g(x))$ exist? Explain your answer. **D2** 2 marks

2M Learning Objective [1.1.2] - Find the rule, domain and range of a composite function (Range doesn't not require splitting to find as the function is easy to draw).

3M Learning Objective [1.2.1] Find a new domain to fix composite functions.

$$g(x) = f(x+1) = \frac{1}{x+1}$$

$$\text{Dom}(g) = \mathbb{R}\setminus\{-1\}, \text{Ran}(g) = \mathbb{R}\setminus\{0\}\mathbf{1M}$$

$$\text{Ran}(g)\ is\ not\ a\ subset\ \text{Dom}(g)\ hence\ g(g(x))\ does\ not\ exist\ \mathbf{1A}$$

c. Consider a new function $h(x)$ with the same rule as $f(x)$ but with domain $(-c, c)\backslash\{0\}$ where $c > 0$. 1 mark

Find the largest possible value of c for which the function $g(h(x))$ exists. **D1**

2M Learning Objective [1.1.2] - Find the rule, domain and range of a composite function (Range doesn't not require splitting to find as the function is easy to draw).

3M Learning Objective [1.2.1] Find a new domain to fix composite functions.

$\text{ran } h \subseteq \text{dom } g$. So we must have $\text{ran } h \subseteq \mathbb{R}\setminus\{-1\}$.

We require $\frac{1}{x} \neq -1 \implies x \neq -1$.

So largest possible value of c is $c = 1$. (1A)

Question 5 (5 marks)

Consider the function:

$$f : [-2, 2] \to \mathbb{R}, f(x) = x^3 - 3x$$

a. Find the rule of the derivative function f'. 1 mark

D1

1M Learning Objective [1.1.1] Find maximal domain and range.

3M Learning Objective [2.1.3] Graph Derivative Functions.

$f'(x) = 3x^2 - 3$

1M Learning Objective [2.3.1] Find general derivatives with functional notation.

b. Find the range of $f'(x)$ for $x \in (0, 2)$. 1 mark

D2

3M Learning Objective [2.1.3] Graph Derivative Functions.

$f'(0) = 3 \cdot 0^2 - 3 = -3$

$f'(2) = 3 \cdot (2^2) - 3 = 3 \cdot 4 - 3 = 12 - 3 = 9$

f' is strictly increasing over the interval $(0,2)$ hence

Range of $f'(x)$ is $(-3, 9)$ *over* $x \in (0,2)$ **1A**

c. Hence, or otherwise, verify that $f(x)$ has a stationary point for some $x \in (0, 2)$. 1 mark

D2

1M Learning Objective [1.1.1] Find maximal domain and range.

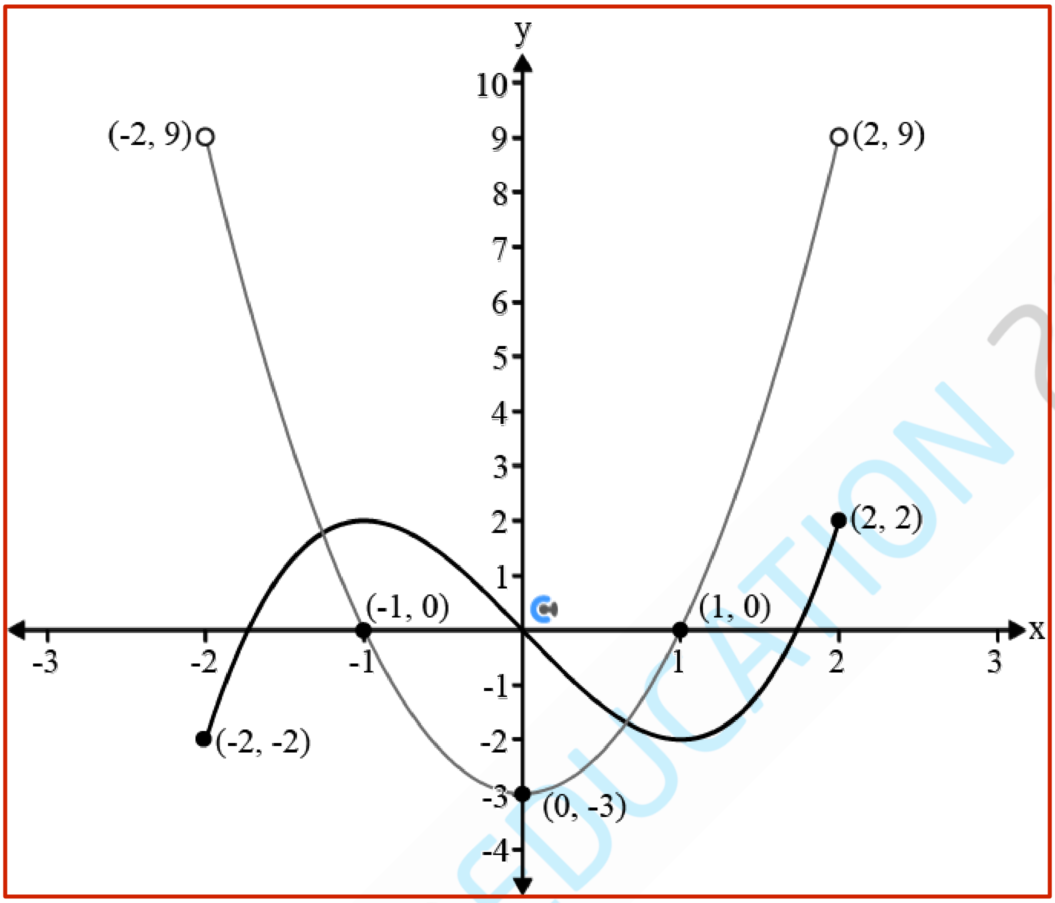
3M Learning Objective [2.1.3] Graph Derivative Functions.

From part (b), the range of $f'(x)$ is $(-3, 9)$ for $x \in (0,2)$. Since $-3 < 0 < 9$, $f'(x)$ must take the value 0 for at least one point in $(0,2)$ **1A**

Answer can be worded in various ways.

d. On a single set of axes, sketch the graph of $y = f'(x)$. Label axial intercepts, endpoints and turning point. 2 marks

D3



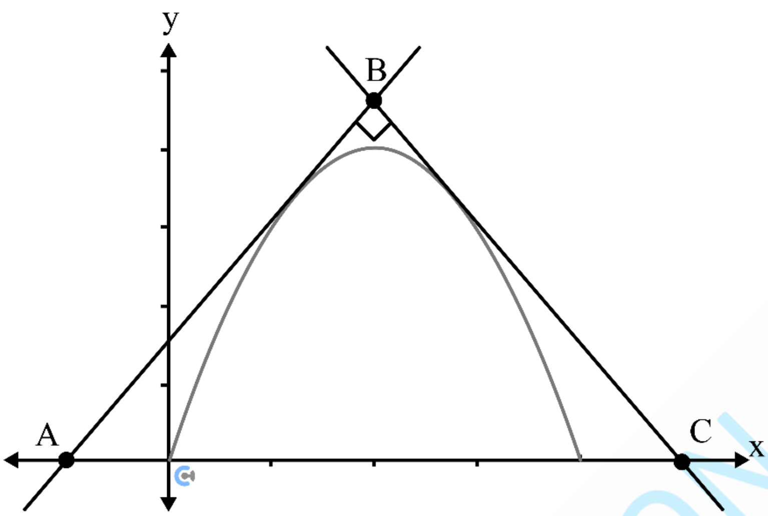
1M Learning Objective [2.3.1] Find general derivatives with functional notation.

1M Learning Objective [1.1.1] Find maximal domain and range.

3M Learning Objective [2.1.3] Graph Derivative Functions.

Question 6 (4 marks)

The graph of $f : [0,4] \to \mathbb{R}, f(x) = 4 - (x-2)^2$ is shown below:



The edges of the right angled triangle ABC are the line segments AB and BC, which are tangent to the graph of f, and the line segment AC, which is part of the horizontal axis, as shown above. Let θ be the angle that AB makes with the positive direction of the horizontal axis, where $45° \le \theta < 90°$.

Learning Objective [2.5.1] Advanced Tangents and Normal Questions.

a. Find the equation of the line through A and B in the form $y = mx + c$ for $\theta = 45°$. 2 marks

D2

$$\theta = 45° \implies \tan(\theta) = 1$$

Let line be $y = x + c$

$$f'(x) = -2(x-2)$$
$$f'(x) = 1$$
$$-2(x-2) = 1 \implies x - 2 = -\frac{1}{2} \implies x = \frac{3}{2} \ \mathbf{1M}$$
$$f(\tfrac{3}{2}) = 4 - (\tfrac{3}{2} - 2)^2 = 4 - (-\tfrac{1}{2})^2 = 4 - \frac{1}{4} = \frac{15}{4}$$
$$\left(\frac{3}{2}, \frac{15}{4}\right): \frac{15}{4} = \frac{3}{2} + c \implies c = \frac{15}{4} - \frac{3}{2} = \frac{9}{4}$$
$$y = x + \frac{9}{4} \ \mathbf{1A}$$

b. Find the coordinates of B when $\theta = 45°$. 2 marks

D3

Learning Objective [2.5.1] Advanced Tangents and Normal Questions.

Slope of BC is -1 (since $AB \perp BC \implies 1 \cdot m_{BC} = -1$)

$$f'(x) = -1 \implies -2(x-2) = -1 \implies x = \frac{5}{2}$$
$$f(\tfrac{5}{2}) = 4 - (\tfrac{5}{2} - 2)^2 = 4 - (\tfrac{1}{2})^2 = 4 - \frac{1}{4} = \frac{15}{4}$$

BC: $y - \frac{15}{4} = -1\left(x - \frac{5}{2}\right) \implies y = -x + \frac{25}{4}$ $\mathbf{1M}$

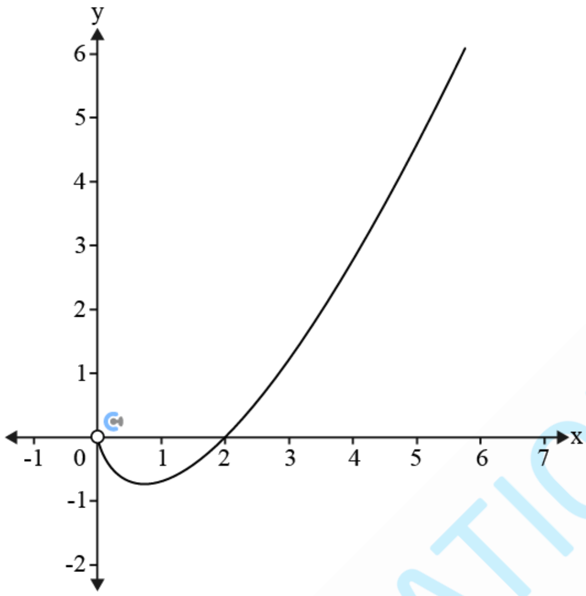
Intersection of AB and BC $\begin{cases} y = x + \frac{9}{4} \\ y = -x + \frac{25}{4} \end{cases} \implies x + \frac{9}{4} = -x + \frac{25}{4} \implies 2x = \frac{16}{4} \implies x = 2$

$$y = 2 + \frac{9}{4} = \frac{17}{4}$$
$$B = \left(2, \frac{17}{4}\right) \ \mathbf{1A}$$

Question 7 (5 marks)

Let $f: (0, \infty) \to R, f(x) = x \log_e\left(\frac{x}{2}\right)$

Part of the graph is shown below, f has a minimum at the point $Q(a, f(a))$:



a. Find the coordinates of Q. 2 marks

D2

2M Learning Objective [2.4.2] Find minimum and maximum.

3M Learning Objective [2.6.2] Find unknowns for number of solutions.

$$f'(x) = 1 + \log_e\left(\frac{x}{2}\right) \ (\mathbf{1M})$$

$$f'(x) = 0 \Rightarrow \frac{x}{2} = e^{-1} \Rightarrow x = \frac{2}{e}$$

$$Therefore, Q\left(\frac{2}{e}, -\frac{2}{e}\right) \ (\mathbf{1A})$$

b. Let $g : (a, \infty) \to \mathbb{R}, g(x) = f(x) + k$ where $k \in \mathbb{R}$.

i. Find the value of k for which the line $y = x$ is tangent to the graph of g. 2 marks

D1

2M Learning Objective [2.4.2] Find minimum and maximum.

3M Learning Objective [2.6.2] Find unknowns for number of solutions.

$$g'(x) = 1 \text{ and } g(x) = x$$

$$1 + \log_e\left(\frac{x}{2}\right) = 1 \Rightarrow x = 2 \ (\mathbf{1M})$$

$$then\ 2 \times \log_e(1) + k = 2 \Rightarrow k = 2 \ (\mathbf{1A})$$

ii. Determine all values of k for which the graphs of $y = g(x)$ and $y = g^{-1}(x)$ do not intersect. 1 mark

D1

2M Learning Objective [2.4.2] Find minimum and maximum.

3M Learning Objective [2.6.2] Find unknowns for number of solutions.

$k > 2$ (**1A**)

Question 8 (5 marks)

Let $f: [0,2] \to \mathbb{R}, f(x) = (x+1)\sqrt{6-x^2}$.

a. Show that $f'(x) = \frac{-2x^2-x+6}{\sqrt{6-x^2}}$. 2 marks

D2

$$u = x+1, v = (6-x^2)^{1/2}$$

$$v' = \frac{1}{2}(6-x^2)^{-\frac{1}{2}} \cdot (-2x) = -\frac{x}{\sqrt{6-x^2}}$$

$$f'(x) = 1 \cdot \sqrt{6-x^2} + (x+1)\left(-\frac{x}{\sqrt{6-x^2}}\right) \mathbf{1M}$$

$$f'(x) = \frac{6-x^2}{\sqrt{6-x^2}} - \frac{x(x+1)}{\sqrt{6-x^2}} \mathbf{1M} = \frac{6-x^2-x^2-x}{\sqrt{6-x^2}} = \frac{-2x^2-x+6}{\sqrt{6-x^2}}.$$

2M Learning Objective [2.3.1] Find general derivatives with functional notation.

3M Learning Objective [2.5.2] Advanced Maximum/Minimum Questions.

b. State the coordinates of the stationary point of the graph of $y = f(x)$. 2 marks

D1

$$-2x^2 - x + 6 = 0 \Leftrightarrow 2x^2 + x - 6 = 0 \Leftrightarrow (x+2)(2x-3) = 0$$

$$x = \frac{3}{2}, x = -2 \quad \mathbf{1M}$$

$x = \frac{3}{2}$ as domain is [0,2]

$$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}+1\right)\sqrt{6-\left(\frac{3}{2}\right)^2} = \frac{5\sqrt{15}}{4}$$

$$\left(\frac{3}{2}, \frac{5\sqrt{15}}{4}\right) \mathbf{1A}$$

2M Learning Objective [2.3.1] Find general derivatives with functional notation.

3M Learning Objective [2.5.2] Advanced Maximum/Minimum

c. State the minimum value of $f(x)$. Hint: We can compare the size of numbers which have square roots by comparing their squares instead. 1 mark

D1

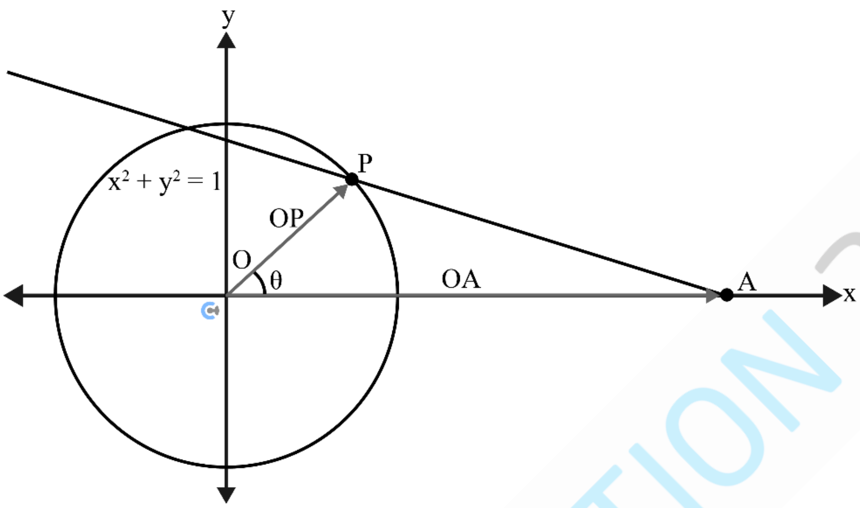
Minimum value is $\sqrt{6}$ **1A**

2M Learning Objective [2.3.1] Find general derivatives with functional notation.

3M Learning Objective [2.5.2] Advanced Maximum/Minimum Questions.

Question 9 (4 marks)

Consider the unit circle $x^2 + y^2 = 1$ centred at the origin O. Let $A = (3, 0)$. For a point P on the half of the circle that lies above the x-axis, let θ (in radians), be the angle between the positive x-axis and the line segment OP. Let $g(\theta)$ be the area of the triangle OAP.



a. Define the function g. 2 marks

D1

3M Learning Objective [2.5.2] Advanced Maximum/Minimum Questions.

$OA = 3, OP = 1$

$$OAP = \frac{1}{2}(OA)(OP)\sin\theta$$

$$g(\theta) = \frac{3}{2}\sin\theta \ \mathbf{1A}$$

Since P lies on the upper half of the unit circle (read question)

Domain: $0 < \theta < \pi$ **1A**

b. Determine the maximum possible area of the triangle OAP and the value of θ at which this occurs. 2 marks

D1

3M Learning Objective [2.5.2] Advanced Maximum/Minimum Questions.

$g(\theta) = \frac{3}{2}\sin\theta$ is maximum when $\theta = \frac{\pi}{2}$ **1A**

Max area is $\frac{3}{2}$ $units^2$ **1A**

Note that this is only because at endpoints area is zero