Write your **student number** in the boxes above. **Letter**

Mathematical Methods ¾

Examination 1 (Tech-Free)

Question and Answer Book

VCE Examination (Term 1 Mock) – April 2025

- Reading time is **15 minutes.**
- Writing time is **1 hour.**

Materials Supplied

- Question and Answer Book of 12 pages.

Instructions

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents

Student's Full Name: ______________________________

Student's Email: ______________________________

Tutor's Name: ______________________________

Marks (Tutor Only): ____________________

Section A

Instructions

- Answer **all** questions in the spaces provided.
- Write your responses in English.

Question 1 (4 marks)

Let $y = \frac{\tan(2x)}{x^3}$.

a. Find $\frac{dy}{dx}$. Give your answer in the simplest form. 2 marks

$$\frac{dy}{dx} = \frac{\frac{1}{\cos^2(2x)} \times 2 \times x^3 - \tan(2x) \times 3x^2}{(x^3)^2}$$

$$= \frac{\frac{2x^3}{\cos^2(2x)} - 3x^2\tan(2x)}{x^6}$$

$$= \frac{2}{x^3\cos^2(2x)} - \frac{\tan(2x)}{x^4}$$

Let $f(x) = x^4 e^{4x-1}$.

b. Evaluate $f'(1)$. 2 marks

$$f'(x) = 4x^3 \cdot e^{4x-1} + x^4 \times e^{4x-1} \times 4$$

$$f'(1) = 4e^3 + 4e^3$$

$$= 8e^3$$

Question 2 (3 marks)

Find the value of $k \in R$ for which the system of linear equations:

$$3x - (k+1)\,y = 2k + 4,$$
$$(k+2)\,x - 10\,y = 4k + 8$$

$$y = \frac{3x}{k+1} - \frac{(2k+4)}{k+1}$$
$$y = \frac{(k+2)x}{10} - \frac{(4k+8)}{10}$$

Has infinitely many solutions. $m_1 = m_2$

$$\frac{3}{k+1} = \frac{k+2}{10}$$
$$30 = k^2 + 3k + 2$$
$$k^2 + 3k - 28 = 0$$
$$(k+7)(k-4) = 0$$
$$\therefore k = -7, 4$$

$c_1 = c_2$

$$\frac{-(2k+4)}{k+1} = \frac{-(4k+8)}{10}$$
$$20k + 40 = 4k^2 + 12k + 8$$
$$4k^2 - 8k - 32 = 0$$
$$k^2 - 2k - 8 = 0$$
$$(k-4)(k+2) = 0$$
$$\therefore k = -2, 4$$

$$\boxed{\therefore k = 4}$$

Question 3 (5 marks)

Consider the quartic polynomial $p(x) = x^4 - 5x^2 + 4$.

a. Express $p(x)$ as a product of linear factors. 2 marks

Let $A = x^2$

$A^2 - 5A + 4$

$= (A-4)(A-1)$

$\therefore\ p(x) = (x^2-4)(x^2-1)$

$\therefore\ p(x) = (x-2)(x+2)(x-1)(x+1)$

b. Sketch the graph of $y = p(x)$, labelling all axial intercepts and turning points. Use the fact that $\frac{\sqrt{10}}{2} \approx 1.6$. 3 marks

$p'(x) = 4x^3 - 10x$

$p'(x) = 0$

$x(4x^2-10) = 0$

$\therefore\ x = 0,\ \pm\sqrt{\frac{5}{2}}$

$x = 0,\ \pm\frac{\sqrt{10}}{2}$

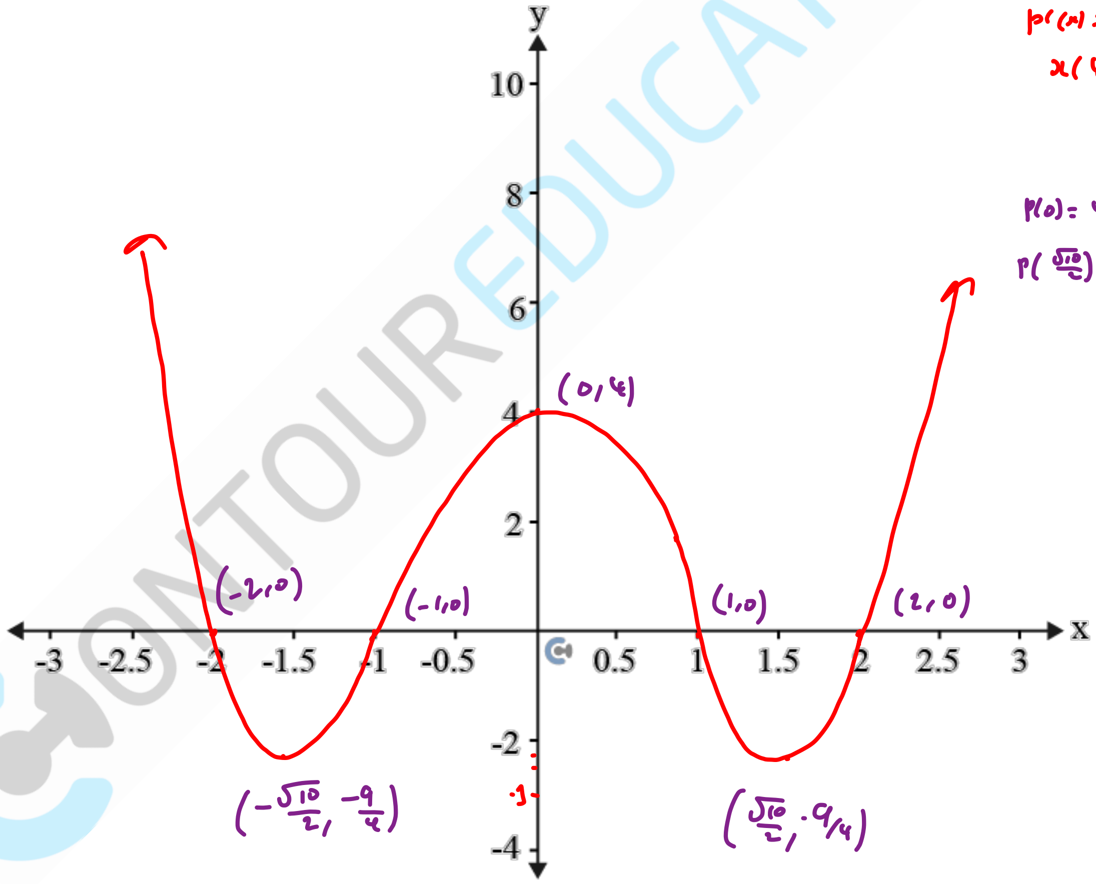
$p(0) = 4$

$p\left(\frac{\sqrt{10}}{2}\right) = \frac{100}{16} - 5\times\frac{10}{4} + 4$

$= \frac{25}{4} - \frac{50}{4} + 4$

$= -\frac{25}{4} + \frac{16}{4}$

$= -9/4$



Question 4 (5 marks)

Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}, f(x) = \frac{1}{x}$ and let $g(x) = f(x+1)$.

a. State the rule of the function $f(f(x))$ and state its domain and range. 2 marks

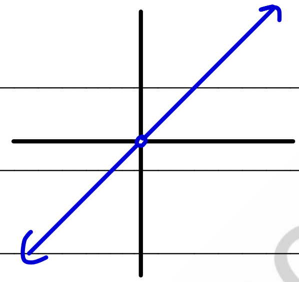
$f(f(x)) = \frac{1}{f(x)}$

$= \frac{1}{1/x}$

$= x$

dom: $R\backslash\{0\}$

ran: $R\backslash\{0\}$



b. Does the function $g(g(x))$ exist? Explain your answer. 2 marks

R|DO

$g(x) = f(x+1) = \frac{1}{x+1}$

dom g: $R\backslash\{-1\}$

ran g: $R\backslash\{0\}$

$R\backslash\{0\} \not\subseteq DO$, ∴ does not exist!

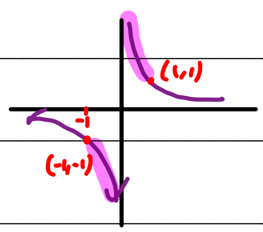
c. Consider a new function $h(x)$ with the same rule as $f(x)$ but with domain $(-c, c)\backslash\{0\}$ where $c > 0$. 1 mark

Find the largest possible value of c for which the function $g(h(x))$ exists.

R|DO

dom g: $R\backslash\{-1\}$

$\therefore c = 1$



Question 5 (5 marks)

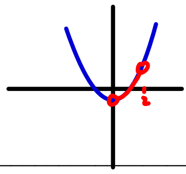
Consider the function:

$$f : [-2, 2] \to \mathbb{R}, f(x) = x^3 - 3x$$

a. Find the rule of the derivative function f'. 1 mark

$f'(x) = 3x^2 - 3$

b. Find the range of $f'(x)$ for $x \in (0, 2)$. 1 mark



$f'(0) = -3$

$f'(2) = 9$

ran: $(-3, 9)$

c. Hence, or otherwise, verify that $f(x)$ has a stationary point for some $x \in (0, 2)$. 1 mark

$f'(x) = 0$

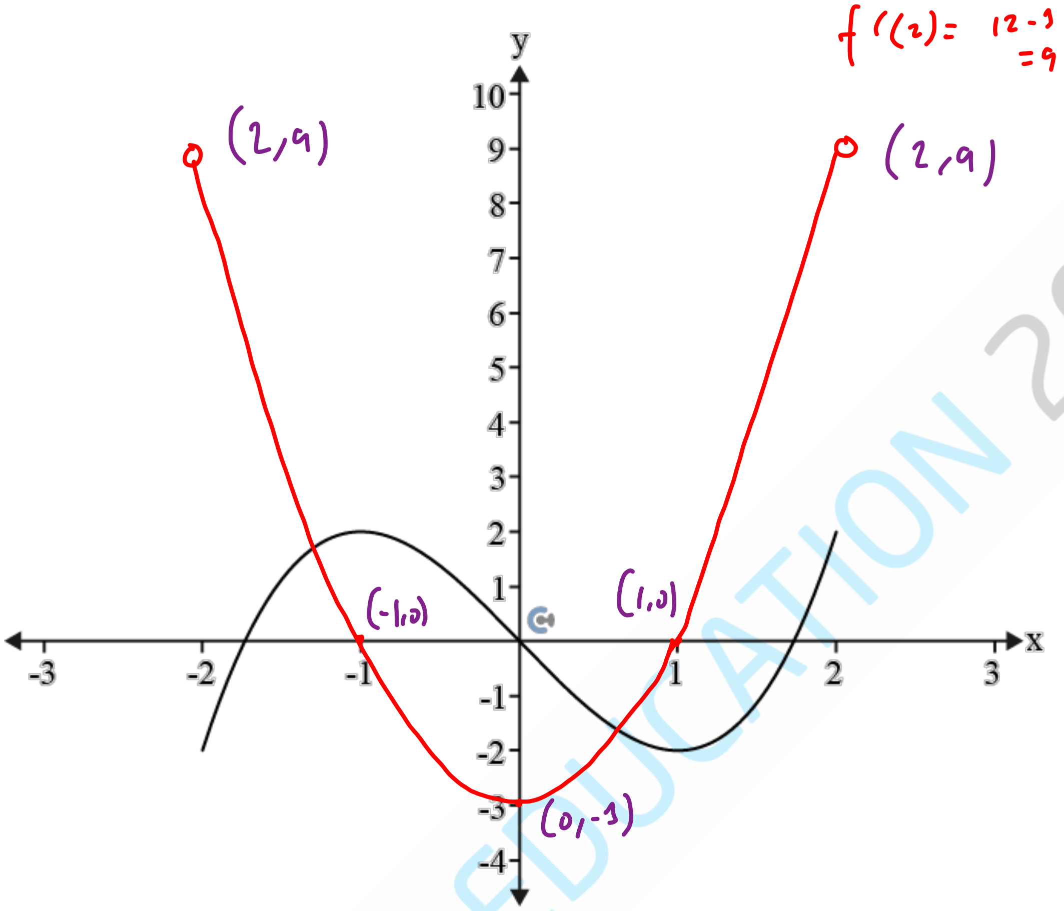
$3x^2 - 3 = 0$

$x^2 = 1$

$x = \pm 1$

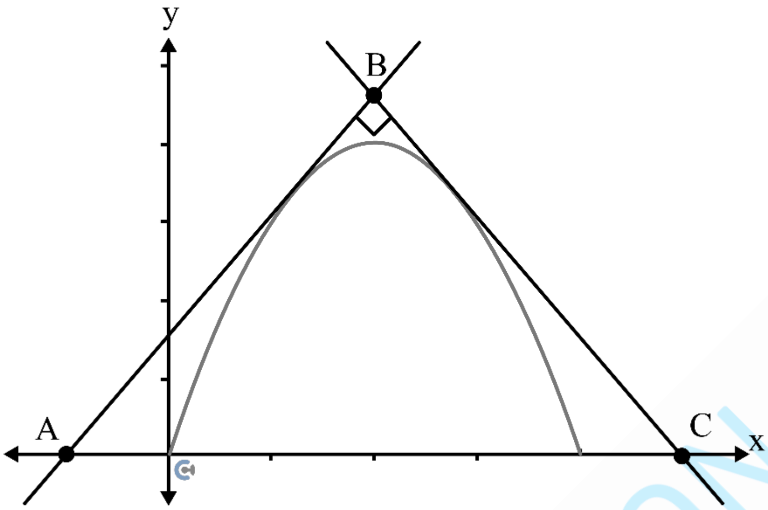
$\therefore$ gradient $= 0$ at $x = 1$, which is in the interval $x \in (0,2)$, hence SP [illegible].

d. On a single set of axes, sketch the graph of $y = f'(x)$. Label axial intercepts, endpoints and turning point. 2 marks



Question 6 (4 marks)

The graph of $f : [0,4] \to \mathbb{R}, f(x) = 4 - (x-2)^2$ is shown below:



The edges of the right angled triangle ABC are the line segments AB and BC, which are tangent to the graph of f, and the line segment AC, which is part of the horizontal axis, as shown above. Let θ be the angle that AB makes with the positive direction of the horizontal axis, where $45° \leq \theta < 90°$.

a. Find the equation of the line through A and B in the form $y = mx + c$ for $\theta = 45°$. 2 marks

$f'(x) = -2(x-2)$

$f'(x) = 1$

$-2(x-2) = 1$

$x - 2 = -\frac{1}{2}$

$x = 3/2$

$f\left(\frac{3}{2}\right) = 4 - \left(\frac{3}{2} - 2\right)^2$

$= 4 - \frac{1}{4}$

$= \frac{15}{4}$

$y = mx + c$

$\therefore y - \frac{15}{4} = 1 \cdot \left(x - \frac{3}{2}\right)$

$y = x + \frac{9}{4}$

b. Find the coordinates of B when $\theta = 45°$. 2 marks

Sub $x = 2$ into $y = x + 9/4$

$\therefore y = 2 + 9/4$

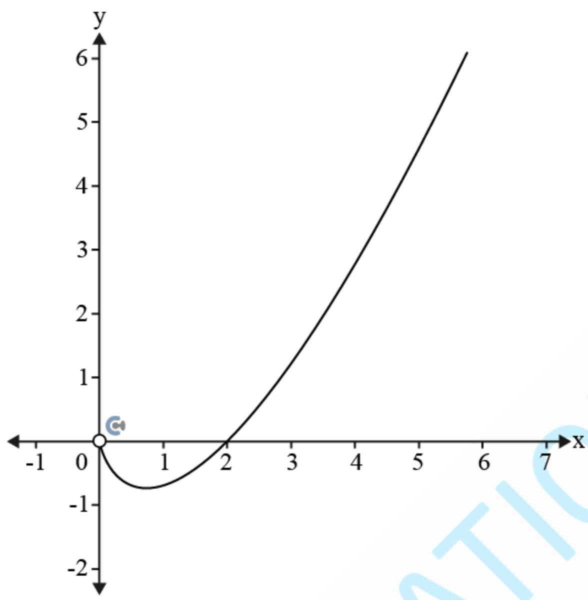
$= \frac{17}{4}$

$\therefore B\left(2, \frac{17}{4}\right)$

Question 7 (5 marks)

Let $f:(0,\infty) \to R, f(x) = x\log_e\left(\frac{x}{2}\right)$

Part of the graph is shown below, f has a minimum at the point $Q(a, f(a))$:



a. Find the coordinates of Q. 2 marks

$f'(x) = \log_e\left(\frac{x}{2}\right) + x \cdot \frac{1}{\frac{x}{2}} \times \frac{1}{2}$

$= \log_e\left(\frac{x}{2}\right) + 1$

$f'(x) = 0$

$\log_e\left(\frac{x}{2}\right) = -1$

$\frac{x}{2} = e^{-1}$

$x = 2e^{-1} = \frac{2}{e}$

$f\left(\frac{2}{e}\right) = \frac{2}{e}\log_e\left(\frac{1}{e}\right)$

$= -\frac{2}{e}$

$\therefore Q\left(\frac{2}{e}, -\frac{2}{e}\right)$

b. Let $g: (a,\infty) \to \mathbb{R}, g(x) = f(x) + k$ where $k \in \mathbb{R}$.

i. Find the value of k for which the line $y = x$ is tangent to the graph of g. 2 marks

$g(x) = x, \quad g'(x) = 1$

$g'(x) = 1;$

$\log_e\left(\frac{x}{2}\right) + 1 = 1$

$\log_e\left(\frac{x}{2}\right) = 0$

$\frac{x}{2} = e^0$

$x = 2$

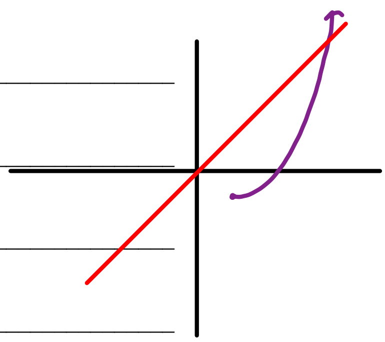
$g(x) = x:$

$2\log_e\left(\frac{x}{2}\right) + k = x$

$\therefore 2\log_e\left(\frac{2}{2}\right) + k = 2$

$\cancel{2\log_e(1)} + k = 2$

$\boxed{\therefore k = 2}$



ii. Determine all values of k for which the graphs of $y = g(x)$ and $y = g^{-1}(x)$ do not intersect. 1 mark

$k > 2$

Question 8 (5 marks)

Let $f : [0,2] \to \mathbb{R}, f(x) = (x+1)\sqrt{6-x^2}$.

a. Show that $f'(x) = \frac{-2x^2-x+6}{\sqrt{6-x^2}}$. 2 marks

$f'(x) = \frac{d}{dx}(x+1) \times \sqrt{6-x^2} + (x+1) \times \frac{d}{dx}\left(\sqrt{6-x^2}\right)$

$= 1 \times \sqrt{6-x^2} + (x+1) \times \frac{d}{dx}\left((6-x^2)^{1/2}\right)$

$= \sqrt{6-x^2} + (x+1) \times \frac{1}{2}(6-x^2)^{-1/2} \times -2x$

$= \sqrt{6-x^2} - \frac{x(x+1)}{\sqrt{6-x^2}}$

$= \frac{6-x^2}{\sqrt{6-x^2}} - \frac{x^2+x}{\sqrt{6-x^2}}$

$= \frac{6-x^2-x^2-x}{\sqrt{6-x^2}}$

$= \frac{-2x^2-x+6}{\sqrt{6-x^2}}$

b. State the coordinates of the stationary point of the graph of $y = f(x)$. 2 marks

$f'(x) = 0$

$-2x^2-x+6=0$

$2x^2+x-6=0$

$(x+2)(2x-3)=0$

$\therefore x = -2, \frac{3}{2}$

$x \in [0,2]$

$\therefore x = 3/2$

$f\left(\frac{3}{2}\right) = \frac{5}{2} \times \sqrt{6-\frac{9}{4}}$

$= \frac{5}{2} \times \frac{\sqrt{15}}{2}$

$= \frac{5\sqrt{15}}{4}$

$\therefore \left(\frac{3}{2}, \frac{5\sqrt{15}}{4}\right)$

TP EP

c. State the minimum value of $f(x)$. Hint: We can compare the size of numbers which have square roots by comparing their squares instead. 1 mark

$\boxed{\text{EP's}}$

$f(0) = \sqrt{6}$

$f(2) = 3\sqrt{2}$

$\boxed{\text{TP}}$

$f\left(\frac{3}{2}\right) = \frac{5\sqrt{15}}{4}$

$(\sqrt{6})^2 = 6$

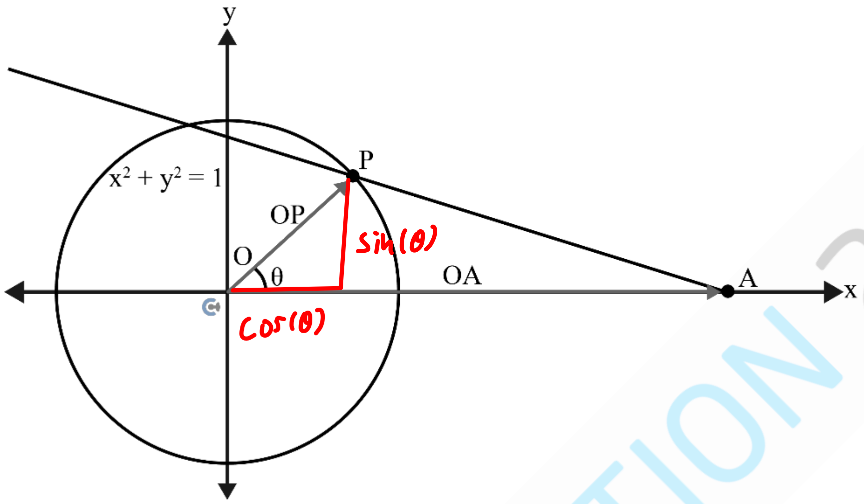
$(3\sqrt{2})^2 = 9 \times 2 = 18$

$\left(\frac{5\sqrt{15}}{4}\right)^2 = \frac{25 \times 15}{16}$

$\therefore \text{Min} = \sqrt{6}$

Question 9 (4 marks)

Consider the unit circle $x^2 + y^2 = 1$ centred at the origin O. Let $A = (3, 0)$. For a point P on the half of the circle that lies above the x-axis, let θ (in radians), be the angle between the positive x-axis and the line segment OP. Let $g(\theta)$ be the area of the triangle OAP.



a. Define the function g. 2 marks

$g(\theta) = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 3 \times \sin(\theta)$

$= \frac{3}{2} \sin(\theta)$

b. Determine the maximum possible area of the triangle OAP and the value of θ at which this occurs. 2 marks

Max $\sin(\theta) = 1$

$\therefore$ Area $= \boxed{\frac{3}{2}}$ when $\theta = \frac{\pi}{2}$