CONTOUREDUCTION

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VCE Mathematical Methods ¾
Term 1 [0.0]
Bound Reference

Cheat Sheet

Maximal Domain and Range

- Inside of a $\log$ must be bigger than 0.
- Inside of a root must be bigger than or equal to 0.
- The denominator cannot be zero.
- The domain of the sum or product of two functions is equal to the intersection of the two domains.

Compositions

- $f(g(x)) = f \circ g(x)$.
- For a composite function to exist, range (output) of inside $\subseteq$ domain (input) of outside.
- The domain of composite is equal to the domain of inside (1^{st}) function.
- The range of composite is a subset of the range of the outside.

Inverse Functions

- f needs to be $1:1$ for f^{-1} to exist.
- The domain of the inverse function equals to range of the original **and** vice versa.
- Symmetrical around $y = x$.
- For intersections of inverses, we can equate the function to $y = x$.
- The composite function of inverses is always given by $f(f^{-1}(x)) = x$.

Fixing Composite Functions

- We restrict the domain of the inside function so its range fits in the domain of the outside function.

Gradient of Inverse Functions

- If the gradient of f at $(a, f(a)) = m$, then the gradient of f^{-1} at $(f(a), a) = \frac{1}{m}$.

Transformation Fundamentals

- The transformed point is called the image and is denoted by (x', y').
- The dilation factor is multiplied by the original coordinate.
- Reflection makes the original coordinates the negative of their original values.
- Translation adds a unit to the original coordinate.
- Transformations should be interpreted when x' and y' are isolated.
- The order of transformation follows the BODMAS order.
- To change the order of transformations, we either factorise or expand.

Cheat Sheet

Find Transformed Functions

- To transform the function, replace its old variables with the new one.
- To find the transformations, simply equate the LHS and RHS after separating the transformations of x and y.

Find Opposite Transformations

- The order is reversed.
- All transformations are in the opposite direction.

Apply Transformations of Functions to Find Their Domain, Range, Transformed Points and Tangents

- Everything moves together as a function.
- Steps:
  1. Find the transformations between two functions.
  2. Apply the same transformations to domain, range, points and tangents.

Find Transformations of the Inverse Functions $f(x)$

- Steps:
  1. Find the transformations between the two original functions.
  2. Inverse the transformations found in $\mathbf{1}$.

Find Multiple Transformations For the Same Functions

- Same transformations can be done differently by either putting it in or out of the $f()$.
- Commonly, look for basic algebra, index and $\log$ laws.

Manipulation of the Functions to Find Appropriate Transformations

- Steps:
  1. Identify the region of x.
  2. Identify the region of y.
  3. Manipulate the function so that all the changes are within the region of x or y.

Midpoint and Distance

- Midpoint of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
- Distance is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.
- Horizontal distance is the distance between x values.
- Vertical distance is the distance between y values.

Find Parallel and Perpendicular Lines

- Parallel lines $m_1 = m_2$.
- Perpendicular lines have $m_1 = -\frac{1}{m_2}$.

Cheat Sheet

Angles Between Lines

- To find the angle between a line and the x-axis we can use the equation $m = \tan(\theta)$.
- To find the angle between two lines, we can use

$$\theta = |tan^{-1}(m_1) - tan^{-1}(m_2)|$$

or

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Find The Unknown Value for Systems of Linear Equations

- Two linear equations have unique solutions if they have different gradients.
- Two linear equations have infinitely many solutions when they have the same gradient and the same constant.
- Two linear equations have no solution when they have the same gradient and different constants.

Find Factored Form of Polynomials

- The Rational Root Theorem **narrows down** the possible roots. If the roots are rational numbers, it must be that any:

$$Potential\ root = \pm\frac{Factors\ of\ constant\ term\ a_0}{Factors\ of\ leading\ coefficient\ a_n}$$

- Sum and difference of cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Graph Factored and Unfactored Polynomials

- Graphs of $a(x-h)^n + k$, where n is an odd positive integer that is not equal to 1:
  - The point (h, k) gives us the stationary point of inflection.
- Graphs of $a(x-h)^n + k$, where n is an even positive integer:
  - The point (h, k) gives us the turning point.
  - These graphs look like a quadratic.

Find a Reflected Point

- The line between a point and its reflection is perpendicular to the line it is reflected in.
- The midpoint of a line and its reflection lie on the line it is reflected in.
- **Steps** for finding the reflection of a point in a line:
  1. Find the perpendicular line passing through the point.
  2. Find the intersection between the original line and the perpendicular line.
  3. Find the reflected point (x, y) by treating the intersection from **2** as the midpoint between the original and reflected point.

Cheat Sheet

Polynomials

- The degree of a polynomial is the polynomial's highest power.
- The roots of a polynomial are its x-intercepts.
- For polynomial long division:

$$\frac{\textbf{Dividend}}{\textbf{Divisor}} = \textbf{Quotient} + \frac{\textbf{Remainder}}{\textbf{Divisor}}$$

- When $P(x)$ is divided by $(x-\alpha)$, the remainder is $P(\alpha)$.
- If $P(\alpha)=0$, then $(x-\alpha)$ is a factor of $P(x)$.
- Steps to graphing factorised polynomials:
  1. Plot x-intercepts.
  2. Determine whether the polynomial is positive or negative.
  3. Use the repeated factors to deduce the shape:
     - Non-Repeated: Only x-intercept.
     - Even Repeated: x-intercept and a turning point.
     - Odd Repeated: x-intercept and a stationary point of inflection.

Odd, Even and Power Functions:

- **Odd Functions:**

$$f(-x) = -f(x)$$

  - Property: Reflecting on the y-axis is the same as reflecting around the x-axis.

- **Even Functions:**

$$f(-x) = f(x)$$

  - Property: It is symmetrical about the y-axis.

- **Power Functions:**

$$y = x^{\frac{n}{m}}$$

  - m: Dictates the number of **tails**.
    - **Odd m = Two tails.**
    - **Even m = One tail.**
  - n: Dictates the **range**.
    - Odd n: Range could be all real.
    - Even n: Range must be non-negative.
  - Power > 1: Looks like a polynomial function.
  - Power < 1: Looks like a root function.

Cheat Sheet

Solve Number of Solutions Questions

- There are no real solutions for a quadratic when $\Delta < 0$.
- There is one real solution for a quadratic when $\Delta = 0$.
- There are two unique real solutions for a quadratic when $\Delta > 0$.
- To find the number of solutions for $f(x) = k$, draw a horizontal line at $y = k$ and count the intersections.

Identify Possible Rule(s) From a Graph

- A turning point x-intercept has $a(n)$ even power on its factor.
- A stationary point of inflection x intercept has $a(n)$ odd power on its factor.
- If the x-intercept passes straight through, the power of the factor is 1.

Instantaneous Rate of Change and Average Rate of Change

- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:
  - $\textit{Average rate of change} = \frac{f(b)-f(a)}{b-a}$
- It is the gradient of the line joining the two points.
- **Instantaneous Rate of Change** is a gradient of a graph at a single point / moment.
- **First Principles** derivative definition:
  - $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$
- **The Product Rule**
  - The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

- **The Quotient Rule**
  - The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

  - Always differentiate the top function first.
- **The Chain Rule**

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

Cheat Sheet

Nature of Stationary Points and Trends (Strictly Increasing and Decreasing)

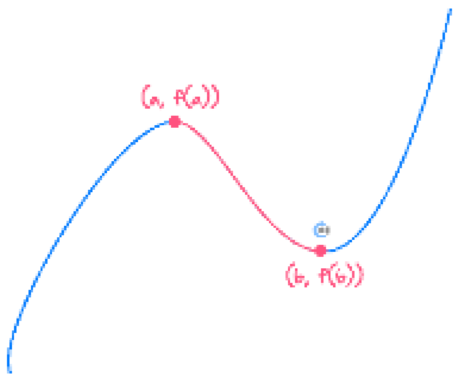
- Point where the gradient of the function is zero.

$$f'(x) = 0, \frac{dy}{dx} = 0$$

- Can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing curve	Stationary Point	∪ - Increasing curve

- Find the gradient of the neighboring points.
- **Strictly Increasing and Strictly Decreasing Functions**



Strictly Increasing: $x \in (-\infty, a] \cup [b, \infty)$

Strictly Decreasing: $x \in [a, b]$

Steps:

1. Find the turning points.
2. Consider the sign of the derivative between / outside the turning points.

Graph Derivative Functions

- **Steps on Sketching the Derivative Function:**

1. Plot x-intercept at the same x-value as the stationary point of the original.
2. Consider the trend of the original function and sketch the derivative.
   - Original is increasing → Derivative is above the x-axis.
   - Original is decreasing → Derivative is below the x-axis

Evaluate Limits and Find Points Where the Function is Not Continuous

- **Limit Definition:**

$$\lim_{x \to a} f(x) = L$$

"The function $f(x)$ approaches L as x approaches a."

- **Validity of Limit:**

$$\lim_{x \to a} f(x) \text{ exists if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Limit is defined when the left limit equals the right limit.

- **Continuity:**

A function f is said to be continuous at a point $x = a$ if:

1. $f(x)$ is defined at $x = a$.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Cheat Sheet

Differentiability and Domain of Derivative

- **Differentiability:**
  - **A function f is said to be differentiable at a point $x = a$ if:**
    1. **$f(x)$ is continuous at $x = a$.**
    2. **$\lim_{x \to a} f'(x)$ exists.**
       - Limit exists when the left and right limits are the same.
       - Gradient on the LHS and RHS must be the same.
- We **cannot** differentiate:
  1. Discontinuous Points.
  2. Sharp Points.
  3. Endpoints.
- **Finding the Derivative of Hybrid Functions**
  1. Simply derive each function.
  2. Reject the values for x that are not differentiable from the domain.

Concavity and Find Inflection Points

- **Second Derivatives**
  - The derivative of the derivative.
  - To get the second derivative, we can **differentiate the original function twice**.

$$\frac{d^2y}{dx^2} = f''(x)$$

- **Concavity**
  - Concave up is when the gradient is increasing.

    $f''(x) > 0$ → Concave Up
  - Concave down is when the gradient is decreasing.

    $f''(x) < 0$ → Concave Down
  - Zero concavity is when the gradient is neither increasing nor decreasing.

    $f''(x) = 0$ → Zero Concavity
- **Points of Inflection**
  - A point at which a curve **changes concavity** is called a **point of inflection**.
- **The Second Derivative Test**
  - Suppose that $f'(a) = 0$ and hence, f has a stationary point at $x = a$. The second derivative test states:
    - **Concave up gives us local minimum.**

      $f''(x) > 0$ → Local Minimum
    - **Concave down gives us local maximum.**

      $f''(x) < 0$ → Local Maximum
    - **Zero concavity gives us a stationary point of inflection.**

      $f''(x) = 0$ → Stationary Point of Inflection

Cheat Sheet

Find Derivatives with Functional Notation

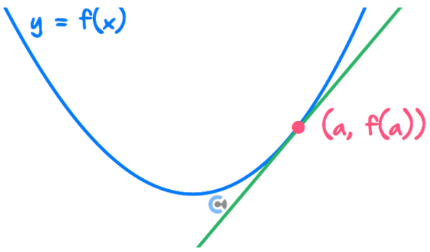
- To derive composite functions like $\sin(f(x))$, apply the chain rule.

Apply Differentiability to Join Two Functions Smoothly

- When two functions join smoothly at a point, the value and derivative of each function are both equal at that point.
- Solve $f(a) = g(a)$ and $f'(a) = g'(a)$ simultaneously.

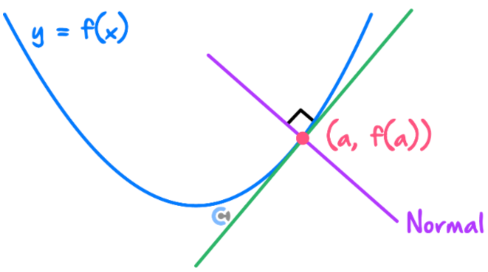
Tangents and Normals

- A **tangent** is a linear line which **just** touches the curve.
- The gradient of a tangent line has to be equal to the gradient of the curve at the intersection.



$$At\ (a, f(a)): \quad m_{tangent} = f'(a)$$

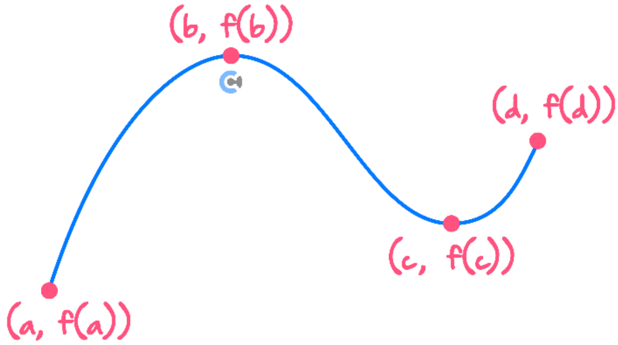
- **Normals:**
  - A **normal** is a linear line which is perpendicular to the tangent.
  - The gradient of a normal line has to equal the negative reciprocal of the gradient of the curve at the intersection.



$$At\ (a, f(a)): \quad m_{normal} = -\frac{1}{f'(a)}$$

Find the Minimum and Maximum

- **Absolute Maximum and Minimum**
  - **Absolute Maxima / Minima** are the overall **largest / smallest** y values for the given domain.
  - **They occur at either** Endpoints or Turning points.



Absolute Min: $f(a)$

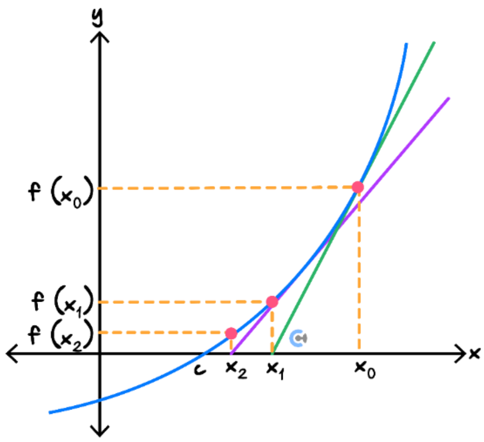
Absolute Max: $f(b)$

- **Steps :**
  1. Find stationary points and endpoints.
  2. Find the largest / lowest y value for max / min.
- **Steps for optimization:**
  1. Construct a function for the subject you want to find the maximum or minimum of.
  2. Find its domain if appropriate.
  3. Find its endpoints and turning points.
  4. Identify maximum or minimum y value.

Cheat Sheet

Newton's Method

- **Newton's Method**
  - It is a method of approximating the x-intercept using tangents.



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- **Steps:**

1. Find the tangent at the x value given.
2. Find the x-intercept of the tangent using an iterative formula.
3. Find the next tangent at the $x = x$-intercept of the previous tangent.
4. Repeat until the value doesn't change by much.

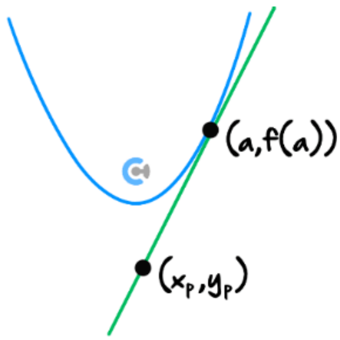
- **Tolerance**: The maximum difference between x_n and x_{n+1} we can have when we stop the iteration.

$$\textit{We stop when } |x_{n+1} - x_n| < \textit{Tolerance}$$

- **Limitation of Newton's Method**
  - **Terminating Sequence**: Occurs when we hit a Stationary point.
  - **Approximating a Wrong Root**: Occurs when we start on the Wrong side.
  - **Oscillating Sequence**: Occurs when we oscillate between two values without getting closer to the real root.

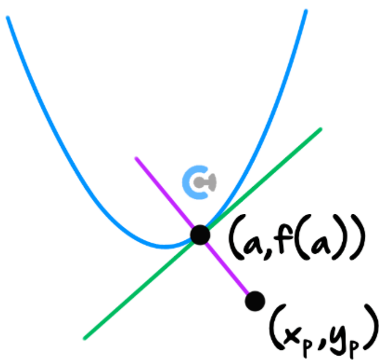
Advanced Tangents and Normal Questions

- Finding tangents / normals to functions, which also pass through a given point
- Tangent of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$f'(a) = \frac{f(a) - y_p}{a - x_p}$$

- Normal of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$-\frac{1}{f'(a)} = \frac{f(a) - y_p}{a - x_p}$$

Advanced Maximum / Minimum Questions

- To find the maximum / minimum instantaneous rate of change, we find the turning point of the derivative function.
- **Families of Functions:** Functions with an unknown.
- They involve understanding and using transformations.
- They involve the use of sliders/manipulate on CAS / technology.

Cheat Sheet

Find Unknowns for Number of Solutions

- For a function to "touch" a line as a tangent:
  - They intersect.

$$f(a) = mx + c$$

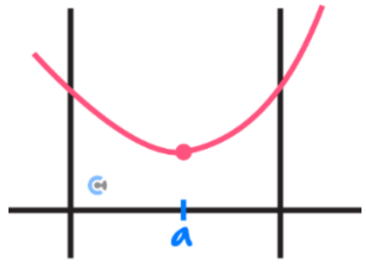
  - With the same gradient.

$$f'^{(a)} = m$$

- We solve these simultaneously.

Find Unknowns for Minimum and Maximum

- **Minimum/maximum at a turning point:**

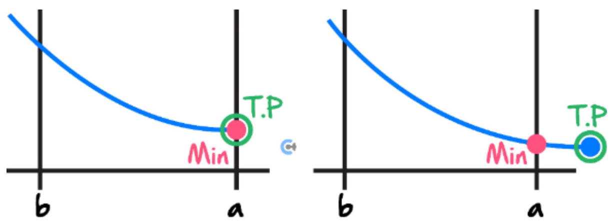


  - To achieve minimum / maximum at $x = a$.

$$f'(a) = 0$$

  - This is only when $x = a$ is not an endpoint.

- **Minimum/maximum at an endpoint:**



  - Step 1: Find the value of the unknown such that the turning point occurs at $x = a$.

$$f'(a) = 0$$

  - Step 2: Find the value of the unknown such that the turning point occurs after/below $x = a$.
  - We can determine whether the unknown can be bigger or smaller than the value achieved in step 1.

$$f'(a \pm something) = 0$$

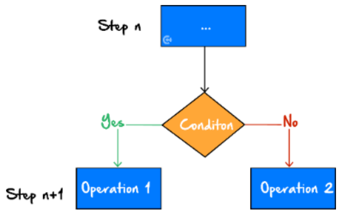
Cheat Sheet

Pseudocode

- **Assigning Variables:**
  - To construct algorithms for more mathematical / complex problems, assigning variables will be useful.

    $A \leftarrow 3$ assigns the **value 3** to the **variable A.**
  - We can also update our variables using the arrow.

    $A \leftarrow A + 3$ assigns the value $\mathbf{A + 3}$ to the **variable A.**
    - Since the value of A was already 3, Its new value will be 6.
- **Selections:**
  - Selections allow us to perform different operations at a given step, depending on a certain condition.



  - We are selectively performing an operation.
  - "If-then"

```
if condition then
    operation
end if
```

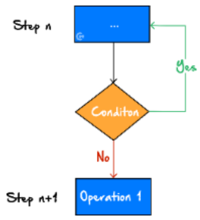
    - **Allows** us to perform an operation only when a certain condition is met.
  - "Else"

```
if condition then
    operation 1
else
    operation 2
end if
```

    - Provides an opportunity to perform an operation only when a certain condition is met.
  - "Else-If"

```
if condition 1 then
    operation 1
else if condition 2 then
    operation 2
else
    operation 3
end if
```

    - Provides an opportunity to add multiple pathways, each with different conditions.
- **Iteration (Loops):**
  - Iteration (a.k.a. looping) allows us to repeat steps in a Controlled way.
  - It is controlled by the condition.
    - E.g., we only loop when a condition is met.



  - **For loops:**

```
for variable from lower bound to upper bound
                    condition
    operation
end for
```

    - Loops for which a variable increases by one each time it loops.
    - The variable gets moved from the lower bound to the upper bound by 1.
  - **While loops:** Loops that do **not** change the value of any variable by default.

```
while condition
    operation
end while
```

Cheat Sheet

Pseudocode for Newton's Method

- A key component of Newton's method is the recursive relationship.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Newton's method requires an input function $f(x)$, the derivative $f'(x)$ and an initial value x_0.

- The number of iterations that Newton's method performs can be limited in our pseudocode.

- The pseudocode can also specify a tolerance for Newton's method where the algorithm terminates if

$$|x_{n+1} - x_n| < \textit{Tolerance}$$