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# VCE Mathematical Methods ½

## Graph of Circular Function Exam Skills [4.5]

## Workbook Solutions

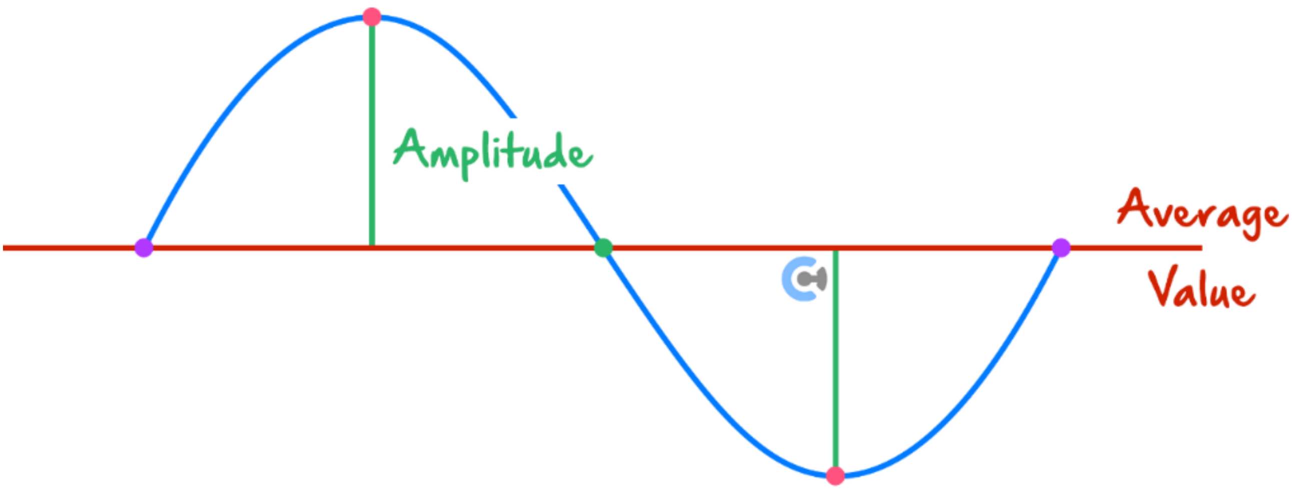
### Outline:

## Section A: Recap

Definition

Amplitude, Period and Average Value

For $y = A\sin/\cos(nx + b) + k$




Consider the sign of our graph

$$\text{Amplitude} = |A|$$

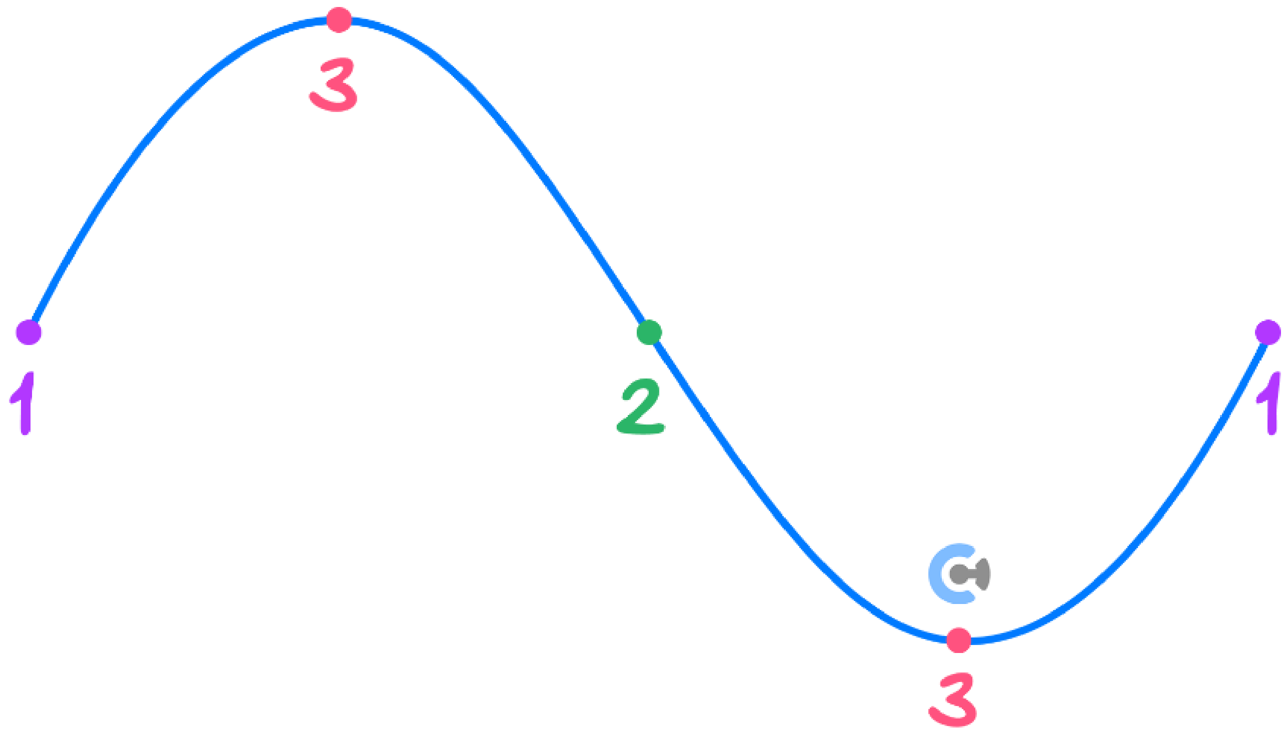
$$\text{Period} = \frac{2\pi}{|n|}$$

$$\text{Average Value} = k$$

Space for Personal Notes

Definition

**Graphing of $\sin$ and $\cos$ Functions**




- **Steps:**

1. Identify: Amplitude, Period, Mean Value and Positive/Negative Shape.
2. Create a "mini-version" of the graph you are about to draw.
3. Start plotting the function from when the angle = 0.
4. Draw the start and end of the periods, and plot the halves (turning points).
5. Find any $x$-intercepts.
6. Join all the points!

Definition

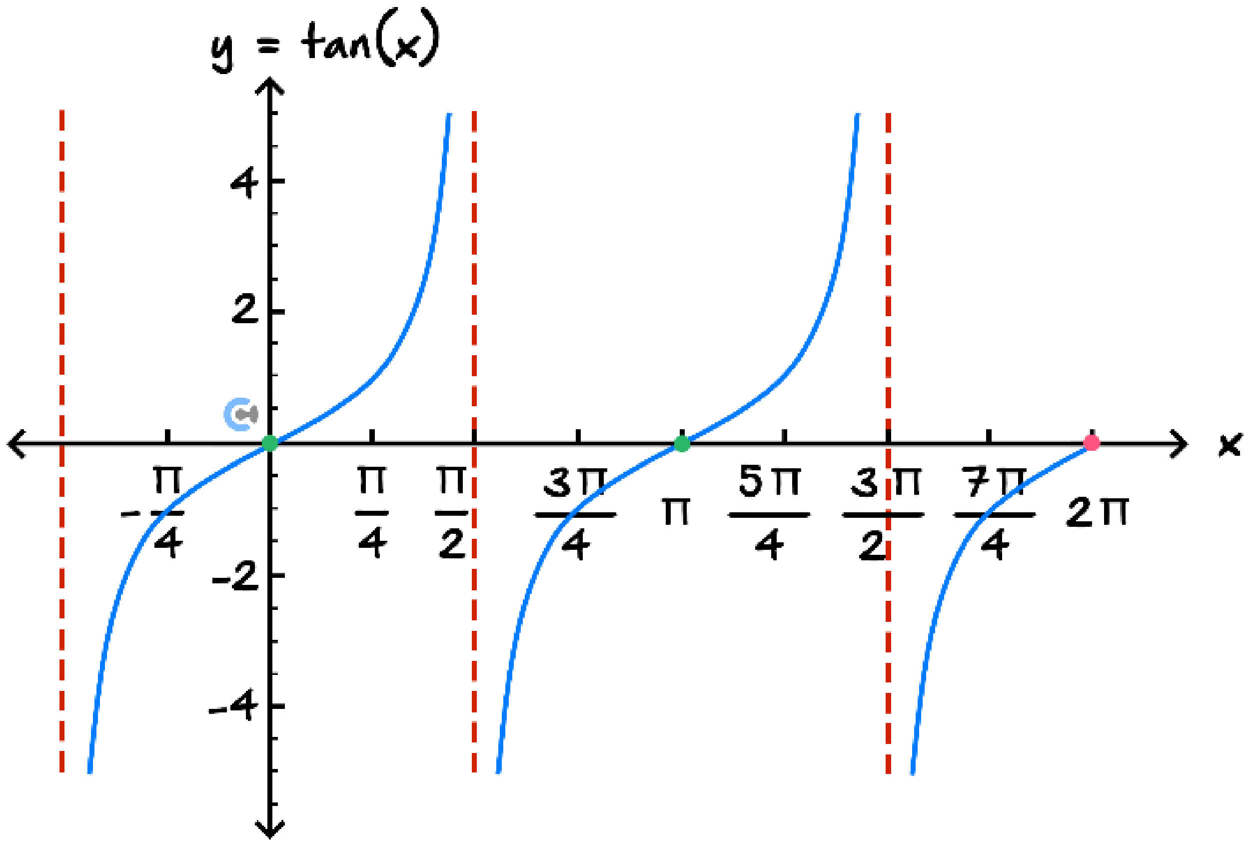
**Finding the Rule**

$$\text{Amplitude } (A) = \frac{max - min}{2}$$

$$\text{Average } (k) = \frac{max + min}{2}$$

Definition

Graph of Tangent

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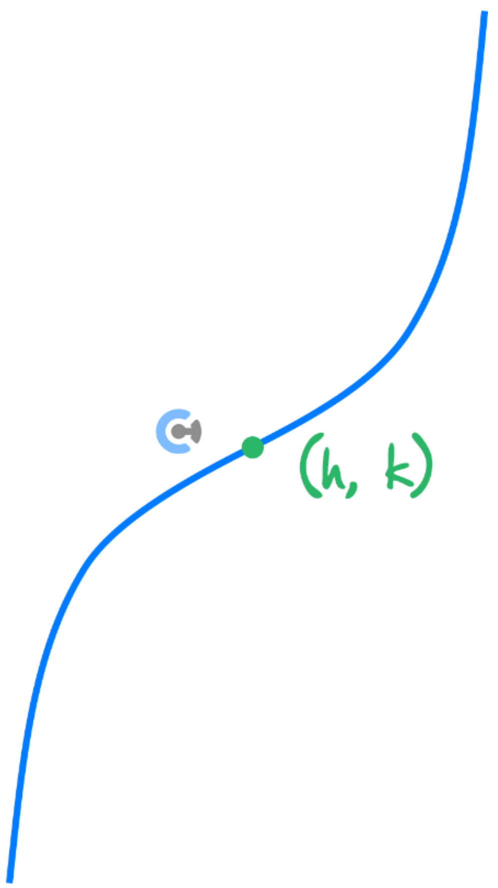
## Steps for Sketching $\tan$ Functions

1. Identify:
   - The period = $\boxed{\frac{\pi}{n}}$.
2. Find the vertical asymptotes by solving for angle = $\boxed{\frac{\pi}{2}}$.
3. Find other vertical asymptotes within the domain by adding the period to the answer from the previous step.
4. Plot the inflection point $(h, k)$. (Midpoint of the two $\boxed{\text{vertical asymptotes}}$.)
   - $x$-value of the inflection point = $x$-value which makes angle = $0$.
   - $y$-value of the inflection point = vertical translation of the function.
5. Find any $\boxed{x\text{-intercepts}}$.
6. Sketch a $\boxed{\text{"cubic-like"}}$ shape.

eg: $\tan(x-h)+k$



Definition

Fraction of Period

$$\boldsymbol{Fraction\ of\ Period} = \frac{\boldsymbol{Duration}}{\boldsymbol{Period}}$$

$$\%\ \boldsymbol{of\ Period} = \frac{\boldsymbol{Duration}}{\boldsymbol{Period}} \times \mathbf{100}\%$$

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## Section B: Warmup Test (16 Marks)

**Question 1** (8 marks)

Consider the function $f : \mathbb{R} \to \mathbb{R}, f(x) = 2 - 4\cos(2x)$.

**a.** Find the general solution to $f(x) = 0$. (2 marks)

$$2 - 4\cos(2x) = 0$$

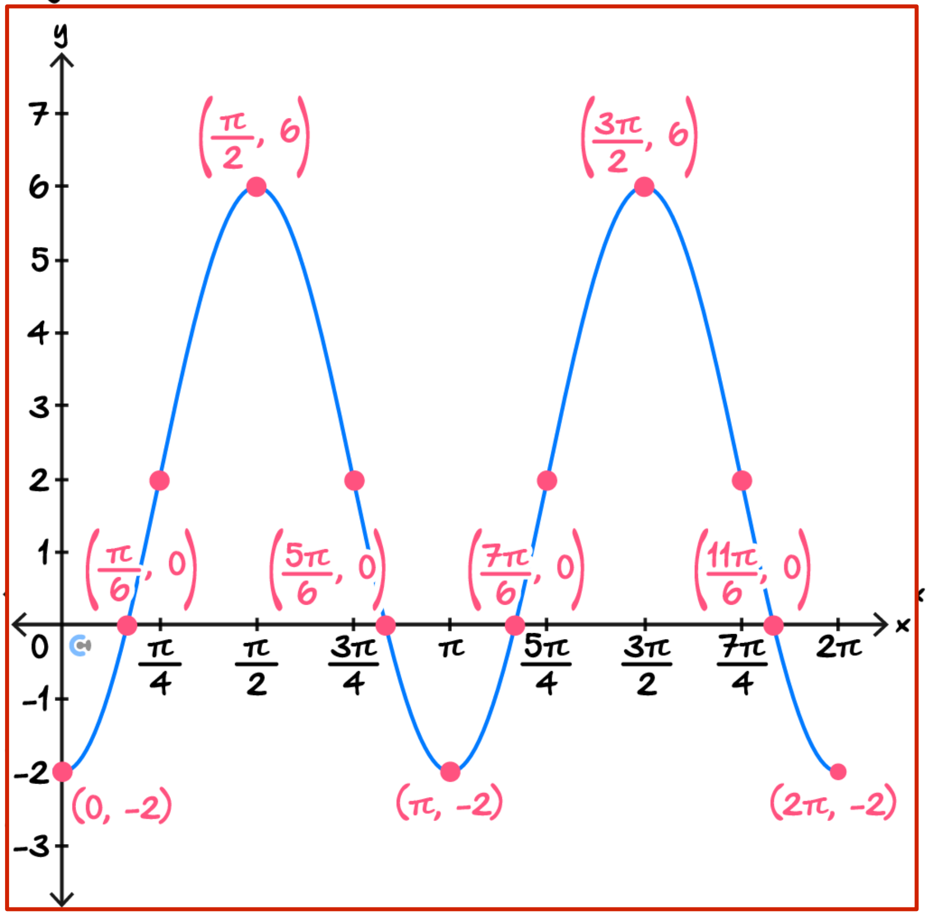
$$\cos(2x) = \frac{1}{2} \quad [1\text{M}]$$

$$2x = \pm\frac{\pi}{3} + 2n\pi$$

$$x = n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z} \quad [1\text{A}]$$

**b.** Sketch the graph of $y = f(x)$ for $x \in [0, 2\pi]$. Label any endpoints, turning points and axes intercepts with coordinates. (3 marks)

[1A shape, 1A turning points, 1A endpoints and intercepts]



**c.** Find the fraction of a period that $f(x)$ is above 4 for. (3 marks)

$f(x) = 4 \implies -4\cos(2x) = 2 \implies \cos(2x) = -\frac{1}{2}$. **[1M]**

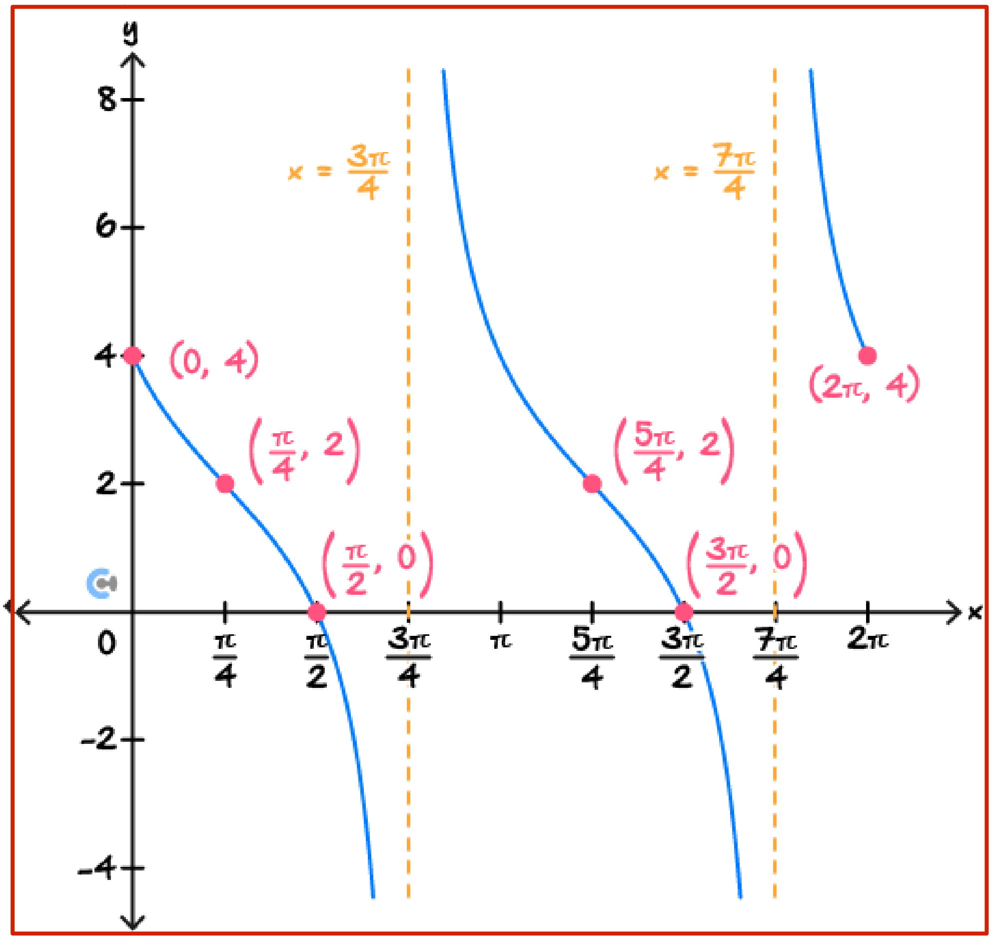
$$\cos(2x) = -\frac{1}{2}$$
$$2x = \frac{2\pi}{3}, \frac{4\pi}{3}$$
$$x = \frac{\pi}{3}, \frac{2\pi}{3} \quad \textbf{[1M]}$$

Period is $\pi$. So fraction of period is $\dfrac{2\pi/3 - \pi/3}{\pi} = \dfrac{1}{3}$ **[1A]** .

**Space for Personal Notes**

**Question 2** (4 marks)

Sketch the graph of $y = -2\tan\left(x - \frac{\pi}{4}\right) + 2$ for $x \in [0, 2\pi]$. Label any asymptotes with equations and endpoints, axes intercepts, and inflection points with coordinates.

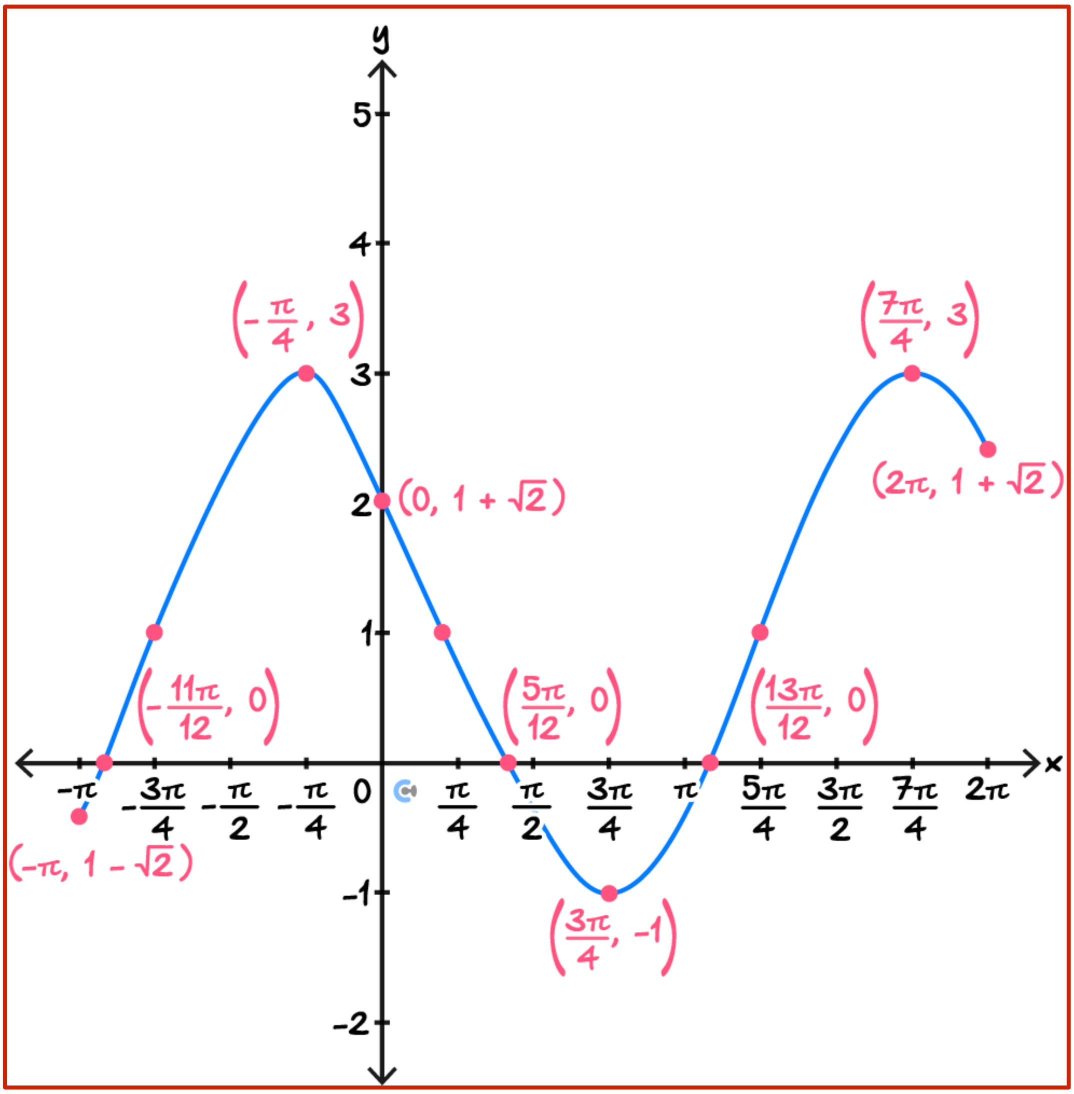



[1M asymptotes, 1M intercepts, 1M shape, 1M infleciton points and endpoints]

Space for Personal Notes

**Question 3** (4 marks)

Sketch the graph of $y = -2\sin\left(x - \frac{\pi}{4}\right) + 1$ for $x \in [-\pi, 2\pi]$. Label any endpoints, turning points and axes intercepts with coordinates.




[1A shape, 1A intercepts, 1A turning points, 1A endpoints]

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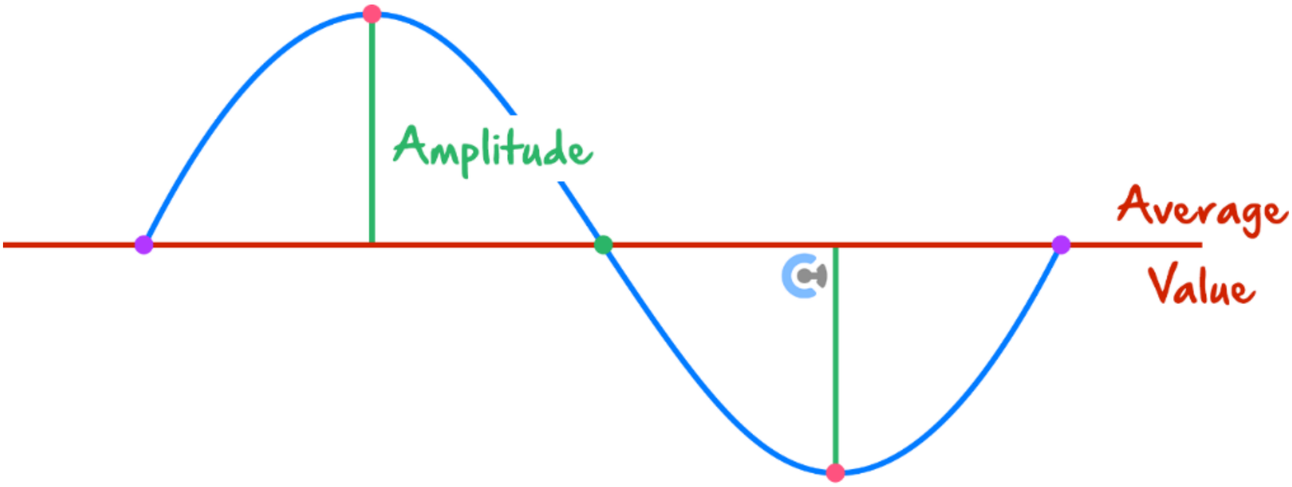
## Section C: Exam Skills

### Sub-Section: Identifying the Correct Graph

REMINDER: Amplitude, Period and Average Value

For $y = A\sin/\cos\,(nx + b) + k$




Consider the sign of our graph

$$\text{Amplitude} = |A|$$

$$\text{Period} = \frac{2\pi}{|n|}$$

$$\text{Average Value} = k$$

REMINDER: Finding the Rule

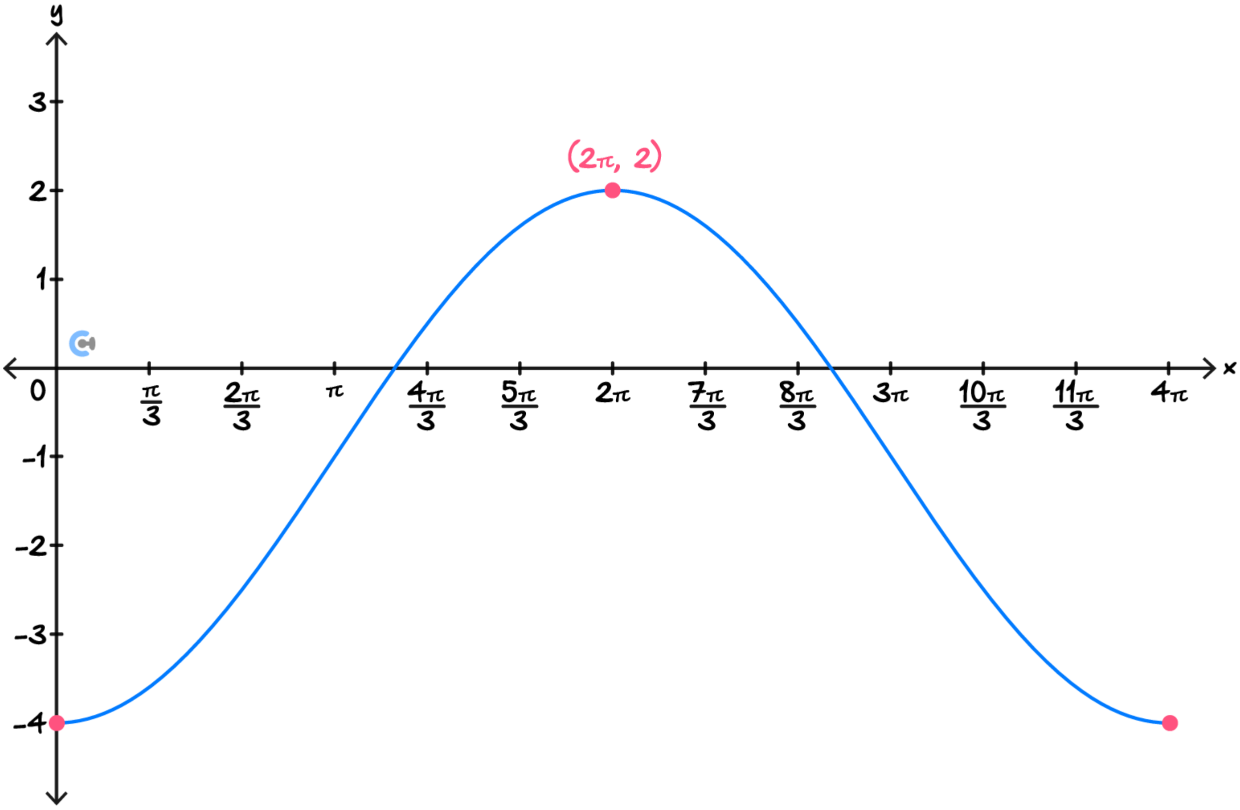
$$\text{Amplitude } (A) = \frac{max - min}{2}$$

$$\text{Average } (k) = \frac{max + min}{2}$$

**Question 4 Walkthrough.**

The graph shown below has a rule of the form $y = a\cos(nx) + k$.

Find a possible rule for the graph.




Period is $4\pi \Rightarrow n = \frac{1}{2}$

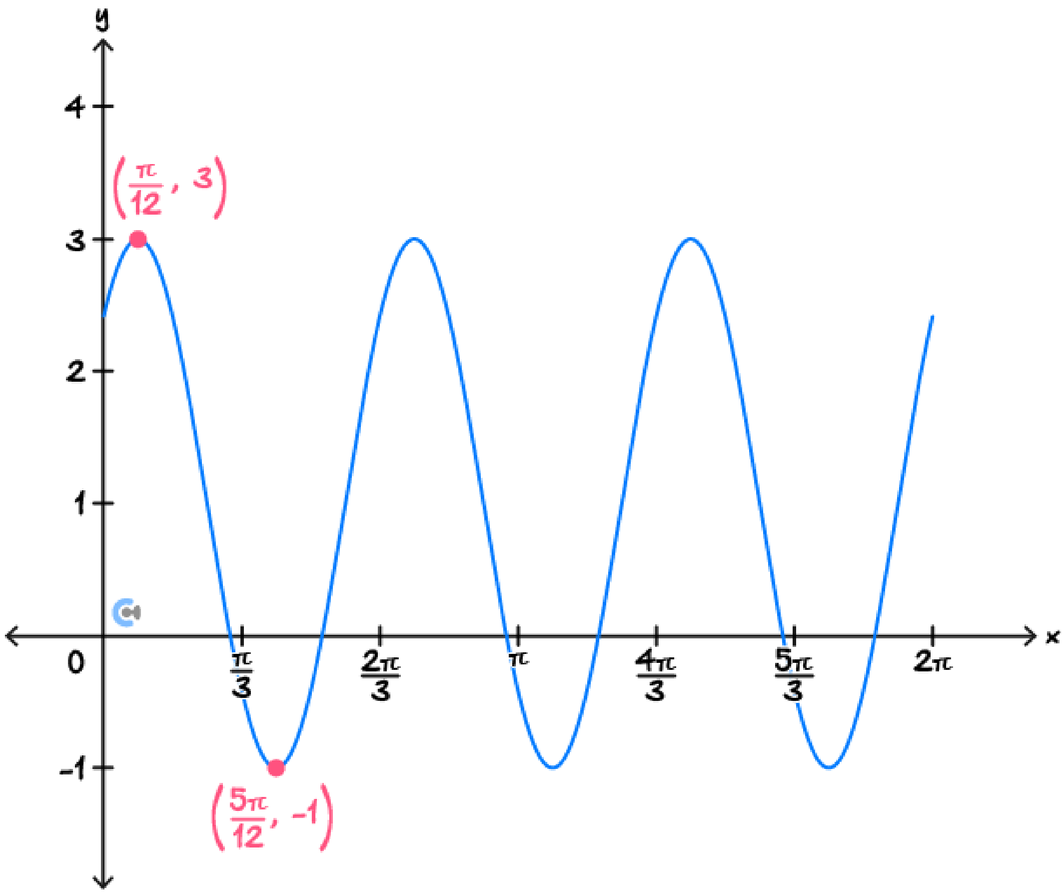
Has negative cos shape.

Average value is $-1$ and range is $[-4,2]$.

$$y = -3\cos\left(\frac{x}{2}\right) - 1$$

**Question 5**

The graph shown below has a rule of the form $y = a\sin(nx + b) + k$.
Find a possible rule for the graph, given that $a, n, b, k$ are all greater than zero.




Period is $2 \times \frac{\pi}{3} = \frac{2\pi}{3} \Rightarrow n = 3$

$$a = 2, k = 1$$

$$y = 2\sin(3x + b) + 1$$

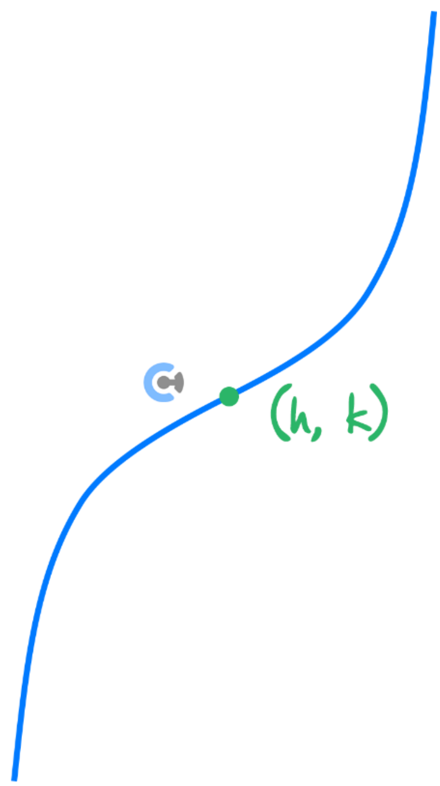
$$3 \times \frac{\pi}{12} + b = \frac{\pi}{2} \Rightarrow b = \frac{\pi}{4}$$

$$y = 2\sin\left(3x + \frac{\pi}{4}\right) + 1$$

**REMINDER: Steps for Sketching $\tan$ Functions**

1. Identify:
   - The period $= \frac{\pi}{n}$.
2. Find the vertical asymptotes by solving for angle $= \frac{\pi}{2}$.
3. Find other vertical asymptotes within the domain by adding the period to answer from the previous step.
   - For instance, for $\tan\left(2x - \frac{\pi}{3}\right)$, solve $2x - \frac{\pi}{3} = \frac{\pi}{2}$ for $x$.
4. Plot the inflection point $(h, k)$. (Midpoint of the two vertical asymptotes.)
   - $x$-value of inflection point $= x$-value which makes angle $= 0$.
   - $y$-value of inflection point $=$ vertical translation of the function.

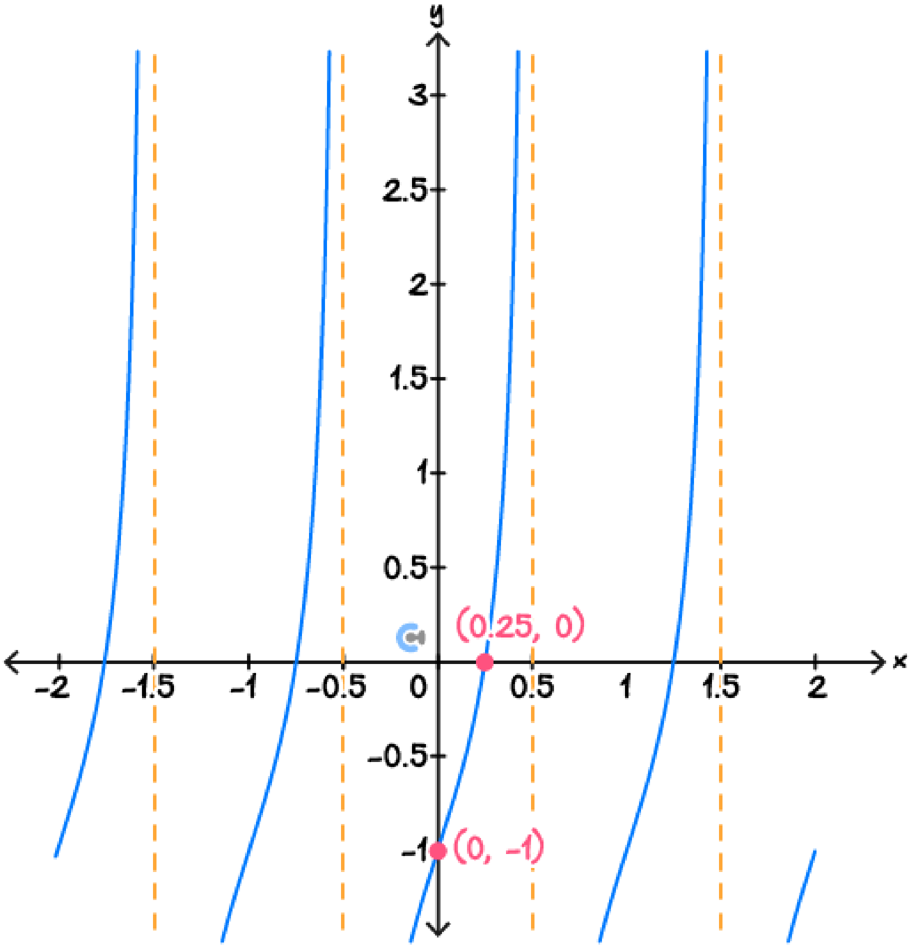
eg: tan(x−h)+k




5. Find any $x$-intercepts.
6. Sketch a "cubic-like" shape.

**Question 6**

The graph shown below has a rule of the form $y = \tan(nx) + k$, where $n, k \in \mathbb{R}$.
Find a possible rule for the graph.




Period is $1 \Rightarrow n = \pi$

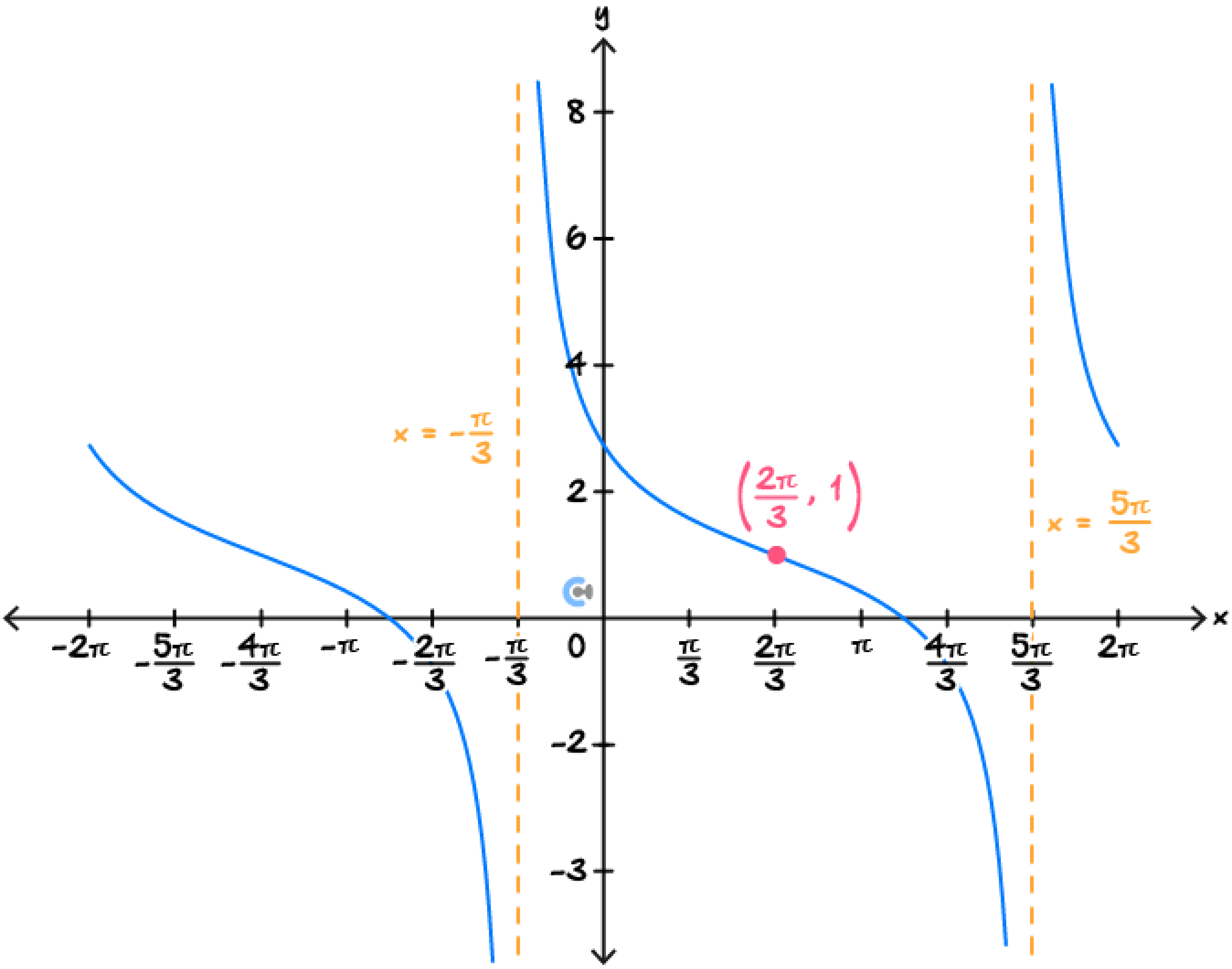
$y = \tan(\pi x) - 1$

**TIP:** Simply check if the asymptotic $x$-value makes inside equal to $90 + 180n$ degrees.

**Question 7**

The graph shown below has a rule of the form $y = -\tan(nx - b) + k$, where $b, n, k > 0$.

Find a possible rule for the graph.




Period is $2\pi \Rightarrow n = \frac{1}{2}$

Inflection point at $\left(\frac{2\pi}{3}, 1\right) \Rightarrow k = 1$

Then shape and point $\Rightarrow a = -1$

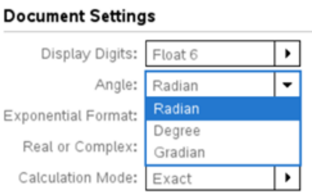
$$\frac{1}{2} \times \frac{2\pi}{3} - b = 0 \Rightarrow b = \frac{\pi}{3}$$

$$y = -\tan\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 1$$

## Section D: Technology Exam Skills

### Calculator Commands: Degrees and Radians

- **TI**
  - Doc → 7 → 2




- **Casio**
  - Change at the bottom of the screen.


- **Mathematica**
  - In radians by default.
  - Write "Degree".

```
In[27]:= Sin[30 Degree]
Out[27]= 1/2
```

### Calculator Commands: Solving Trigonometric Functions.

- **TI**
  - solve($trig(..) = a, x$) | domain restriction.
  - | is under control equal.

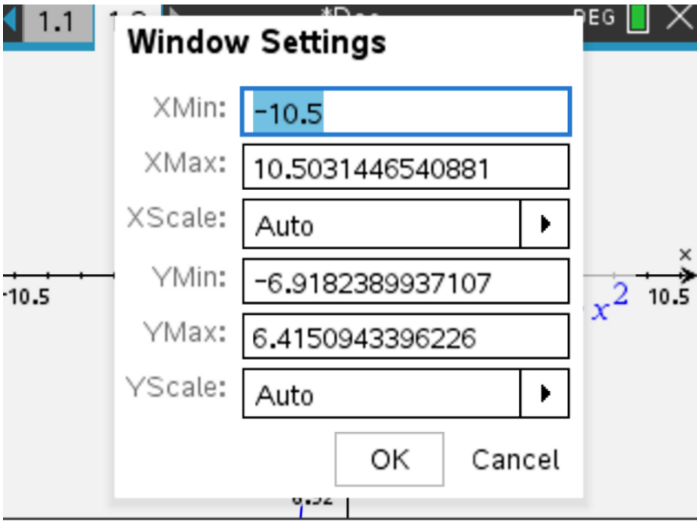
- **Casio**
  - solve($trig(..) = a, x$) | domain restriction.
  - | is under maths 3.

- **Mathematica**
  - Solve[$trig$[] $==$ $a$ && domain restriction, $x$].

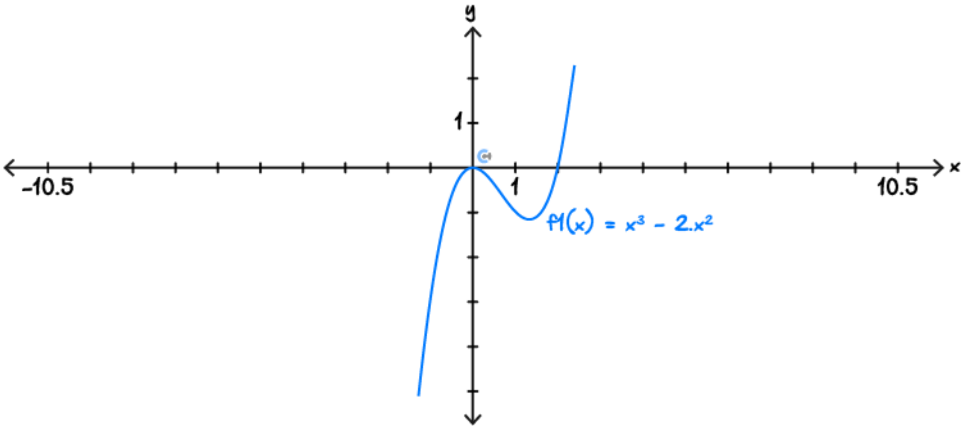
### Space for Personal Notes

<u>Calculator Commands:</u> Graphing

- Open a graph page and plot your function.
- Zoom settings: Menu → $4$ (window/zoom) → $1$ enter your $x$ and $y$-ranges.

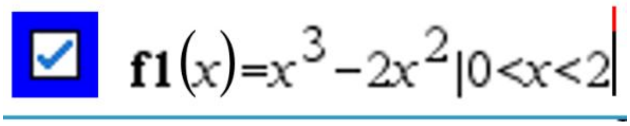



- Can also click the axis numbers on the graph and alter them directly.

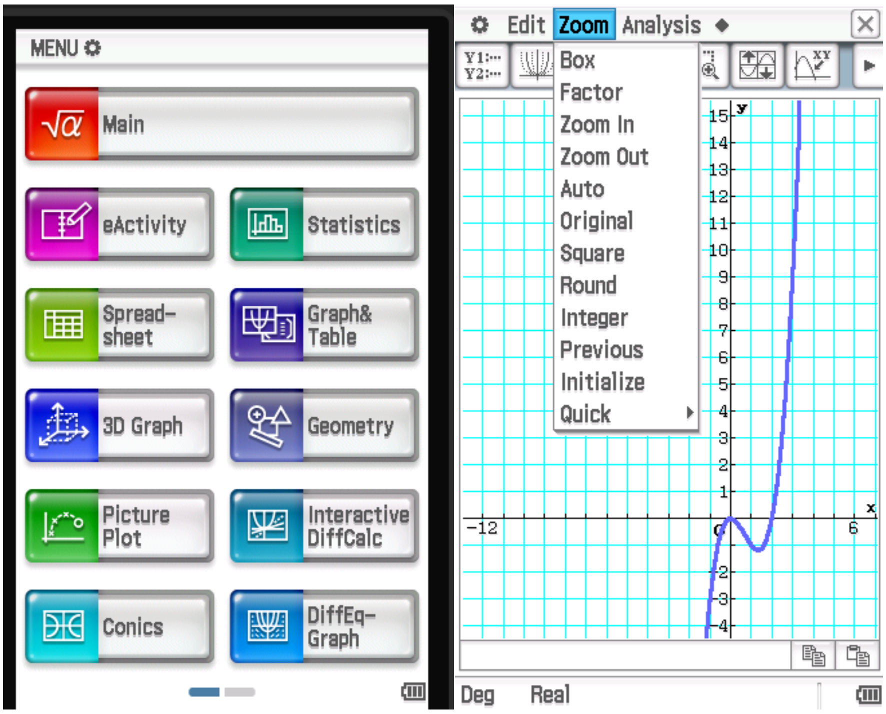



- Menu→ $6$ (Analyse) to find $min/max$ $x$ and $y$-intercepts.
- Restrict domain to $0 < x < 2$ use the bar can get it from $\text{ctrl}+=$

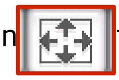
| | | |
|---|---|---|
| > | < | ≠ |
| ≥ | ≤ | \| |

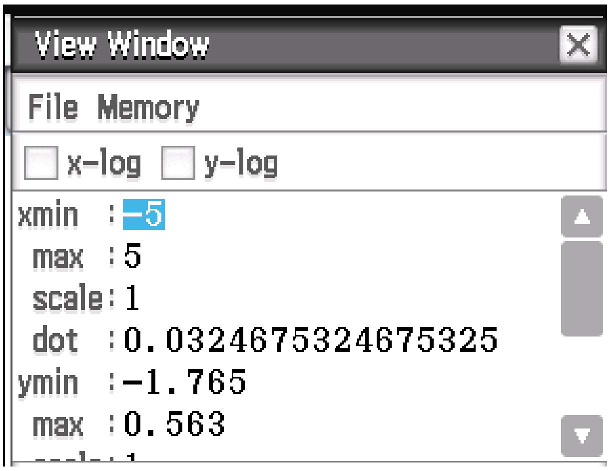


- **Casio:** Click Graph & Table, and enter the function.




- Analysis→ G-Solve to find intercepts.

- Use this button  to set the view window.




- Use | to restrict domain → find it in Math 3.

$$y1=x^3-2\cdot x^2 \mid 0<x<2$$

- **Mathematica:** Plot[function, {$x$, $x$min, $x$max}, PlotRange → {$y$min, $y$max}]

  - PlotRange is optional but can be used to make the scale appropriate for the question.

## Section E: Exam 2 (30 Marks)

**Question 8** (1 mark)

The $x$-intercepts for $y = 3\sin\frac{x}{3} + 3$, where $x \in [-3\pi, 5\pi]$, are equal to:

**A.** $-2\pi, 0, 2\pi$

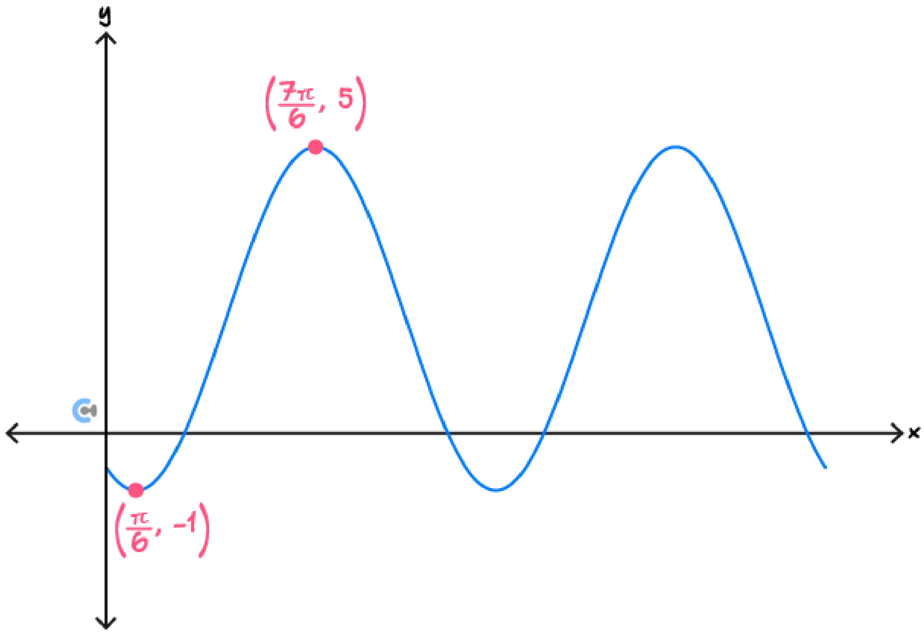
**B.** $-\frac{3\pi}{2}, \frac{9\pi}{2}$

**C.** $-\pi, 0, \pi$

**D.** $-\frac{3\pi}{2}, \frac{3\pi}{2}$

**Question 9** (1 mark)

The graph shown below could have the rule:




**A.** $-3\cos\left(x + \frac{\pi}{3}\right) + 2$

**B.** $3\sin\left(x + \frac{\pi}{3}\right) + 1$

**C.** $-3\sin\left(x + \frac{\pi}{3}\right) + 2$

**D.** $3\cos\left(x + \frac{\pi}{6}\right) + 2$

**Question 10** (1 mark)

Let $f : \left(\frac{\pi}{6}, \frac{5\pi}{6}\right) \to \mathbb{R}$, where $f(x) = 3\cos(2x)$.

The range of $f$ is:

**A.** $[-3,3]$

**B.** $\left[-3, \frac{3}{2}\right)$

**C.** $\left[-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right]$

**D.** $\left(-3, \frac{3}{2}\right)$

**Question 11** (1 mark)

Let $f : [0, a] \to \mathbb{R}$, where $f(x) = \cos(3x - \pi)$. If the inverse function $f^{-1}$ exists, then the largest value that a can take is:

**A.** $\frac{\pi}{6}$

**B.** $\frac{\pi}{3}$

**C.** $\frac{\pi}{2}$

**D.** $\pi$

**Question 12** (1 mark)

Let $g : \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \to \mathbb{R}$, with $g(x) = \tan(3x)$. The graph of $y = g(x)$ is transformed by a dilation by a factor of 4 from the $x$-axis, followed by a reflection in the $x$-axis. The resulting function $h$ is given by:

**A.** $h: \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \to \mathbb{R}, h(x) = -4\tan(3x)$

**B.** $h: \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right) \to \mathbb{R}, h(x) = -4\tan(3x)$

**C.** $h: \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \to \mathbb{R}, h(x) = \frac{1}{4}\tan(3x)$

**D.** $h: \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \to \mathbb{R}, h(x) = -\frac{1}{4}\tan(3x)$

**Question 13** (1 mark)

Given the function $f : \left[0, \frac{4\pi}{k}\right] \to \mathbb{R}$, where $f(x)\ a + b\sin(kx)$, and $a, b, k \in R$, which of the following statements is **FALSE**?

**A.** $f$ has a maximum at $y = a + b$ and occurs when $x = \frac{\pi}{2k}$.

**B.** $f$ has a minimum at $y = a - b$ and occurs when $x = \frac{3\pi}{2k}$.

**C.** The range of $f$ is $[a - b, a + b]$.

**D.** The graph of $y = f(x)$ consists of three full cycles.

Space for Personal Notes

**Question 14** (12 marks)

A theme park ride features a pirate ship that swings back and forth in a circular motion. The lowest point of the ride is 2 metres above the ground, and the highest point is 14 metres above the ground. The height of the top of the ship follows a sinusoidal pattern with time. The ship completes one full swing every 20 seconds, starting at its lowest point when $t = 0$.

**a.** Show that the height $h(t)$, in metres, of the top of the ship above the ground after $t$ seconds can be modelled by: (2 marks)

$$h(t) \ = \ 8 - 6\cos\left(\frac{\pi t}{10}\right)$$

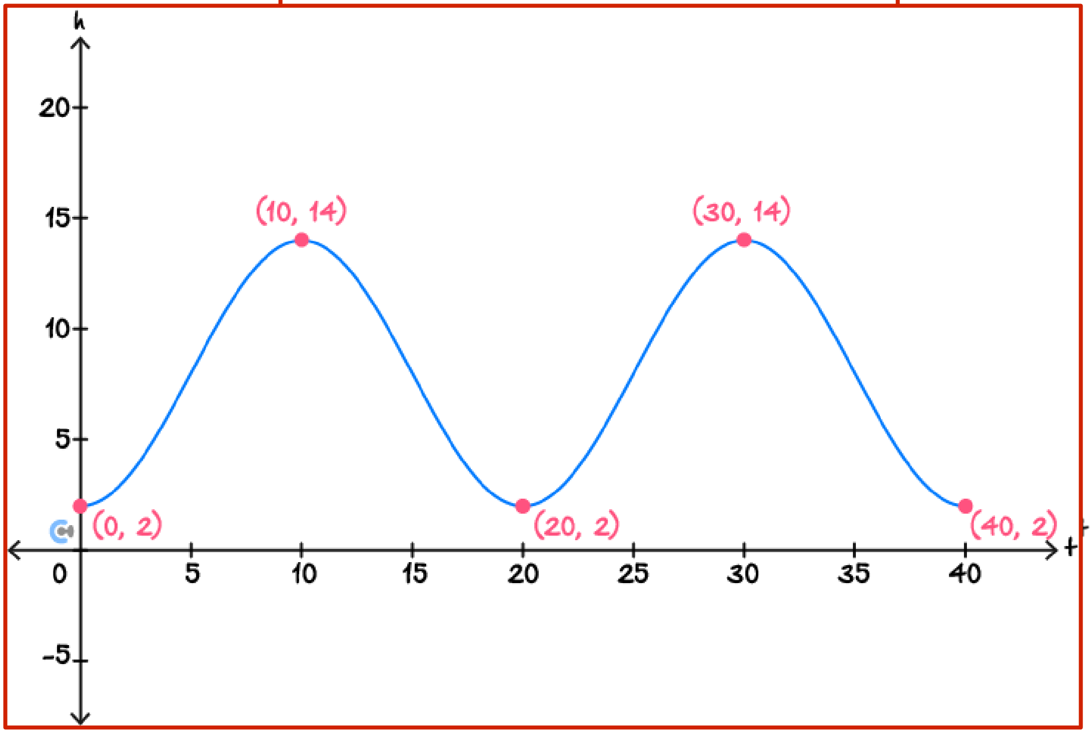
The midline is halfway between 2 and 14, so it's at 8. The amplitude is $14 - 8 = 6$. **[1M]** Cosine starts at a maximum, so to start at the minimum we use $-\cos$. The period is 20 seconds, so:

$$\text{Period} = \frac{2\pi}{20} = \frac{\pi}{10} \quad \textbf{[1M]}$$

So the rule is $h(t) = 8 - 6\cos\left(\frac{\pi t}{10}\right)$.

**b.** Sketch the graph of $h(t)$ for $t \in [0,40]$ on the axes below. Label any maxima and minima with coordinates. (3 marks).

**[1A shape, 1A max and mins, 1A endpoints]**




**c.** Determine the height of the top of the ship 5 seconds after the ride begins. (1 mark)

$h(5) = 8$ metres.

**d.** Find the first time after the ride begins when the top of the ship reaches a height of 10 metres. Give your answer in seconds correct to two decimal places. (2 marks)

Solve $h(t) = 10$ **[1M]**
$t = 6.08$ seconds. **[1A]**

**e.** For how many seconds in the first 60 seconds is the top of the ship higher than 12 metres? (3 marks)

Solve $h(t) = 12 \implies t = 7.323, 12.677$ **[1M]**
Then $12.677 - 7.323 = 5.354$. **[1M]**
Over 60 seconds there are three full periods.
So total time above 12 metres is $3 \times 5.354 = 16.06$ seconds. **[1A]**

**f.** The operator adjusts the ride so it starts at its maximum height instead. Write a new rule for $h(t)$ under this condition. (1 mark)

$h(t) = 8 + 6\cos\left(\frac{\pi t}{10}\right)$. **[1A]**

**Space for Personal Notes**

**Question 15** (12 marks)

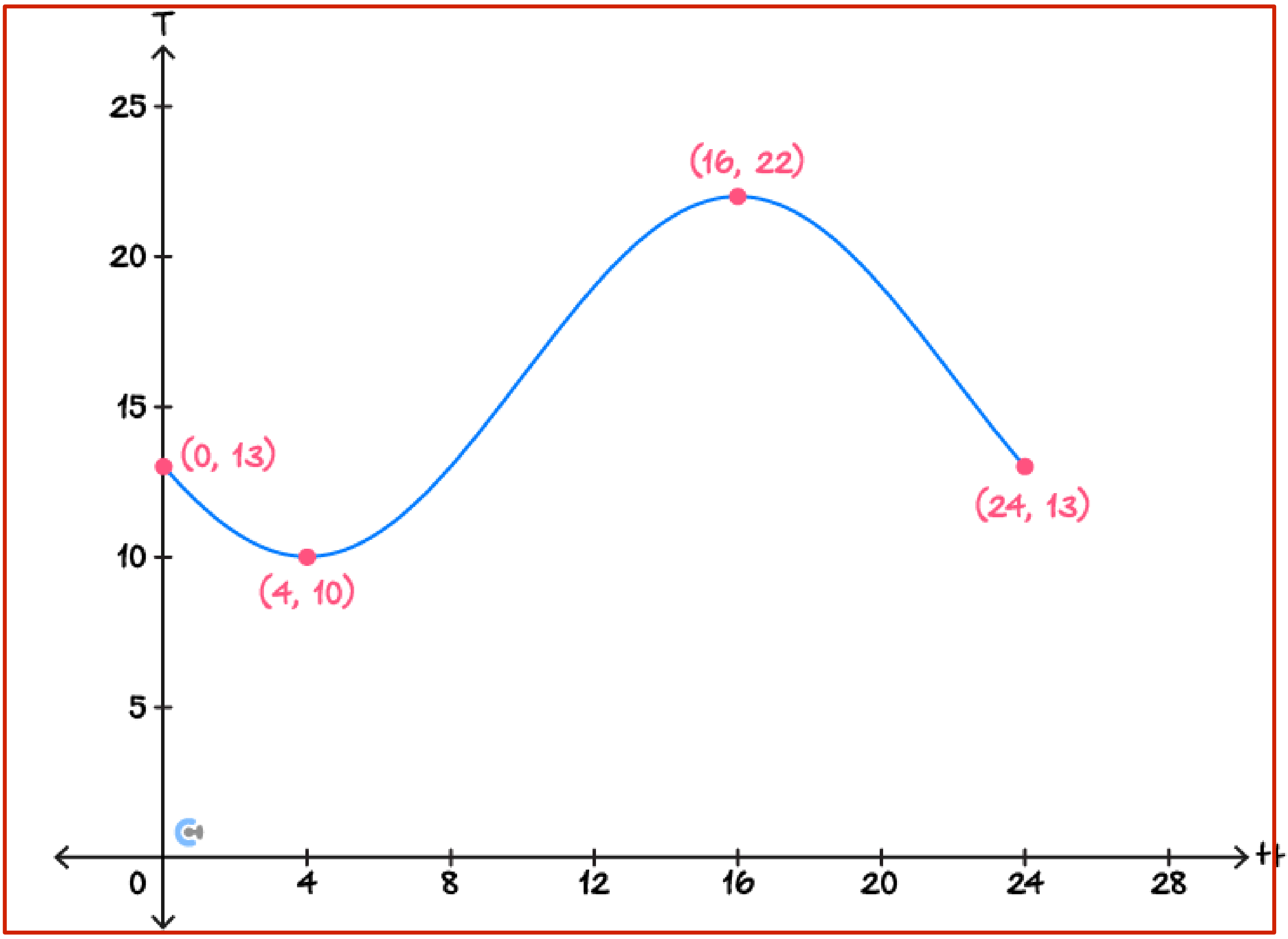
In an underground mine, the temperature varies periodically over a 24-hour cycle and follows a sinusoidal pattern.

The temperature $T(t)$, in degrees Celsius, at $t$ hours after midnight is given by the rule:

$$T(t) = 16 + 6\sin\left(\frac{\pi}{12}(t-10)\right)$$

**a.** Sketch a graph of $T(t)$ for $0 \le t \le 24$. Label any endpoints and minima and maxima with coordinates. (3 marks)

[1A shape, 1A turning points, 1A endpoints]




**b.** Find an equivalent rule for $T(t)$ in the form $a - b\cos\left(\frac{\pi}{12}(t+c)\right)$, where $a, b, c \in [0,24]$. (2 marks)

Rule must be of the form $16 - 6\cos\left(\frac{\pi}{12}(t+c)\right)$. **[1M]**
By looking at the shape we see our graph can be obtained by translated $T(t) = 16 - 6\cos\left(\frac{\pi}{12}t\right)$, 4 units to the right.
The period is 24 so a translation 4 units to the right is the same as a translation 20 units to the left. Thus

$$T(t) = 16 - 6\cos\left(\frac{\pi}{12}(t+20)\right) \quad \textbf{[1A]}$$

**c.** Determine the times during the day when the temperature is exactly 16°C. (2 marks)

Solve $T(t) = 16 \implies t = 10, 22$ **[1M]**
So the times are 10:00 am and 10:00 pm **[1A]** .

**d.** Calculate the amount of time in the 24-hour period that the temperature is above 18°C. Give your answer in hours correct to the nearest minute. (2 marks)

Solve $T(t) = 18 \implies t = 11.298, 20.702.$ **[1M]**
So above for $20.702 - 11.298 = 9.40384$ hours which is 9 hours 24 minutes. **[1A]**

**e.** The average temperature in the mine is too warm for the workers to work at maximum efficiency.
The ventiallation system is adjusted so that now the temperature in the mine is given by $T_1(t) = T(t) - k$, where $k \in R$.
Find the value of $k$ if the temperature in the mine is below 15°C for 80% of the time. (3 marks)

Over a 24 hour period must be below 15° for $0.8 \times 24 = 19.2$ hours. **[1M]**
Let $T_2(t) = K + 6\sin\left(\frac{\pi}{12}(t - 10)\right)$.

Let $t_1$ and $t_2$ be two solutions to $T_2(t) = 15$, where $t_2 > t_1$.

We solve $T_2(t) = 15 \implies t_1 = 10 + \frac{12}{\pi}\arcsin\left(\frac{15-K}{6}\right), t_2 = 22 - \frac{12}{\pi}\arcsin\left(\frac{15-K}{6}\right)$.
**[1M]**

By considering the shape of the graph, we require that $t_2 - t_1 = 24 - 19.2$. So solve

$$12 - \frac{24}{\pi}\arcsin\left(\frac{15-K}{6}\right) = 4.8$$

$$K = 10.1459$$

Therefore $k = 16 - K = 5.85$ **[1A]**

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## VCE Mathematical Methods ½

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