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VCE Mathematical Methods ½
Exponentials [5.1]
Homework Solutions

Admin Info & Homework Outline:

Student Name	
Questions You Need Help For	
Compulsory Questions	Pg 2 – Pg 20
Supplementary Questions	Pg 21 – Pg 33

Section A: Compulsory Questions

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Learning Objective: [5.1.1] - Basics of exponentials

Key Takeaways

- Exponentials

$$base \times \cdots \times base = base^{power}$$

  - Exponentiation is a stacked multiplication.
  - The power represents the number of bases we are multiplying.

- Index Laws

$$a^x \times a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{xy}$$

$$a^0 = 1$$

$$(a \cdot b)^x = a^x \cdot b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$a^{-x} = \frac{1}{a^x}$$

$$a^{\frac{1}{x}} = \sqrt[x]{a}$$

- Inequalities for Exponentials

$$\text{For } a^x < a^y$$

- Flip the inequality sign when the base is less than 1.

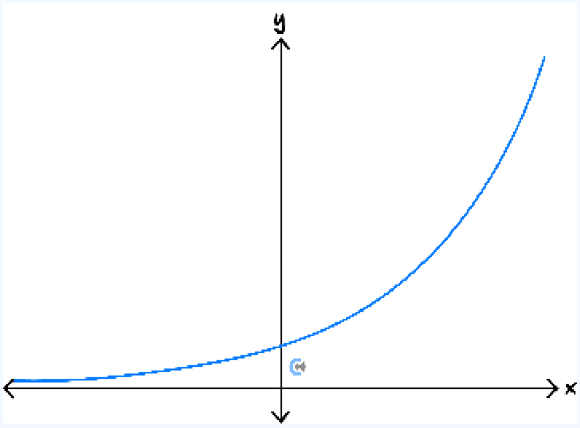
$$\text{If } a > 1 \text{ then } x < y$$

$$\text{If } 0 < a < 1 \text{ then } x > y$$

Learning Objective: [5.1.2] – Graph exponentials

Key Takeaways

- Exponential Functions



$$a^x \text{ where } a > 1$$

- Domain of the exponential function is R.
- Range of the exponential function is R^+.

- Graphs of Exponential Functions

$$y = a\ base^{b(x-h)} + k$$

- The horizontal asymptote is always given by $y = k$.
- **Steps to take when sketching an exponential:**
  1. Find corresponding asymptotes.
  2. Plot x and y-intercepts (if they exist).
  3. Sketch the curve.
  4. Always follow these steps as they minimise potential mistakes.

Learning Objective: [5.1.3] - Solve hidden quadratics of exponentials

Key Takeaways

- Hidden Quadratics

$$af(x)^2 + bf(x) + c = 0$$

$$Let\ A = f(x)$$

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Sub-Section [5.1.1]: Basics of Exponentials

Question 1

Evaluate $8^{-\frac{2}{3}}$.

$8^{\frac{-2}{3}}$ $\frac{1}{4}$

Question 2

Solve $49^x = 7$ for x.

Step-by-step:

- $49 = 7^2$, so $49^x = (7^2)^x = 7^{2x}$
- So $7^{2x} = 7^1$

Now equate exponents:

$$2x = 1 \Rightarrow x = \frac{1}{2}$$

$$\boxed{x = \frac{1}{2}}$$

Question 3

Solve the equation $2^{3x-3} = 8^{2-x}$ for x.

Step-by-step:

- $8 = 2^3$, so $8^{2-x} = (2^3)^{2-x} = 2^{3(2-x)} = 2^{6-3x}$

Now equation becomes:

$$2^{3x-3} = 2^{6-3x}$$

Equate the powers:

$$3x - 3 = 6 - 3x \Rightarrow 6x = 9 \Rightarrow x = \frac{3}{2}$$

$$\boxed{x = \frac{3}{2}}$$

Question 4

Find the set of values of x for which $9^{x+1} \geq 27^{x-5}$

$$3^{2x+2} \geq 3^{3x-15}$$

$$2x + 2 \geq 3x - 15$$

$$x \leq 17$$

Question 5

Simplify the following, writing your answer in the positive index form.

a. $\sqrt{25x^5y^4}$

$$5x^{\frac{5}{2}}y^2$$

b. $3^n \times 9^{n+1} \times 27^n + 2$

$$3^{6n+2} + 2$$

c. $\frac{(-3x^2y^3)^2}{(2xy)^3}$

$$\frac{(-3 \cdot x^2 \cdot y^3)^2}{(2 \cdot x \cdot y)^3} \quad \frac{9 \cdot x \cdot y^3}{8}$$

d. $\frac{x^3yz^{-2} \times (2x^3y^{-2}z)^2}{xyz^{-1}}$

Step-by-step:

First simplify the squared part:

- $(2x^3y^{-2}z)^2 = 4x^6y^{-4}z^2$

Now multiply numerator:

$$x^3yz^{-2} \times 4x^6y^{-4}z^2 = 4x^9y^{-3}z^0 = 4x^9y^{-3}$$

Now divide by denominator xyz^{-1}:

- $x^9/x = x^8$
- $y^{-3}/y = y^{-4}$
- $z^0/z^{-1} = z^1$

Final answer:

$$\boxed{4x^8y^{-4}z} \Rightarrow \boxed{\frac{4x^8z}{y^4}}$$

Question 6

Evaluate $3^{-4} \times 27^{-\frac{1}{3}} \times 2^2$

$$3^{-4} \cdot 27^{\frac{-1}{3}} \cdot 2^2 \qquad \frac{4}{243}$$

Question 7

Simplify $\frac{12^{-5} \times 2^3 \times 9^{-8}}{6^2}$.

Step-by-step:

- $12 = 2^2 \times 3 \rightarrow$ so $12^{-5} = (2^2 \times 3)^{-5} = 2^{-10} \times 3^{-5}$
- $9^{-8} = (3^2)^{-8} = 3^{-16}$
- $6^2 = (2 \times 3)^2 = 2^2 \times 3^2$

Now putting everything together:

$$\frac{2^{-10} \times 3^{-5} \times 2^3 \times 3^{-16}}{2^2 \times 3^2} = \frac{2^{-7} \times 3^{-21}}{2^2 \times 3^2} = 2^{-7-2} \times 3^{-21-2} = 2^{-9} \times 3^{-23}$$

So the answer is:

$$\boxed{\frac{1}{2^9 \cdot 3^{23}}}$$

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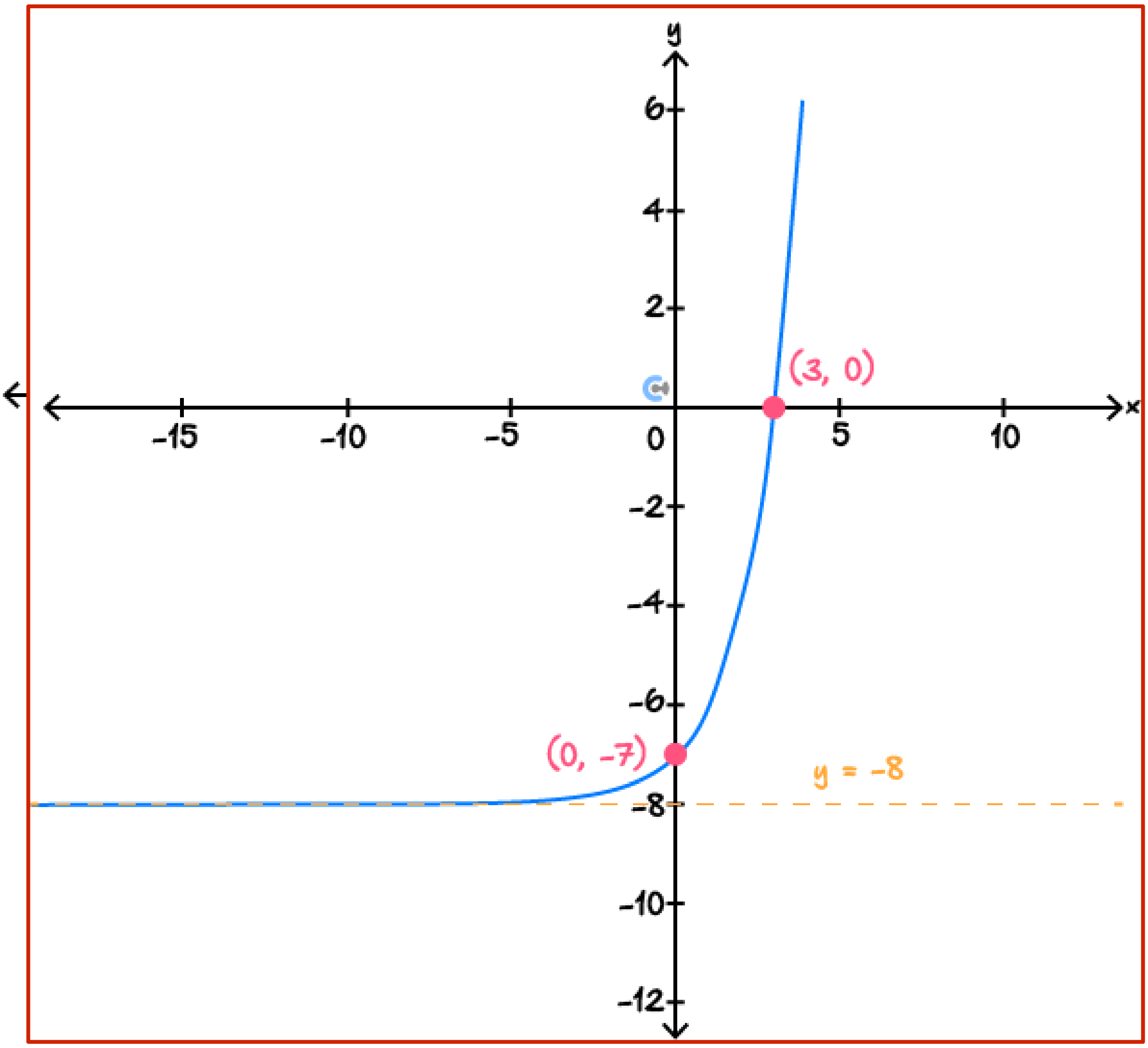
Sub-Section [5.1.2]: Graphs of Exponentials

Question 8

Sketch the graphs of each of the following, labelling all intercept coordinates and asymptotes with equations.

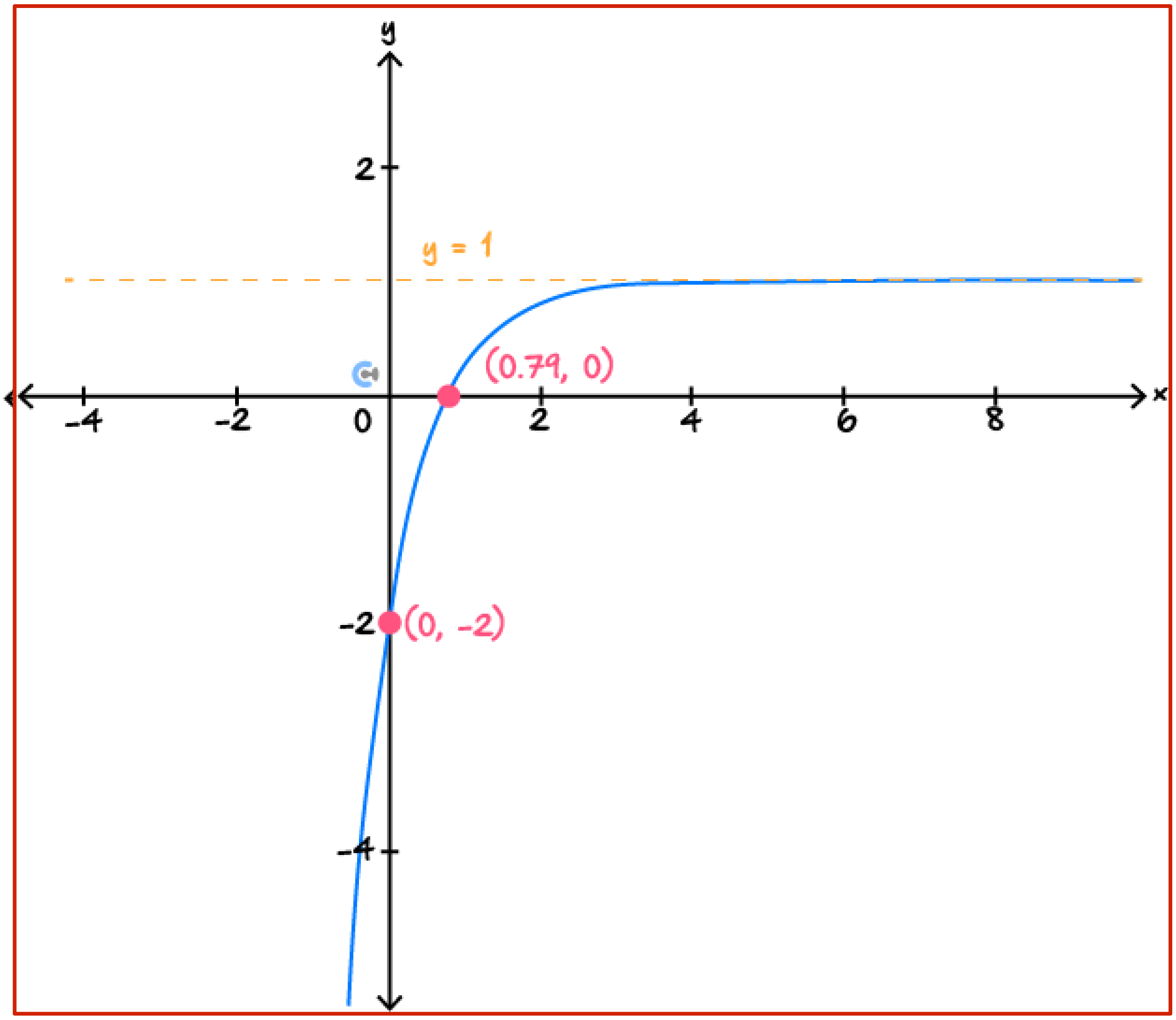
a. $y = 2^x - 8$



Asymptote: $y = -8$, $x - intercept$ $(3,0)$

b. $y = -3 \times 4^{-x} + 1$

Useful information : $4^{-0.79} = \frac{1}{3}$



Asymptote: $y = 1, x - in\ (0.79{,}0)$

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Sub-Section [5.1.3]: Solving Exponential Equations with Hidden Quadratics

Question 9

a. Solve the equation $4^x - 15 \times 2^x = 16$ for x.

Step-by-step:

- Note: $4^x = (2^2)^x = 2^{2x}$

Let's set $y = 2^x$, then:

- $4^x = y^2$
- $2^x = y$

So the equation becomes:

$$y^2 - 15y = 16 \Rightarrow y^2 - 15y - 16 = 0$$

Use the quadratic formula:

$$y = \frac{15 \pm \sqrt{(-15)^2 + 4 \cdot 1 \cdot 16}}{2} = \frac{15 \pm \sqrt{225 + 64}}{2} = \frac{15 \pm \sqrt{289}}{2} = \frac{15 \pm 17}{2}$$

So:

- $y = \frac{32}{2} = 16$
- $y = \frac{-2}{2} = -1$ (but we discard this, since $2^x > 0$)

Now solve:

$$2^x = 16 \Rightarrow x = 4$$

b. Solve the equation $2^{2x} + 4 \bullet 2^x - 32 = 0$ for x.

Let $y = 2^x$, so $2^{2x} = y^2$. The equation becomes:

$$y^2 + 4y - 32 = 0$$

Solve the quadratic:

$$(y + 8)(y - 4) = 0 \Rightarrow y = -8 \text{ or } y = 4$$

But $y = 2^x > 0$, so $y = -8$ is not valid.

So:

$$2^x = 4 \Rightarrow x = 2$$

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Sub-Section: Problem Solving

Question 10

Solve $3^x = 6 + 27 \bullet 3^{-x}$ for x.

Let $y = 3^x$, so $3^{-x} = \frac{1}{y}$.

Substitute into the equation:

$$y = 6 + 27 \cdot \frac{1}{y}$$

Multiply both sides by y:

$$y^2 = 6y + 27 \Rightarrow y^2 - 6y - 27 = 0$$

Solve using the quadratic formula:

$$y = \frac{6 \pm \sqrt{(-6)^2 + 4 \cdot 27}}{2} = \frac{6 \pm \sqrt{36 + 108}}{2} = \frac{6 \pm \sqrt{144}}{2} = \frac{6 \pm 12}{2}$$

$$y = 9 \quad \text{or} \quad y = -3$$

But $y = 3^x > 0$, so reject $y = -3$.

$$3^x = 9 \Rightarrow x = 2$$

Question 11

Solve $5 \bullet 3^{-2x} = 11 \bullet 3^{-x} + 12$.

Let $y = 3^{-x}$, so $3^{-2x} = (3^{-x})^2 = y^2$

Now substitute:

$$5y^2 = 11y + 12 \Rightarrow 5y^2 - 11y - 12 = 0$$

Solve using the quadratic formula:

$$y = \frac{11 \pm \sqrt{(-11)^2 + 4 \cdot 5 \cdot 12}}{2 \cdot 5} = \frac{11 \pm \sqrt{121 + 240}}{10} = \frac{11 \pm \sqrt{361}}{10} = \frac{11 \pm 19}{10}$$

$$y = \frac{30}{10} = 3 \quad \text{or} \quad y = \frac{-8}{10} = -0.8$$

Since $y = 3^{-x} > 0$, reject $y = -0.8$

So:

$$3^{-x} = 3 \Rightarrow -x = 1 \Rightarrow x = -1$$

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Question 12

a. If the function $f(x) = 2^x$, describe the transformations that map $f(x)$ onto $-3f\left(\frac{x}{2}\right) + 5$.

- Dilation by a factor of 2 from y-axis.
- Dilation by a factor of 3 from x-axis.
- Reflection in x-axis.
- Translation up by 5 units.

b. State the rule for the function $-3f\left(\frac{x}{2}\right) + 5$. Write down its maximal domain and the corresponding range.

$$f\left(\frac{x}{2}\right) = 2^{x/2} \Rightarrow -3f\left(\frac{x}{2}\right) + 5 = -3 \cdot 2^{x/2} + 5$$

Domain: R, Range: $(-\infty, 5)$

Now consider another function $g(x)$.

c. If $g(x)$ in the form of $a \bullet b^x + c$ and passes through points $(1, 7)$ and $(2, 17)$, and has an asymptote at $y = -3$, find the value(s) of a, b and c.

Given $g(x) = a \cdot b^x + c$, passes through:

- $(1, 7)$
- $(2, 17)$

and has horizontal asymptote at $y = -3 \Rightarrow c = -3$

So:

$$g(x) = a \cdot b^x - 3$$

Plug in points:

From (1, 7):

$$7 = ab - 3 \Rightarrow ab = 10 \quad (1)$$

From (2, 17):

$$17 = ab^2 - 3 \Rightarrow ab^2 = 20 \quad (2)$$

Divide (2) by (1):

$$\frac{ab^2}{ab} = \frac{20}{10} \Rightarrow b = 2$$

Sub into (1):

$$a \cdot 2 = 10 \Rightarrow a = 5$$

✅ **Values:**

$$a = 5, \quad b = 2, \quad c = -3$$

d. Find the **point of intersection** between $-3f\left(\frac{x}{2}\right)+5$ and $g(x)$.

We want to solve:

$$-3\cdot 2^{x/2}+5=5\cdot 2^{x}-3$$

Let's set $y=2^{x/2}\Rightarrow 2^{x}=y^{2}$

Now substitute:

$$-3y+5=5y^{2}-3\Rightarrow 5y^{2}+3y-8=0$$

Solve using quadratic formula:

$$y=\frac{-3\pm\sqrt{3^{2}+4\cdot 5\cdot 8}}{2\cdot 5}=\frac{-3\pm\sqrt{169}}{10}=\frac{-3\pm 13}{10}\Rightarrow y=1\quad\text{(positive root)}$$

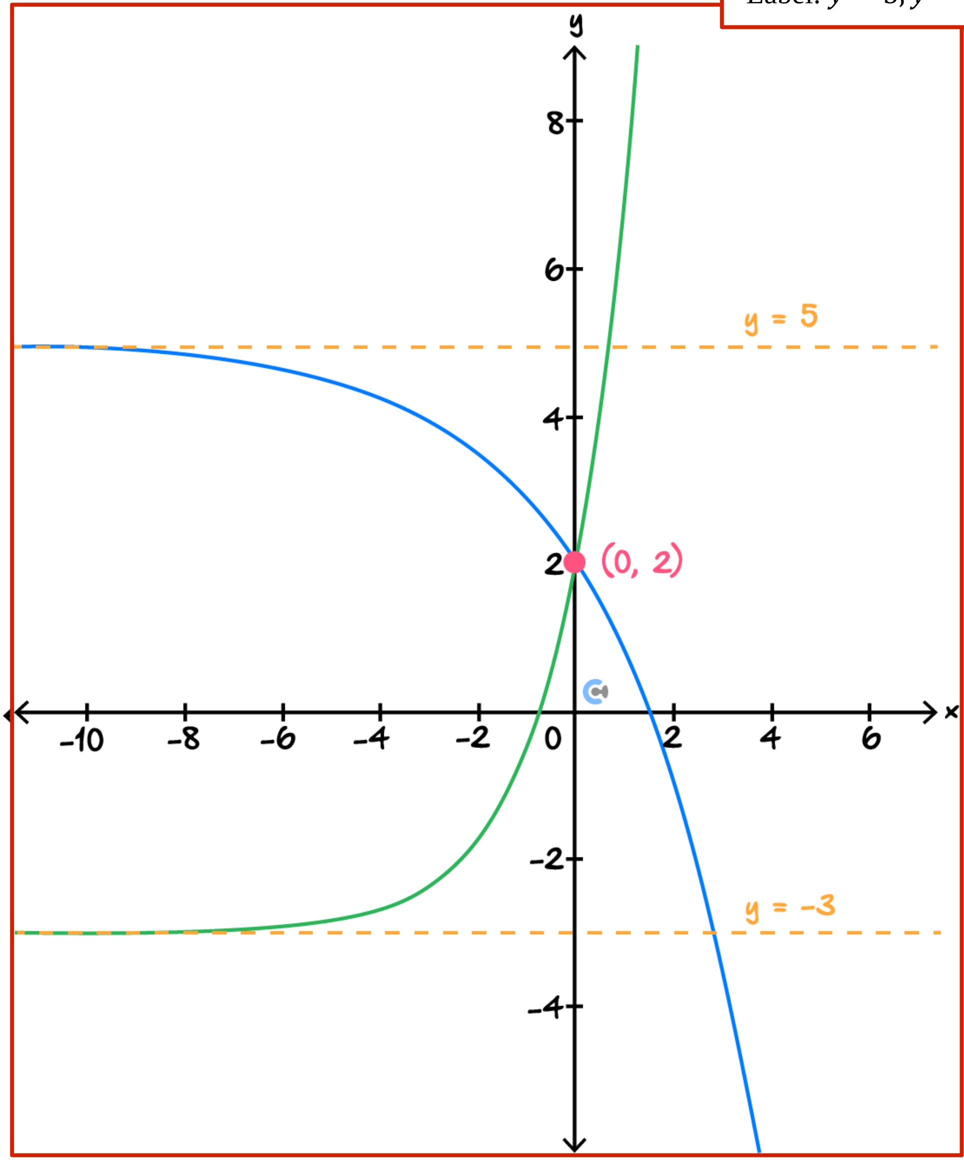
Now back-substitute:

$$2^{x/2}=1\Rightarrow x/2=0\Rightarrow x=0$$

$(0,2)$

e. Sketch the graph of $-3f\left(\frac{x}{2}\right)+5$ and $g(x)$. Label any **asymptotes** with their equations and the ***y*-intercepts** with their coordinates.

Label: $y=5, y=-3$ $(0,2)$



f. If you do not want any intersections between $-3f\left(\frac{x}{2}\right)+5$ and $g(x)$, what translation should you apply to $g(x)$?

Apply a vertical translation of at least 8 units up to $g(x)$.

Question 13 Tech-Active.

The value of a piece of equipment initially valued at $40000 depreciates at a rate of 8% per annum.

a. Find the value of the equipment after 1 year.

$$\text{Value after 1 year} = 40000 \times 0.92 = \boxed{36800}$$

b. Find the value of the equipment after n years.

$$\text{Value after } n \text{ years} = 40000 \times (0.92)^n$$

c. After how many years will the value of the equipment be less than $20000? Will the value of the equipment ever reach $0?

solve$\left(40000\cdot(0.92)^n<20000,n\right)$ $\quad n>8.31295$

After **9 years**, value is less than $20,000.

The value will **never reach $0**. (The value **gets closer to zero but never actually reaches it**.)

Question 14 Tech-Active.

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 13% per annum while that of the red kangaroos decreases at 7% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

- Initially:
  Let grey kangaroos = $G = 1$ unit
  Then red kangaroos = $R = 10$ units
- Grey kangaroo population increases by **13% per annum** → multiplies by 1.13^n
- Red kangaroo population decreases by **7% per annum** → multiplies by 0.93^n

We want the **populations to reverse**, so grey kangaroos become 10 times more than red ones:

$$1.13^n = 10 \cdot 0.93^n$$

$\text{solve}\left((1.13)^n=10\cdot(0.93)^n,n\right)$ $\quad n=11.821$

After 12 years.

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Sub-Section: The Tech-Free 'Final Boss'

Question 15

Consider the following functions:

$$h(x) = -2 \bullet 3^{x-1} + \frac{2}{3}, p(x) = \frac{1}{2} \bullet 3^{2x} - \frac{1}{2}$$

a. Describe the transformation(s) applied to the base function $y = 3^x$ to obtain:

i. 3^{x-2}, using dilation only.

- Dilation by a factor of $\frac{1}{9}$ from x-axis.

ii. $h(x)$.

- Dilation by a factor of 2 from x-axis.
- Reflection in x-axis.
- Translation 1 unit to the right, $\frac{2}{3}$ unit up.

iii. $p(x)$.

- Dilation by a factor of $\frac{1}{2}$ from y-axis.
- Dilation by a factor of $\frac{1}{2}$ from x-axis.
- Transition $\frac{1}{2}$ unit down.

b. Solve $h(x) > p(x)$ for x.

Start by setting the two functions equal:

$$-2 \cdot 3^{x-1} + \frac{2}{3} = \frac{1}{2} \cdot 3^{2x} - \frac{1}{2}$$

Multiply both sides by 6 (to eliminate fractions):

$$6\left(-2 \cdot 3^{x-1} + \frac{2}{3}\right) = 6\left(\frac{1}{2} \cdot 3^{2x} - \frac{1}{2}\right)$$

$$-12 \cdot 3^{x-1} + 4 = 3 \cdot 3^{2x} - 3$$

Recall $3^{x-1} = \frac{1}{3} \cdot 3^x$, so:

$$-12 \cdot \frac{1}{3} \cdot 3^x + 4 = 3 \cdot 3^{2x} - 3 \Rightarrow -4 \cdot 3^x + 4 = 3 \cdot 3^{2x} - 3$$

Bring everything to one side:

$$-4 \cdot 3^x - 3 \cdot 3^{2x} + 7 = 0$$

Now let $y = 3^x$, so $3^{2x} = y^2$:

$$-4y - 3y^2 + 7 = 0 \quad \Rightarrow \quad -3y^2 - 4y + 7 = 0$$

Multiply through by -1:

$$3y^2 + 4y - 7 = 0$$

Solve using the quadratic formula:

$$y = \frac{-4 \pm \sqrt{4^2 - 4(3)(-7)}}{2(3)} = \frac{-4 \pm \sqrt{16 + 84}}{6} = \frac{-4 \pm \sqrt{100}}{6} = \frac{-4 \pm 10}{6}$$

So:

- $y = \frac{6}{6} = 1 \Rightarrow 3^x = 1 \Rightarrow x = 0$
- $y = \frac{-14}{6}$ is negative → discard, since $3^x > 0$

We want:

$$-4 \cdot 3^x + 4 > 3 \cdot 3^{2x} - 3 \Rightarrow -4y + 4 > 3y^2 - 3 \Rightarrow 3y^2 + 4y - 7 < 0$$

Solve:

$$3y^2 + 4y - 7 < 0$$

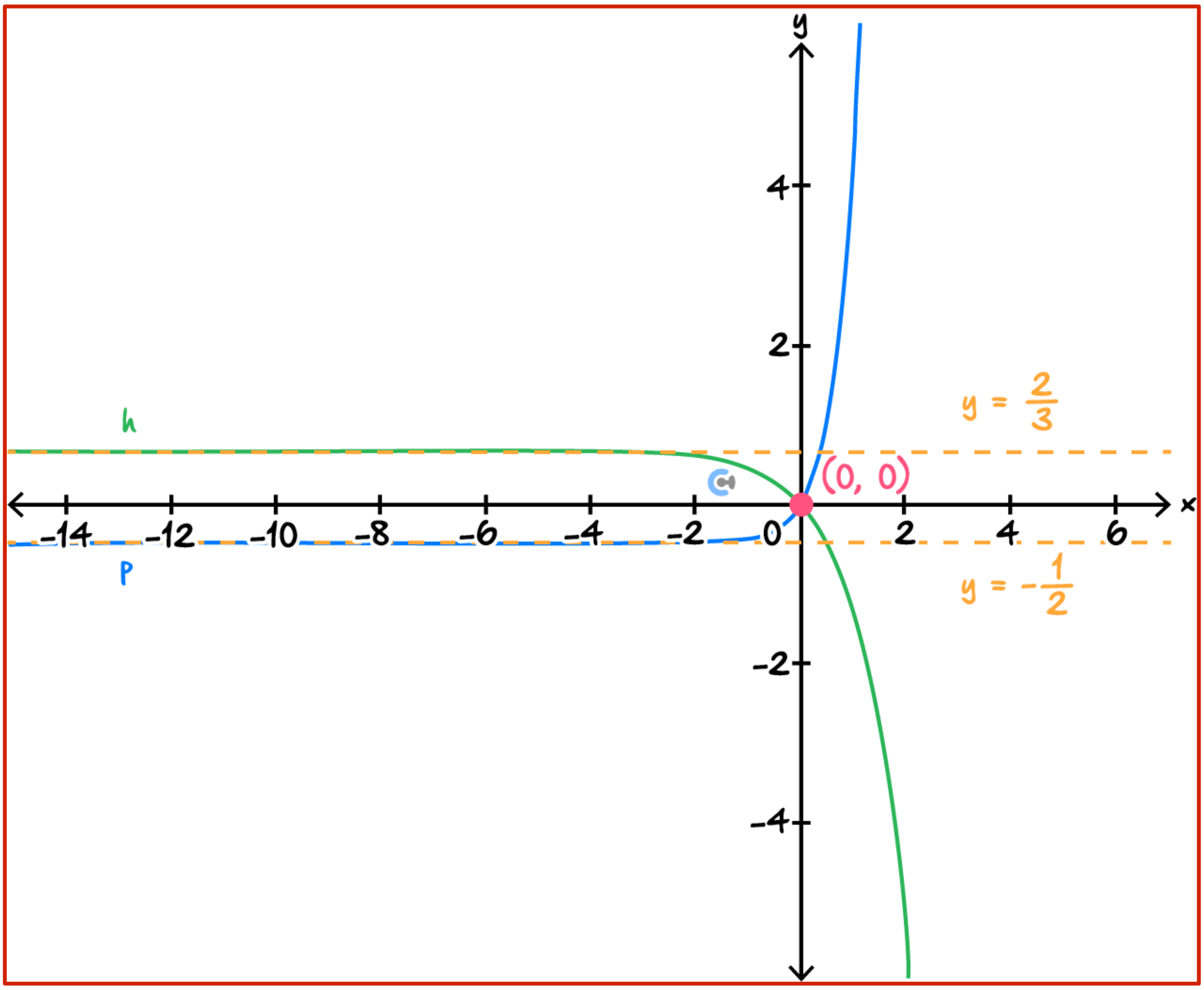
We already found the roots: $y = 1$ and $y = -\frac{7}{3}$

We only consider $y > 0$, so test the sign of the quadratic in the interval $0 < y < 1$.

Because the parabola opens upwards, it's negative between the two roots. Therefore:

$$0 < y < 1 \Rightarrow 3^x < 1 \Rightarrow x < 0$$

c. On the same axes, sketch the graphs of $h(x)$ and $p(x)$. Label all **asymptotes** and **y-intercepts**.



Label $(0,0), y = \frac{2}{3}, y = -\frac{1}{2}$

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Sub-Section: The Tech-Active 'Final Boss'

Question 16

Liam recently started a new job at a hospital in the eastern suburbs of Melbourne. As more residents have moved into the area over time, peak-hour traffic has worsened, and the time it takes Liam to commute has increased significantly. The travel time is found to increase exponentially with the number of cars on the road.

A model of the amount of time, in minutes, it takes Liam to get to work is $T(n) = a \bullet 2^{(kn)}$ where T is the time (in minutes), and the number of drivers is n (in hundreds of thousands); a and k are constants.

When Liam first started commuting, it used to take him 39 minutes, and there were 120,000 drivers on the road. Recent transport data shows that 10 years later, the same trip now takes 2 hours and 36 minutes, with 360,000 drivers on the road.

a. Show that $k = \frac{5}{6}$ and $a = \frac{39}{2}$.

From the model:

$$T(n) = a \cdot 2^{kn}$$

We can write the equations:

1. $39 = a \cdot 2^{1.2k}$
2. $156 = a \cdot 2^{3.6k}$

Divide equation 2 by equation 1:

$$\frac{156}{39} = \frac{a \cdot 2^{3.6k}}{a \cdot 2^{1.2k}} = 2^{(3.6k-1.2k)} = 2^{2.4k} \Rightarrow 4 = 2^{2.4k} \Rightarrow 2^2 = 2^{2.4k} \Rightarrow 2.4k = 2 \Rightarrow k = \frac{5}{6}$$

Now plug $k = \frac{5}{6}$ into Equation 1:

$$39 = a \cdot 2^{1.2 \cdot \frac{5}{6}} = a \cdot 2^1 = 2a \Rightarrow a = \frac{39}{2}$$

b. At this rate, if the number of drivers is expected to reach 450,000 in the future, how long, to the nearest minute, will the same trip take?

$$T(n) = \frac{39}{2} \cdot 2^{\frac{5}{6} \cdot 4.5} = \frac{39}{2} \cdot 2^{3.75} = \frac{39}{2} \cdot 13.4543 \approx 19.5 \cdot 13.4543 \approx 262.36 \text{ minutes}$$

262 minutes

c. How many cars are predicted to be on the road if a trip is expected to take 3 hours? Give your answer to the nearest 5000 drivers.

$$\text{solve}\left(\frac{39}{2}\cdot 2^{\frac{5}{6}\cdot n}=180,n\right) \quad n=3.84774$$

385000 cars

d. How much longer (in minutes) is the trip expected to take if the number of drivers increases from 760,000 to 900,000?

$$\frac{39}{2}\cdot 2^{\frac{5}{6}\cdot 9}-\frac{39}{2}\cdot 2^{\frac{5}{6}\cdot 7.6} \quad 1957.5$$

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Section B: Supplementary Questions

Sub-Section: Exam 1 (Tech-Free)

Question 17

Find the set of values of t for which $4 \times 2^{0.2t} > 2$.

Find the set of values of t for which

$$4 \times 2^{0.2t} > 2$$

Step 1: Divide both sides by 2:

$$2 \cdot 2^{0.2t} > 1 \Rightarrow 2^{1+0.2t} > 1$$

Now use logarithmic reasoning:

Since $2^0 = 1$, and exponential functions are increasing, we solve:

$$1 + 0.2t > 0 \Rightarrow 0.2t > -1 \Rightarrow t > -5$$

Question 18

Solve for x where $256^{-x} = \frac{4^x}{16^{3-x}}$.

Solve for x where

$$256^{-x} = \frac{4^x}{16^{3-x}}$$

Step 1: Express everything as powers of 2:

- $256 = 2^8$
- $4 = 2^2$
- $16 = 2^4$

So the equation becomes:

$$(2^8)^{-x} = \frac{(2^2)^x}{(2^4)^{3-x}} \Rightarrow 2^{-8x} = \frac{2^{2x}}{2^{4(3-x)}}$$

Step 2: Simplify the right-hand side:

$$\frac{2^{2x}}{2^{12-4x}} = 2^{2x-(12-4x)} = 2^{6x-12}$$

So we now have:

$$2^{-8x} = 2^{6x-12}$$

Step 3: Equate exponents:

$$-8x = 6x - 12 \Rightarrow -14x = -12 \Rightarrow x = \frac{6}{7}$$

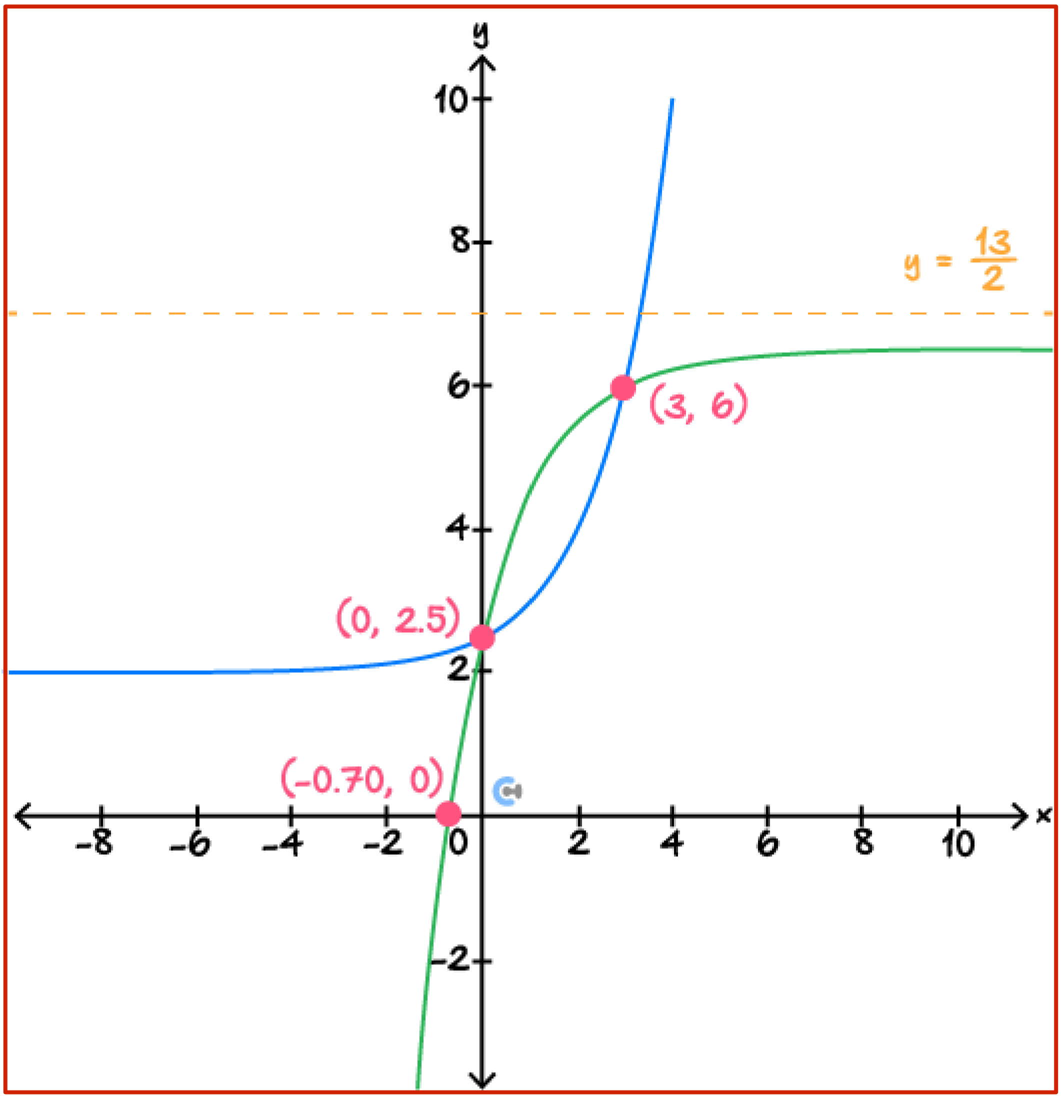
Question 19

Simplify $\frac{x}{1+x^{-1}} + \frac{x}{x^{-1}-1}$.

$$\frac{-2x^2}{x^2-1}$$

Question 20

The graph of the function defined by $f: (-\infty, 4] \to R, f(x) = 2^{x+b} + c$ is shown below.



a. Use algebra to show that $b = -1$ and $c = 2$.

We are told that the graph of $f(x)$ has a horizontal asymptote at $y = 2$, so:

$$\Rightarrow c = 2$$

Assume the point $(4, 10)$ lies on the graph:

$$f(4) = 2^{4+b} + 2 = 10$$
$$2^{4+b} = 8 \Rightarrow 2^{4+b} = 2^3 \Rightarrow 4 + b = 3 \Rightarrow b = -1$$

b. Determine the range of $f(x)$.

$(2,10]$

c. Does the inverse function $f^{-1}(x)$ exist? Explain why.

Yes, because $f(x)$ is one-to-one on its domain.

Now consider $g(x) = -16 \bullet 2^{-x-2} + \frac{13}{2}$.

d. Find the set of values for x such that $g(x) > f(x)$.

Given:

$$2^{x-1} + 2 = -16 \cdot 2^{-x-2} + \frac{13}{2}$$

Multiply both sides by 2:

$$2^x + 4 = -32 \cdot 2^{-x-2} + 13$$

Simplify:

$$2^x + 2^{3-x} = 9$$

Let $y = 2^x$, so $2^{-x} = \frac{1}{y}$:

$$y + \frac{8}{y} = 9$$

Multiply both sides by y:

$$y^2 - 9y + 8 = 0$$

Solve the quadratic:

$$y = 8 \quad \text{or} \quad y = 1$$

Back-substitute $y = 2^x$:

$$2^x = 8 \Rightarrow x = 3$$
$$2^x = 1 \Rightarrow x = 0$$

Final answer: $x \in (0,3)$

e. On the same set of axes above, sketch $g(x)$. Label asymptote, intercepts and point of intersection(s).

Useful information: $2^{0.70} = \frac{13}{8}$

Also label $(3,6), (-0.70,0)$

Smooth decreasing curve

Asymptote at $y = \frac{13}{2}$

Pass through $(0, 2.5)$

f. Describe the sequence of transformations applied to $g(x)$ to obtain 2^x.

From $g(x) = -16 \bullet 2^{-x-2} + \frac{13}{2}$ to 2^x:

- **Translation $\frac{13}{2}$ unit down, 2 units to the right.**
- **Reflection in x-axis and y-axis.**
- **Dilation by a factor of $\frac{1}{16}$ from x-axis.**

Sub-Section: Exam 2 (Tech-Active)

Question 21

If $3^{\frac{1}{x}} = \frac{1}{9}$ **then:**

A. $x = -2$

B. $x = 2$

C. $x = -\frac{1}{2}$

D. $x = \frac{1}{2}$

Question 22

Let $g(x) = 2^x, x \in R$. Which one of the following equations is true for all positive real values of x?

A. $g(x) + g(y) = 2^{x+y}$

B. $g(2x) = 2g(x)$

C. $2g(2x) = 4g(x)$

D. $g(x)g(y) = 2^{x+y}$

E. $g(x) - g(y) = 2^{2x+y}$

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Question 23

The table below shows the mass m, in grams, of a substance at a time t hours.

t	1	2	3	4	5	6
m	8.2	6.0	4.3	3.1	2.3	1.6

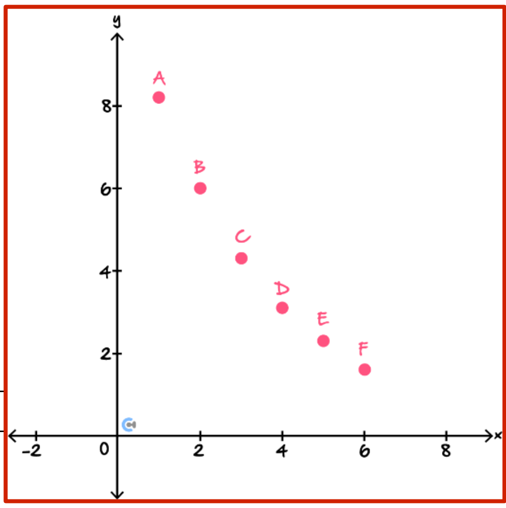
The mass and time would be best modelled using:

A. A linear function.

B. An exponential function.

C. A power function.

D. A circular function.



Question 24

The size of a population of rabbits is determined by the rule $P = 7200 \times 3^{0.3t} - 500$, where P is the size of the population t years after January 2016.

a. Find the size of the population when:

i. $t = 0$

$$P = 7200 \times 3^{0.3 \cdot 0} - 500 = 7200 \times 1 - 500 = 6700$$

ii. $t = 20$

$$P = 7200 \times 3^{0.3 \cdot 20} - 500 = 7200 \times 3^6 - 500$$

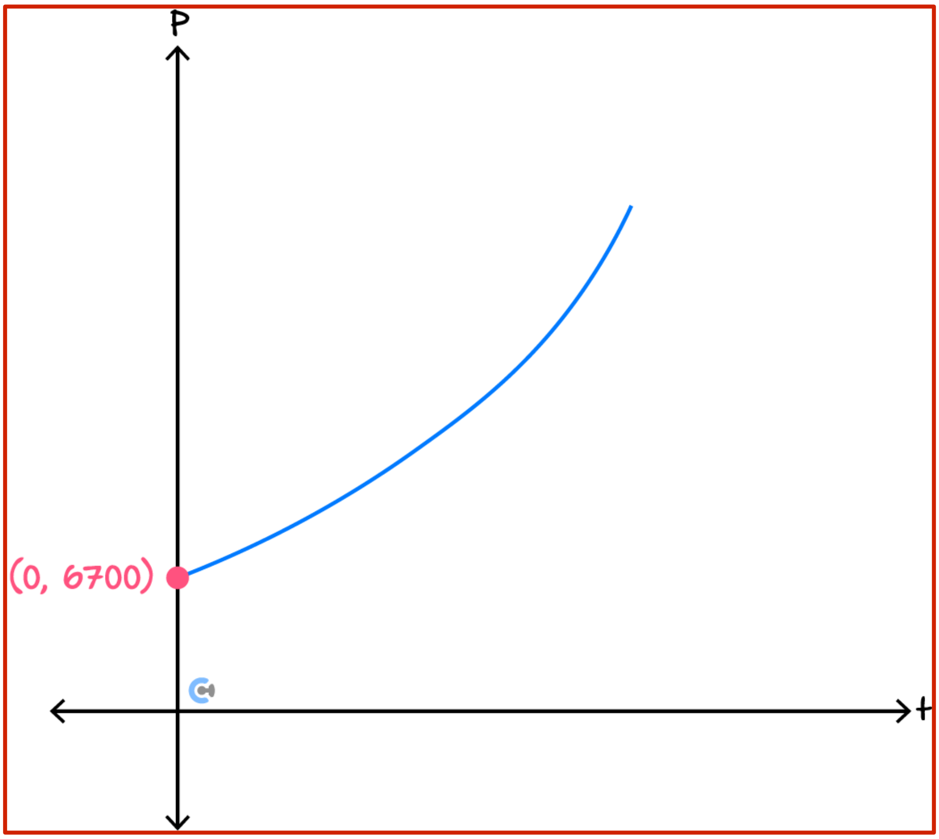
$$3^6 = 729 \Rightarrow P = 7200 \times 729 - 500 = 5,248,800 - 500 = \boxed{5,248,300}$$

b. After how many years does the size of the population exceed 1000000?

$\text{solve}\left(7200 \cdot 3^{0.3 \cdot x} - 500 > 1000000, x\right)$ $\quad x > 14.9709$

After 15 years.

c. Sketch the graph of P against t. Label y-intercept of the graph.



y-intercept: At $t = 0, P = 6700$

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Question 25

The half-life of plutonium-239 is 24,000 years. A geologist discovers a rock sample containing 10 grams of plutonium-239. How long will it take until only 10% of the original plutonium remains in the rock? Give your answer to the nearest year.

Hint: The equation that describes how the mass of plutonium changes over time looks like: $A = A_0 \times 2^{kt}$, where A_0 and k are constants.

Given $\frac{1}{2}A_0 = A_0 \times 2^{24000k}$

$$2^{-1} = 2^{24000k}$$

$$k = \frac{-1}{24000}$$

$$\Rightarrow A = A_0 \times 2^{-\frac{t}{24000}}$$

$$0.1 A_0 = A_0 \times 2^{\frac{-t}{24000}}$$

$$0.1 = 2^{\frac{-t}{24000}}$$

CAS solve: $t = 79726$ years.

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Question 26

A wooden sculpture discovered in an ancient shipwreck contains 40% of the original carbon-14 it once had. Carbon-14 is a radioactive isotope with a half-life of 5730 years.

How old is the sculpture? Give your answer to the nearest year.

Hint: Similar to the previous question, the equation used to model this problem looks like: $A = A_0 \times 2^{kt}$, where A_0 and k are constants.

Given that $\frac{1}{2}A_0 = A_0 \times 2^{5730k}$

$$2^{-1} = 2^{5730k}$$

$$k = \frac{-1}{5730}$$

$\Rightarrow A = A_0 \times 2^{-\frac{t}{5730}}$

To find when $A = 0.4A_0$:

$$0.4A_0 = A_0 \times 2^{\frac{-t}{5730}}$$

$$0.4 = 2^{-\frac{t}{5730}}$$

$t = 7574.64\ldots \approx 7575$ years.

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Question 27

The population of a town increases by 6% every year. In January 2016, the population was 5500.

a. Find the population of the town:

i. In January 2017.

$$P = 5500 \times (1.06)^1 = 5500 \times 1.06 = \boxed{5830}$$

ii. After n years from 2016.

$$\boxed{P(n) = 5500 \times (1.06)^n}$$

b. Find the year in which the population will reach 11000.

$$\text{solve}\left(5500 \cdot (1.06)^n = 11000, n\right) \quad n = 11.8957$$

In 2016 + 12 = 2028

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Question 28

After a conservation program was introduced in a wildlife sanctuary on **1st January 2015**, the population of **birds** was modelled by:

$$B(t) = 50 \times 2^{0.4t}$$

Where t is the number of months after the start of the program.

At the same time, the number of **squirrels** was modelled by

$$S(t) = 400 \times 4^{-0.1t}$$

Find

a. The number of **birds** and **squirrels** in the sanctuary on **1st January 2015**.

$$B(0) = 50 \times 2^{0.4 \cdot 0} = 50 \times 1 = \boxed{50}$$

$$S(0) = 400 \times 4^{-0.1 \cdot 0} = 400 \times 1 = \boxed{400}$$

b. Which of the two animals will have the highest numbers on **1st April 2016**?

$$B(15) = 50 \times 2^{0.4 \cdot 15} = 50 \times 2^6 = 50 \times 64 = \boxed{3200}$$

$$S(15) = 400 \times 4^{-0.1 \cdot 15} = 400 \times 4^{-1.5} = 400 \times \frac{1}{4^{1.5}} = 400 \times \frac{1}{8} = \boxed{50}$$

Birds are more numerous (3200 vs. 50)

c. The **date** when the bird population will equal the squirrel population.

$\text{solve}\left(50 \cdot 2^{0.4 \cdot t} = 400 \cdot 4^{-0.1 \cdot t}, t\right)$ $\quad t=5.$

After 5 months = 1st June 2015

If the number of squirrels falls below 25, they are at risk of extinction in the sanctuary.

d. According to the model, will this happen? If so, **when** will it happen?

$\text{solve}\left(400 \cdot 4^{-0.1 \cdot t} < 25, t\right) \quad t > 20.$

Yes, squirrels fall below 25 **after 20 months** → 1st September 2016

Question 29

Alex takes a bottle of juice out of the fridge at **9: 00 AM**. The juice starts at a cold temperature of **4**°C and begins to warm up in a room. The room temperature is constant, and the juice's temperature increase is modelled by the equation:

$$T(t) = a \bullet \left(\frac{9}{4}\right)^{-kt}$$

Where, $T(t)$ is the **difference** in temperature (in °C) between the juice and the **ambient room temperature of 22**°C, t is the number of minutes since **9: 00 AM**, and a and k are constants.

a. Determine the value of the constant a.

When $t = 0$, difference in temperature $= 18$.

$T(0) = a \cdot \left(\frac{9}{4}\right)^0 = a \cdot 1 = 18 \Rightarrow \boxed{a = 18}$

b. If the juice reaches **14**°C at **9: 20 AM**, determine the value of k.

$\text{solve}\left(18 \cdot \left(\frac{9}{4}\right)^{-k \cdot t} = 22 - 14, k\right) \Big| t = 20 \quad k = \frac{1}{20}$

c. By the time Alex remembers the juice, it is **9: 45 AM**. Using your values of a and k, calculate the juice's temperature at that time, to the **nearest degree**.

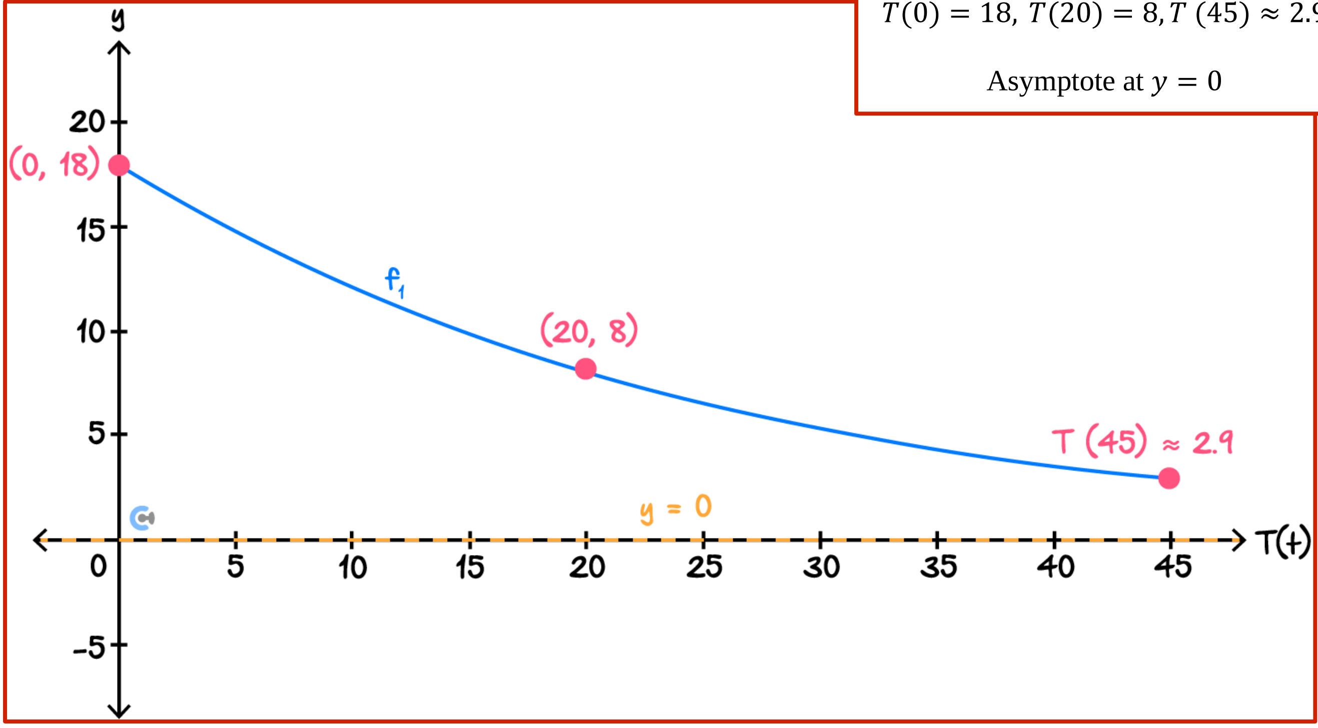
$18\cdot\left(\frac{9}{4}\right)^{\frac{-1}{20}\cdot t}\Big\vert t=45$	$\frac{32\cdot\sqrt{6}}{27}$
$22-\frac{32\cdot\sqrt{6}}{27}$	19.0969

19 degrees

d. Sketch the graph of $T(t)$ over 45 minutes, clearly labelling start and endpoints.

$T(0) = 18,\ T(20) = 8, T\ (45) \approx 2.9$

Asymptote at $y = 0$



At 9: 45 AM, Alex puts the juice in the fridge again. The **cooling** of the juice is now modelled by:

$$C(t) = 4 + A \times 2^{-\frac{t-45}{10}}$$

Where $C(t)$ is the juice temperature at time t, and t is the number of minutes since **9: 00 AM**.

e. Find the value of A, correct to 1 decimal place.

Juice temperature at $t = 45$ is 19.1 degree, so

$C(45) = 19.1 = 4 + A \bullet 2^0 \rightarrow A = 15.1$

f. What is the appropriate **domain** for $C(t)$? Justify your answer.

Domain: $t \geq 45$

Before $t = 45$, the juice is warming, not cooling. The function models cooling **after** fridge insertion.

g. For how long (to the nearest minute) is the juice **above 14**°C?

$\text{solve}\left(4+15.1\cdot 2^{\frac{-(t-45)}{10}}>14,t\right)$	$t<50.9455$
$45-20+50.9455-45$	30.9455

31 minutes

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