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VCE Mathematical Methods ½
Circular Functions I [4.1]
Workbook

Outline:

Learning Objectives:

- ☐ **MM12 [4.1.1]** - Evaluate Exact Values for Sine, Cosine, and Tangent
- ☐ **MM12 [4.1.2]** - Applying Pythagorean Identity to Evaluate Trigonometric Functions
- ☐ **MM12 [4.1.3]** - Apply Supplementary and Complementary Relationship to Evaluate Trigonometric Functions
- ☐ **MM12 [4.1.4]** - Solve Particular Solution for Trigonometric Functions

Section A: Introduction to Circular Functions

Sub-Section: Radians and Degrees

180°

Radians and Degrees

degrees $\xrightarrow{\times \frac{\pi}{180}}$ rad

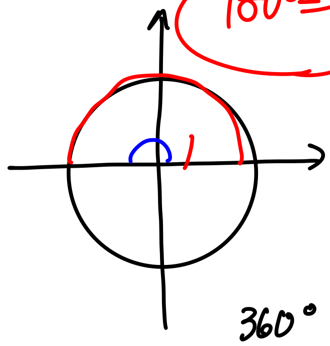
rad $\xrightarrow{\times \frac{180}{\pi}}$ degrees

$$1^c = \left(\frac{180}{\pi}\right)^o$$

$$1^o = \left(\frac{\pi}{180}\right)^c$$

$$180^o = \pi^c$$

$180° = \pi$



$360° = 2\pi$

Question 1

$\pi = 180°$

$1 = \frac{180}{\pi}$

a. Find $\left(\frac{\pi}{4}\right)^c$ in degrees:

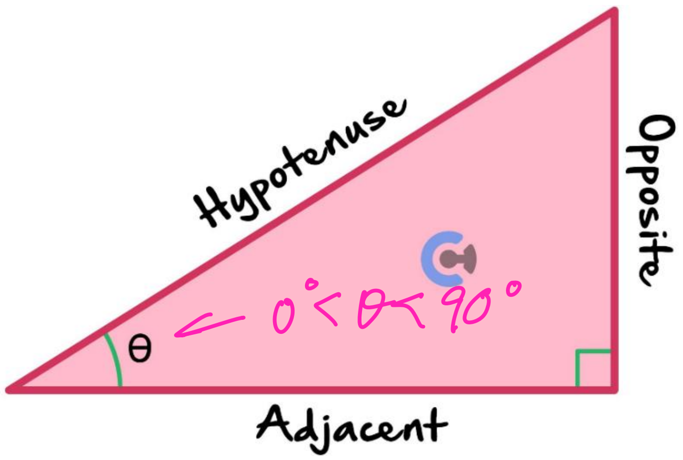
$\frac{\pi}{4} \times \frac{180°}{\pi} = 45°$

b. Find 12^o in radians:

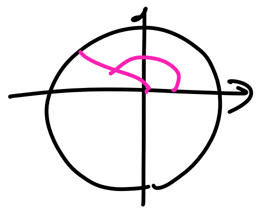
$12 \times \frac{\pi}{180} = \frac{\pi}{15}$

Sub-Section: Unit Circle

REMINDER: Don't forget!



$\theta \in R$



sin = O/H

cos = A/H

tan = O/A

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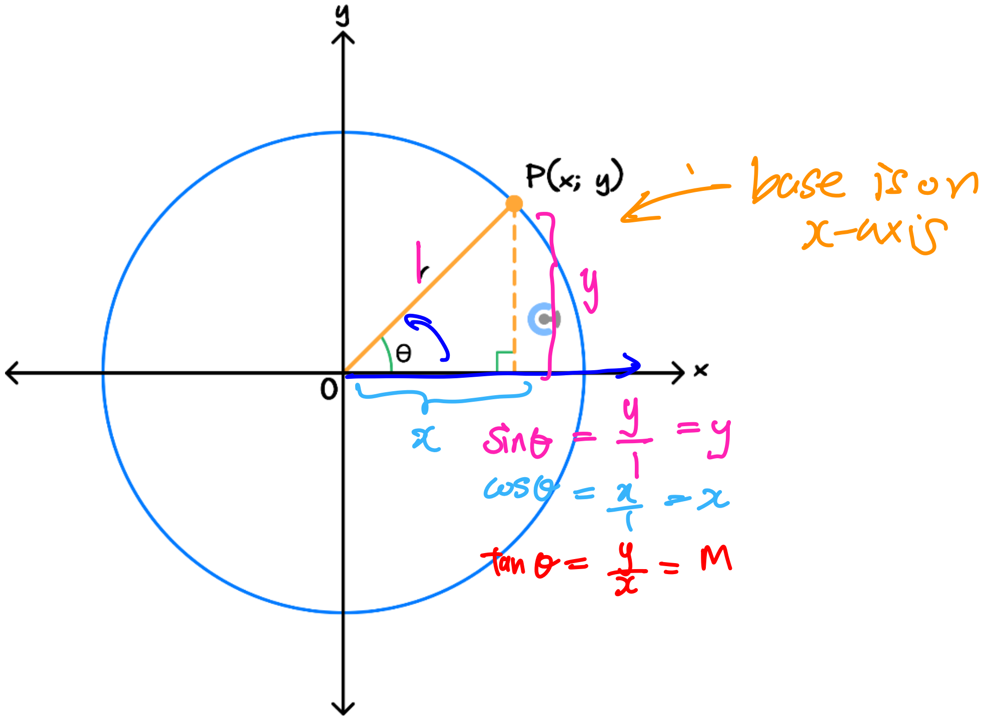
What is a unit circle and how do we use it?

Exploration: Unit Circle

- The unit circle is simply a circle of radius 1.
- Angles are measured from the +ve x-axis anticlockwise.
- It can be divided into **four quadrants:**



- We can use the elementary definition of the trigonometric functions.
- Select the option below!

$$\sin(\theta) = [X\ Value, Y\ Value, Gradient]$$

$$\cos(\theta) = [X\ Value, Y\ Value, Gradient]$$

$$\tan(\theta) = [X\ Value, Y\ Value, Gradient]$$

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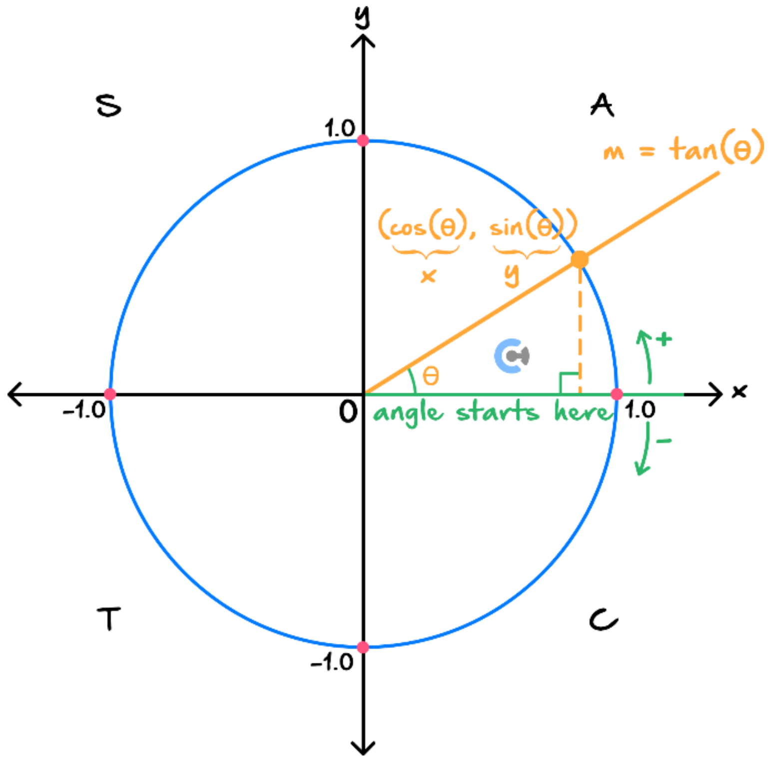
Definition

Unit Circle

- The unit circle is simply a circle of radius 1.



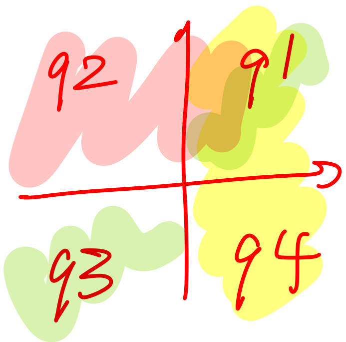
$$\sin(\theta) = y$$

$$\cos(\theta) = x$$

$$\tan(\theta) = gradient$$

Discussion: For which quadrant is cos, sin and tangent positive?

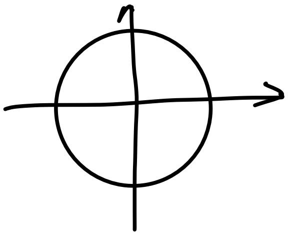
x



Sub-Section: Period

how often repeats

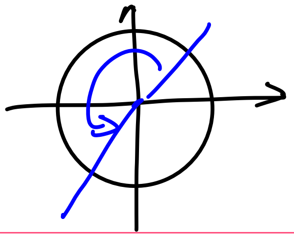
Discussion: What would happen if we rotate 360° to the value of $\sin$ and $\cos$?



Same!

sin / cos : repeats every 360°

Discussion: What would happen if we rotate 180° to the value of $\tan$?



period

REMINDER: Don't forget! Transformation

$$f(x) \rightarrow f(nx)$$

Dilation by factor $\frac{1}{n}$ from the y-axis.

Period of a Trigonometric Function

dilation

Period of $\sin(nx)$ and $\cos(nx)$ functions $= \frac{2\pi}{n}$

Period of $\tan(nx)$ functions $= \frac{\pi}{n}$

where n = coefficient of x and $n > 0$

Question 2

Find the period of each of the following trigonometric functions:

a. $p(x) = \tan\left(\frac{x}{2}\right)$

$n = \frac{1}{2}$

period $= \frac{\pi}{n} = \pi \div \frac{1}{2} = \pi \times 2$

$= 2\pi$

b. $q(x) = \cos\left(\frac{3}{2}x + \frac{\pi}{3}\right)$

$n = \frac{3}{2}$

$\frac{2\pi}{n} = 2\pi \div \frac{3}{2} = 2\pi \times \frac{2}{3} = \frac{4\pi}{3}$

Question 3 Extension.

Find the period of the following trigonometric function:

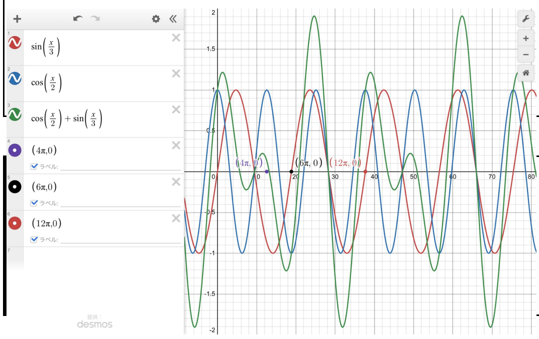
$$f(x) = \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{3}\right)$$

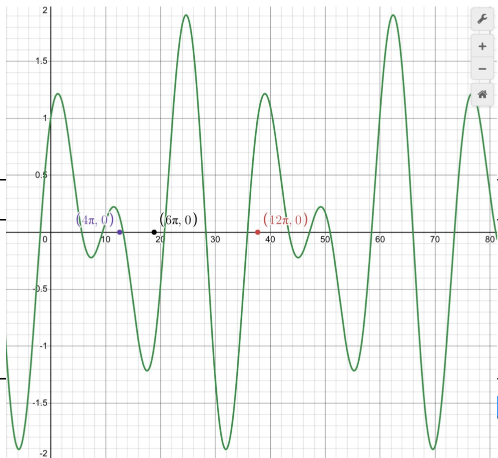
period $= \frac{2\pi}{1/2} = 4\pi$

period $= \frac{2\pi}{1/3} = 6\pi$

period $= \operatorname{lcm}(4\pi, 6\pi)$

$= 12\pi$

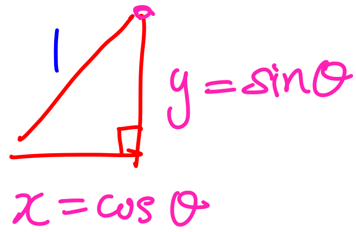




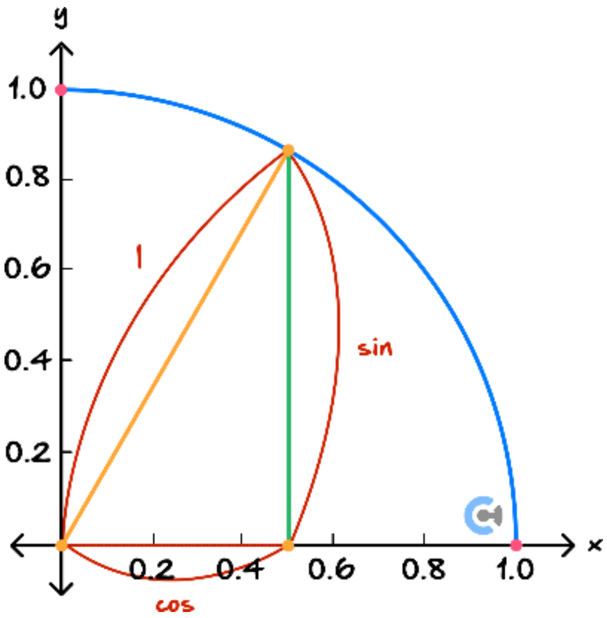
Sub-Section: Pythagorean Identities

Discussion: What does $\sin^2(\theta) + \cos^2(\theta)$ equal to?

$= 1$



Pythagorean Identities



$$\sin^2(\theta) + \cos^2(\theta) = 1$$

- Can be used for finding one trigonometry function by using the other.

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Question 4

Find the value of $\cos(x)$ given that $\sin(x) = \frac{2}{3}$ and x is first quadrant.

Pythagorean Identity

$\sin^2 x + \cos^2 x = 1$

$\left(\frac{2}{3}\right)^2 + \cos^2 x = 1$

$\frac{4}{9} + \cos^2 x = 1$

$\cos^2 x = 1 - \frac{4}{9}$

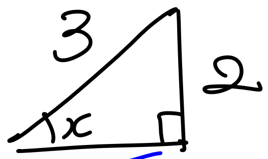
$\cos^2 x = \frac{5}{9}$

$\cos x = \pm\frac{\sqrt{5}}{3}$

x x-value is (+ve) $\cos x = \frac{\sqrt{5}}{3}$

Triangle Method

$\sin(x) = \frac{2}{3} \rightarrow \frac{O}{H}$



$\sqrt{5}$ ← pythag

$\cos(x) = \frac{A}{H} = \pm\frac{\sqrt{5}}{3}$

quad 1 $\cos x$ +ve

$\cos(x) = \frac{\sqrt{5}}{3}$

① triangle

② +− then reject

NOTE: Consider the quadrant to determine signs.

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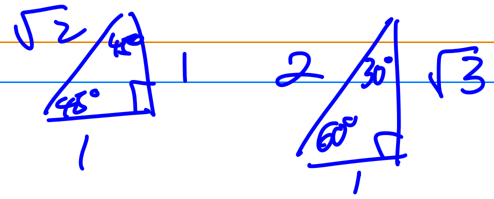
Sub-Section: Exact Values



The Exact Values Table

x	0 (0^o)	$\frac{\pi}{6}$ (30^o)	$\frac{\pi}{4}$ (45^o)	$\frac{\pi}{3}$ (60^o)	$\frac{\pi}{2}$ (90^o)
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

$\hookrightarrow \tan x = \frac{\sin x}{\cos x}$

TIP: Use the fact that $\sin$ is the y value, $\cos$ is the x value and the tangent is the gradient to remember the values well!

$180° = \pi$

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1. $\sqrt{\ }$
2. $\div 2$

x	$0°$ 0	$30°$ $\frac{\pi}{6}$	$45°$ $\frac{\pi}{4}$	$60°$ $\frac{\pi}{3}$	$90°$ $\frac{\pi}{2}$
$\sin x$	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2} = 1$
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

$\tan x = \frac{y}{x} = \frac{\sin x}{\cos x}$

Question 5

Without looking at the exact value table, evaluate the following.

a. $\sin\left(\frac{\pi}{6}\right)$

$= \frac{1}{2}$

b. $\cos\left(\frac{\pi}{3}\right)$

$= \frac{1}{2}$

c. $\sin\left(\frac{\pi}{2}\right)$

$= 1$

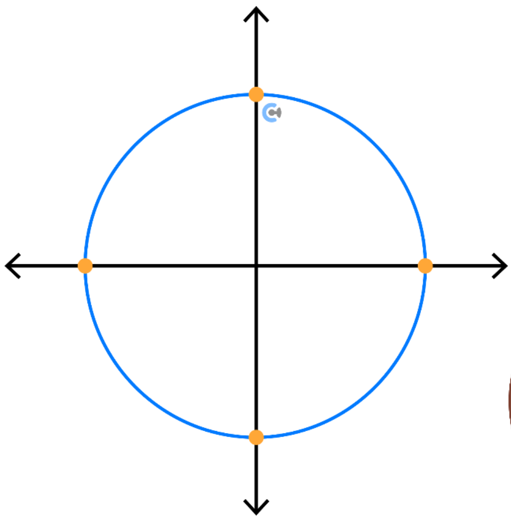
d. $\tan\left(\frac{\pi}{6}\right)$

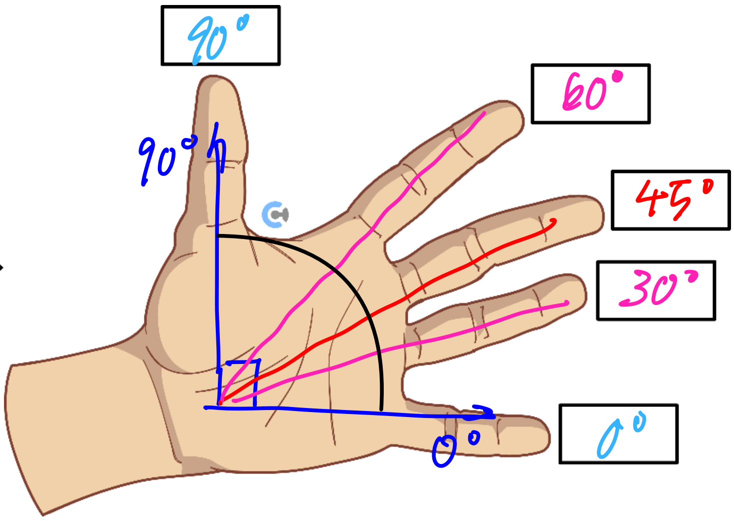
$= \frac{\sqrt{3}}{3} \quad \left(\frac{1}{\sqrt{3}}\right)$

Is there a quick method to remember the exact values?

<u>Exploration:</u> Exact values

- Hold your left palm up facing yourself.
- Imagine the thumb to align with the positive y-axis, so the thumb is $90°$.
- Imagine the pinky to align with the positive x-axis, so the pinky is $0°$.
- Can you guess what angles the index finger, middle finger and ring finger represent?
- Label the angles on a unit circle as well as the finger.





$$\sin(\theta) = \frac{\sqrt{(\text{the number of fingers below})}}{2}$$

$$\cos(\theta) = \frac{\sqrt{(\text{the number of fingers above})}}{2}$$

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Active Recall: The Exact Values Table

x	$0\ (0^o)$	$\frac{\pi}{6}\ (30^o)$	$\frac{\pi}{4}\ (45^o)$	$\frac{\pi}{3}\ (60^o)$	$\frac{\pi}{2}\ (90^o)$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

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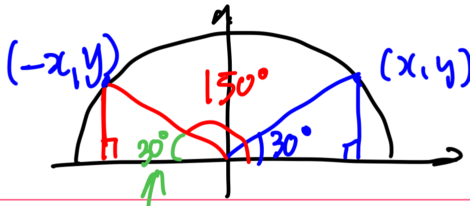
Section B: Symmetry

Sub-Section: Supplementary Relationships

↳ 180°

Discussion: Given that $\sin(30) = \frac{1}{2}$, how do we solve for $\sin(150)$?

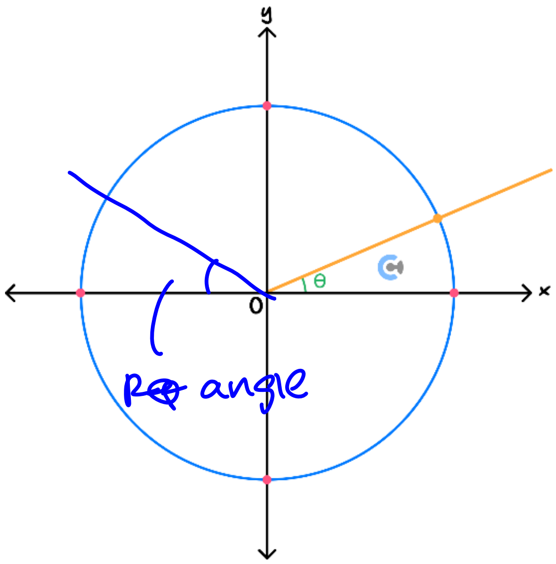


Reference angle

What does Reflection in the y-axis look like?

Exploration: Reflection in y-axis

- Consider the unit circle.

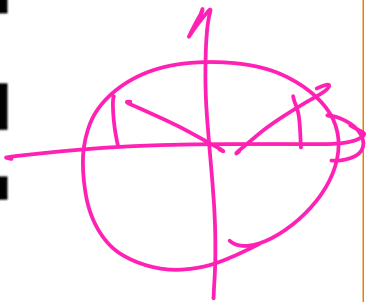


- Reflect the angle around the y-axis on the unit circle above.
- What is the angle in terms of θ?

(180°)

$\pi - \theta$

- How does that affect the sine / cosine / tangent functions?
- Scan the QR code below and have a look!



$$\cos(\pi - \theta) = +/- \cos(\theta)$$

$$\sin(\pi - \theta) = +/- \sin(\theta)$$

$$\tan(\pi - \theta) = +/- \tan(\theta) = \frac{\sin\theta}{\cos\theta} = \frac{+}{-} = -$$

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Question 6 walkthrough

QRS

a. Find the angle at which $\frac{5\pi}{6}$ corresponds to in the first quadrant.

quadrant

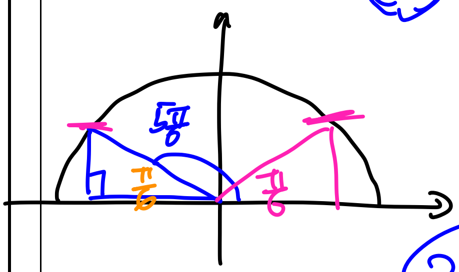
ref angle

sign

Ref angle

$\pi - \frac{5\pi}{6}$

$= \frac{\pi}{6}$



2nd quad

b. What would $\sin\left(\frac{5\pi}{6}\right)$ equal to given that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$?

↳ quad 2

↳ sin = y-value

↳ +ve

$\sin\frac{5\pi}{6} = +\sin\frac{\pi}{6} = +\frac{1}{2}$

c. What would $\cos\left(\frac{5\pi}{6}\right)$ equal to given that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$?

↳ quad 2

↳ cos = x-value

↳ −ve

$\cos\frac{5\pi}{6} = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

d. What would $\tan\left(\frac{5\pi}{6}\right)$ equal to given that $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$?

↳ quad 2

↳ tan = m

↳ −ve

$\tan\frac{5\pi}{6} = -\tan\left(\frac{\pi}{6}\right)$

$= -\frac{1}{\sqrt{3}}$

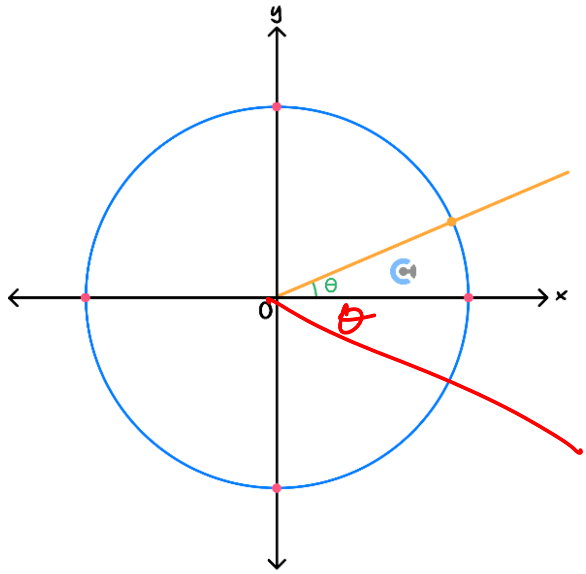
NOTE: Visualise on the unit circle.

ALSO NOTE: In the second quadrant, the y value $(\sin)$ is positive.

What does a Reflection in the x-axis look like?

<u>Exploration</u>: Reflection in x-axis

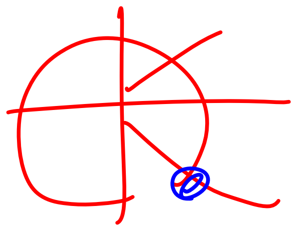
- Consider the unit circle.



$2\pi - \theta$

$-\theta$

- Reflect the angle around the x-axis on the unit circle above.
- What is the angle in terms of θ?
- How does that affect the sine / cosine / tangent functions?
- Scan the QR code below and have a look!



x $\quad \cos(-\theta) = +/- \cos(\theta)$

y $\quad \sin(-\theta) = +/- \sin(\theta)$

$\tan(-\theta) = +/- \tan(\theta)$

Question 7

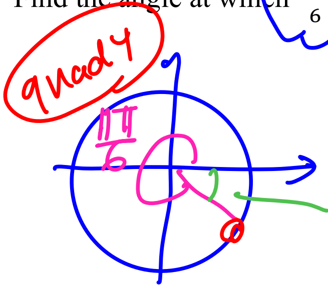
$2\pi - \frac{\pi}{6}$

QRS

quad Ref

a. Find the angle at which $\frac{11\pi}{6}$ corresponds to in the first quadrant.

$180° = \pi$



Ref

$2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$

b. What would $\sin\left(\frac{11\pi}{6}\right)$ equal to given that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$?

$= -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

c. What would $\cos\left(\frac{11\pi}{6}\right)$ equal to given that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$?

↳ +ve

$= +\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

d. What would $\tan\left(\frac{11\pi}{6}\right)$ equal to given that $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$?

↳ −ve

$= -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

NOTE: In the fourth quadrant, the x value $(\cos)$ is positive.

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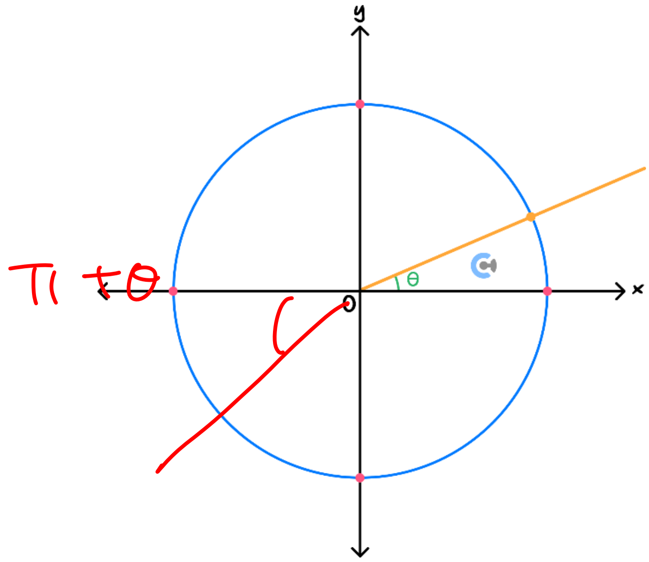
What does Reflection in both axes look like?

Exploration: Reflection in both axes

- Consider the unit circle.



- Reflect the angle around both axes on the unit circle above.
- What is the angle in terms of θ?
- How does that affect the sine / cosine / tangent functions?
- Scan the QR code below and have a look!

$$\cos(\pi + \theta) = +/- \cos(\theta)$$

$$\sin(\pi + \theta) = +/- \sin(\theta)$$

$$\tan(\pi + \theta) = +/- \tan(\theta)$$

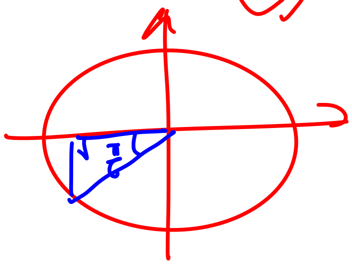
Question 8

$\frac{7\pi}{6} = \pi + \frac{\pi}{6}$

$> \pi$

a. Find the angle which $\frac{7\pi}{6}$ corresponds to in the first quadrant.

quad 3



Ref $= \frac{\pi}{6}$

b. What would $\sin\left(\frac{7\pi}{6}\right)$ equal to given that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$?

$= -\sin\frac{\pi}{6}$

$= -\frac{1}{2}$

c. What would $\cos\left(\frac{7\pi}{6}\right)$ equal to given that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$?

$= -\cos\frac{\pi}{6}$

$= -\frac{\sqrt{3}}{2}$

d. What would $\tan\left(\frac{7\pi}{6}\right)$ equal to given that $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$?

$= \tan\frac{\pi}{6}$

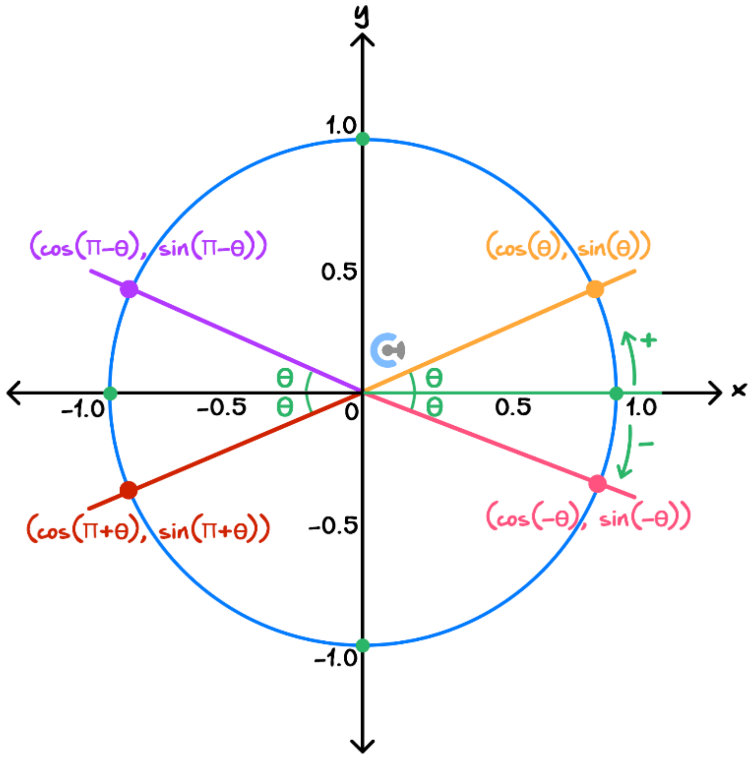
$= \frac{\sqrt{3}}{3}$ $\left(\frac{1}{\sqrt{3}}\right)$

NOTE: In the third quadrant, the gradient (tangent) is positive.

Let's summarise

Definition

Supplementary Relationships



- Simply look at the quadrant to find the correct sign.
  - Second Quadrant $(\pi - \theta)$

$$\cos(\pi - \theta) = -\cos(\theta)$$

$$\sin(\pi - \theta) = +\sin(\theta)$$

$$\tan(\pi - \theta) = -\tan(\theta)$$

  - Third Quadrant $(\pi + \theta)$

$$\cos(\pi + \theta) = -\cos(\theta)$$

$$\sin(\pi + \theta) = -\sin(\theta)$$

$$\tan(\pi + \theta) = +\tan(\theta)$$

Fourth Quadrant $(-\boldsymbol{\theta})$

$$\cos(-\theta) = +\cos(\theta)$$

$$\sin(-\theta) = -\sin(\theta)$$

$$\tan(-\theta) = -\tan(\theta)$$

Steps

1. Equate to $\cos$ / $\sin$ / $\tan(\theta)$.
2. Determine the sign $(\pm)$ by considering the quadrant.

Try the following question!

TIP: Simply draw the unit circle.

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Sin | All
S | A
sin (+) | sin (+)
cos (−) | cos (+)
tan (−) | tan (+)

sin (−) | sin (−)
cos (−) | cos (+)
tan (+) | tan (−)
T | C
Tan | Cos

ASTC

Question 9 Walkthrough.

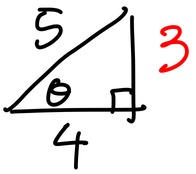
If $\cos(\theta) = 0.8$ where θ is a first quadrant angle, evaluate the following.

a. $\cos(\pi + \theta)$

Q : quad 3
R : θ
S : $-$ve

$\cos(\pi + \theta) = -\cos\theta$
$= -0.8$

$\cos\theta = 0.8 = \frac{4}{5}$



b. $\sin(\pi - \theta)$

Q: quad 2
R: θ
S: $+$ve

$\sin(\pi - \theta) = +\sin(\theta)$
$= \frac{3}{5}$

c. $\tan(\pi + \theta)$

Q: quad 3
R: θ
S: $+$ve

$\tan(\pi + \theta) = +\tan(\theta)$
$= \frac{3}{4}$

Space for Personal Notes

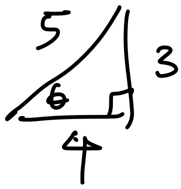
Question 10

If $\sin(\theta) = 0.6$ where θ is a first quadrant angle, evaluate the following.

a. $\sin(\pi + \theta)$

$= -\sin\theta$

$= -0.6$



b. $\cos(\pi + \theta)$

$= -\cos\theta$

$= -\frac{4}{5}$

c. $\tan(\pi - \theta)$

$= -\tan\theta$

$= -\frac{3}{4}$

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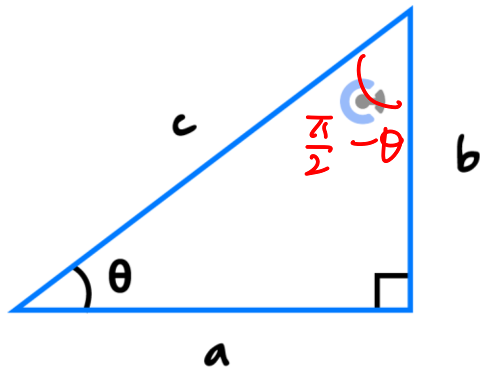
Sub-Section: Complementary Relationships

Let's have a look at this right-angle triangle.

Exploration: Understanding Complementary Relationships

- Take a look at this right-angle triangle:



- What does $\sin(\theta)$ equal to?

$$\sin(\theta) = \underline{\quad \frac{b}{c} \quad}$$

- On the triangle above, label $\frac{\pi}{2} - \theta$.

  What does $\cos\left(\frac{\pi}{2} - \theta\right)$ equal to?

$$\cos\left(\frac{\pi}{2} - \theta\right) = \underline{\quad \frac{b}{c} \quad}$$

- What do you notice?

$$\sin(\theta) = \underline{\quad \cos\left(\frac{\pi}{2} - \theta\right) \quad}$$

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<u>Discussion:</u> Given that $\sin$ and $\cos$ swaps, what would happen to the value of $\tan$?

$\frac{\sin}{\cos}$ $\frac{\cos}{\sin}$

$\frac{1}{\tan\theta}$

<u>Exploration</u>: Understanding Complementary Relationships

- We swap x: $\cos$ and y: $\sin$ of the unit circle.

$$\sin\left(\frac{\pi}{2}-\theta\right) \to \cos(\theta)$$

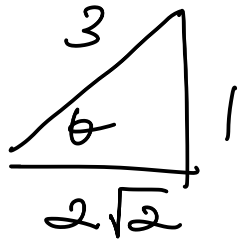
$$\cos\left(\frac{\pi}{2}-\theta\right) \to \sin(\theta)$$

$$\tan\left(\frac{\pi}{2}-\theta\right) \to \frac{1}{\tan(\theta)}$$

Space for Personal Notes

Question 11

If $\sin(\theta) = \frac{1}{3}$ where θ is a first quadrant angles, evaluate the following.



a. $\cos\left(\frac{\pi}{2} - \theta\right)$ +ve

$= +\sin\theta$

$= \frac{1}{3}$

b. $\sin\left(\frac{\pi}{2} - \theta\right)$

$= \cos\theta = \frac{2\sqrt{2}}{3}$

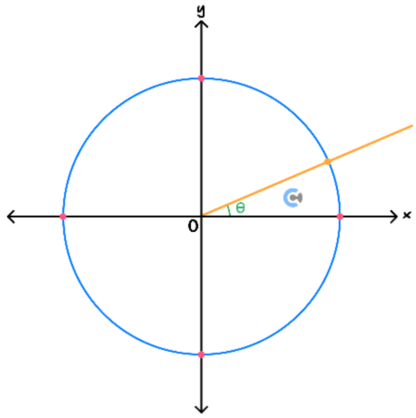
c. $\tan\left(\frac{\pi}{2} - \theta\right)$

$= \frac{1}{\tan\theta} = \frac{1}{\frac{1}{2\sqrt{2}}} = 2\sqrt{2}$

Now let's take a look at more complementary relationships in other quadrants.

<u>Exploration</u>: Generalising Complementary Relationships

- Take a look at the unit circle below.



- Mark the angle $\frac{\pi}{2} - \theta$. Which quadrant is it in?
- Mark the angle $\frac{\pi}{2} + \theta$. Which quadrant is it in?
- Mark the angle $\frac{3\pi}{2} - \theta$. Which quadrant is it in?
- Mark the angle $\frac{3\pi}{2} + \theta$. Which quadrant is it in?

First Quadrant ________________.

quad 1 quad 1 "treat it as quad"

$$\cos\left(\frac{\pi}{2} - \theta\right) = +/- \sin(\theta)$$

+ ⟶ +

$$\sin\left(\frac{\pi}{2} - \theta\right) = +/- \cos(\theta)$$

+ ⟶ +

$$\tan\left(\frac{\pi}{2} - \theta\right) = +/- \frac{1}{\tan(\theta)}$$

+ ⟶ +

- Second Quadrant ________________.

quad 2 quad 1

$$\cos\left(\frac{\pi}{2}+\theta\right) = +/-\sin(\theta)$$

-ve +ve

$$\sin\left(\frac{\pi}{2}+\theta\right) = +/-\cos(\theta)$$

+ve → +ve

$$\tan\left(\frac{\pi}{2}+\theta\right) = +/-\frac{1}{\tan(\theta)}$$

-ve → +ve

- Third Quadrant ________________.

$$\cos\left(\frac{3\pi}{2}-\theta\right) = +/-\sin(\theta)$$

$$\sin\left(\frac{3\pi}{2}-\theta\right) = +/-\cos(\theta)$$

$$\tan\left(\frac{3\pi}{2}-\theta\right) = +/-\frac{1}{\tan(\theta)}$$

- Fourth Quadrant ________________.

$$\cos\left(\frac{3\pi}{2}+\theta\right) = +/- \sin(\theta)$$

$$\sin\left(\frac{3\pi}{2}+\theta\right) = +/- \cos(\theta)$$

$$\tan\left(\frac{3\pi}{2}+\theta\right) = +/- \frac{1}{\tan(\theta)}$$

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Definition

Complementary Relationships

Consider the quadrant for signs.

- First Quadrant $\left(\frac{\pi}{2} - \theta\right)$

$$\cos\left(\frac{\pi}{2} - \theta\right) = +\sin(\theta)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = +\cos(\theta)$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = +\frac{1}{\tan(\theta)}$$

- Second Quadrant $\left(\frac{\pi}{2} + \theta\right)$

$$\sin\left(\frac{\pi}{2} + \theta\right) = +\cos(\theta)$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\frac{1}{\tan(\theta)}$$

- Third Quadrant $\left(\frac{3\pi}{2} - \theta\right)$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos(\theta)$$

$$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin(\theta)$$

$$\tan\left(\frac{3\pi}{2} - \theta\right) = \frac{1}{\tan(\theta)}$$

- Fourth Quadrant $\left(\frac{3\pi}{2}+\theta\right)$

$$\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos(\theta)$$

$$\cos\left(\frac{3\pi}{2}+\theta\right) = +\sin(\theta)$$

$$\tan\left(\frac{3\pi}{2}+\theta\right) = -\frac{1}{\tan(\theta)}$$

- **Steps**

1. Note complementary relationship by identifying a vertical angle $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$.
2. Equate to the opposite trigonometric function $\cos/\sin/\frac{1}{\tan(\theta)}$.
3. Determine the sign $(\pm)$ by considering the quadrant.

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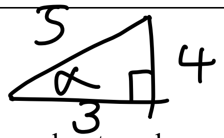
complementary

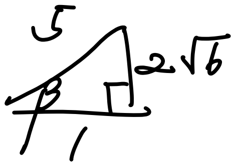
$\frac{\pi}{2} \pm \theta$

$\frac{3\pi}{2} \pm \theta$

Question 12 Walkthrough.



If $\sin(\alpha) = \frac{4}{5}$ and $\cos(\beta) = \frac{1}{5}$ where α, β are first quadrant angles, evaluate the following.



quad 1

a. $\cos\left(\frac{\pi}{2} - \alpha\right)$ +ve

$= +\sin(\alpha)$

quad 1

+ve

$= \frac{4}{5}$

+ve

+ve

b. $\sin\left(\frac{\pi}{2} + \alpha\right)$ $= + \cos(\alpha)$

$= \frac{3}{5}$

−ve

quad 3

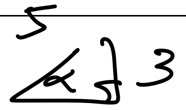
quad 4

c. $\sin\left(\frac{3\pi}{2} - \alpha\right) + \cos\left(\frac{3\pi}{2} + \beta\right)$ +ve

$= -\cos(\alpha) + \sin(\beta)$

$= -\frac{3}{5} + \frac{2\sqrt{6}}{5}$

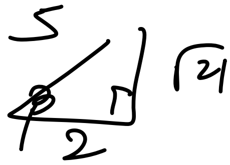
Question 13



If $\sin(\alpha) = 0.6$ and $\cos(\beta) = 0.4$ where α, β are first quadrant angles, evaluate the following.

quad 2

a. $\cos\left(\frac{\pi}{2} + \alpha\right)$ $-ve$



$= -\sin(\alpha)$

$= -0.6$

b. $\sin\left(\frac{\pi}{2} - \alpha\right)$ $= +\cos(\alpha)$

$= \frac{4}{5}$

$+ \mid \times$ quad 4 quad 3 $\times \mid +$

c. $\sin\left(\frac{3\pi}{2} + \alpha\right) - \cos\left(\frac{3\pi}{2} - \beta\right)$ $-ve$ $-ve$

$= -\cos(\alpha) - (-\sin(\beta))$

$= -\frac{4}{5} + \frac{\sqrt{21}}{5}$

Discussion: How can we tell apart from supplementary relationship and complementary relationships?

π $\frac{\pi}{2}$ $\frac{3\pi}{2}$

2π

Supplementary vs Complementary

Supplementary: $trig(Horizontal\ Angle \pm \theta)$

Complementary: $trig(Vertical\ Angle \pm \theta)$

Question 14 HW

If $\sin(\alpha) = 0.6$ and $\cos(\beta) = 0.4$ where α, β are first quadrant angles, evaluate the following.

a. $\sin(\pi + \alpha)$

b. $\sin\left(\frac{\pi}{2} + \beta\right)$

c. $\sin(\pi - \alpha) - \sin\left(\frac{3\pi}{2} - \beta\right)$

Section C: Solving Trigonometric Equations

Sub-Section: Particular Solutions

Active Recall: Period of Trigonometric Function

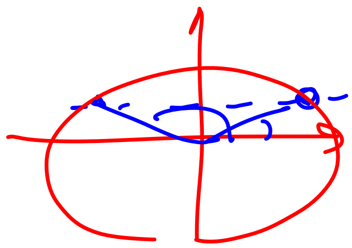
Period of $\sin(nx)$ and $\cos(nx)$ functions = $\frac{2\pi}{n}$

Period of $\tan(nx)$ functions = $\frac{\pi}{n}$

where n = coefficient of x and $n > 0$

Discussion: How often would the solution to $\sin(x) = \frac{1}{2}$ repeat?

every period



Particular Solutions

- Solving trigonometric equation **for finite solutions.**
- **Steps**

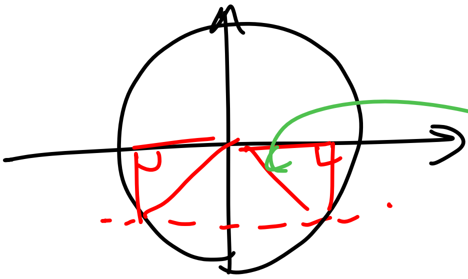
1. Make the trigonometric function the subject. sin = ____ cos = ____
2. Find the necessary angle for one period. Pet → 2 angles
3. Solve for x by equating the necessary angles to the inside of the trigonometric functions.
4. Add and subtract the period to find all other solutions in the domain.

Question 15 Walkthrough.

Solve the following equations for x over the domain specified.

$$2\sin(2x) + \sqrt{3} = 0 \text{ for } x \in [0, 2\pi]$$

$\sin(2x) = -\frac{\sqrt{3}}{2}$



quad 3 & 4

ref (table)

$\frac{\pi}{3}$

$2x = \pi + \frac{\pi}{3}, -\frac{\pi}{3}$

$2x = \frac{4\pi}{3}, -\frac{\pi}{3}$

$\pm$period $= \frac{2\pi}{2} = \pi$

$x = \frac{2\pi}{3}, -\frac{\pi}{6}$

$x \in [0, 2\pi]$

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$x = \cancel{-\frac{\pi}{3}}, \cancel{-\frac{7\pi}{6}}, \frac{2\pi}{3}, \cancel{-\frac{\pi}{6}}, \frac{5\pi}{3}, \frac{5\pi}{6}, \cancel{\frac{8\pi}{3}}, \frac{11\pi}{6}$

$-\pi$ $+\pi$ $+\pi$

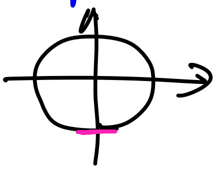
Question 16

Solve the following equations for x over the domains specified.

a. $\sin(4x) = -1$ for $x \in [-\pi, \pi]$.

① Solution for 1 period

$4x = \frac{3\pi}{2}$



$x = \frac{3\pi}{8}$

② Add / Subtract periods

period $= \frac{2\pi}{4} = \frac{\pi}{2}$

$x = -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$

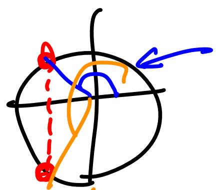
b. $2\cos\left(2x - \frac{\pi}{2}\right) + 1 = 0$ for $x \in \boxed{[0, 2\pi]}$. $\Rightarrow \cos\left(2x - \frac{\pi}{2}\right) = -\frac{1}{2}$

Ref $= \frac{\pi}{3}$

↳ in triangle

$2x - \frac{\pi}{2}$

Sols for 1 cycle (period)



$\pi - \frac{\pi}{3}$

$\pi + \frac{\pi}{3}$

$2x - \frac{\pi}{2} = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$2x - \frac{\pi}{2} = \frac{2\pi}{3}, \frac{4\pi}{3}$

$2x = \frac{5\pi}{3}, \frac{7\pi}{3}$

$x = \frac{5\pi}{6}, \frac{7\pi}{6}$

± period

$\pm \frac{2\pi}{n} = \pi$

more period

$\cancel{-\frac{\pi}{6}}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \cancel{\frac{13\pi}{6}}$

$-\pi$

Question 17 Extension.

$\sin^2 x + \cos^2 x = 1$

$\cos^2(2x) = 1 - \sin^2(2x)$

Solve the following equation for x over the domain specified.

$$\sin^2(2x) - 9\sin(2x) - 5\cos^2(2x) + 8 = 0 \text{ for } x \in (0, \pi)$$

$\sin^2(2x) - 9\sin(2x) - 5(1 - \sin^2(2x)) + 8 = 0$

$6\sin^2(2x) - 9\sin(2x) + 3 = 0$

Let $A = \sin(2x)$

$6A^2 - 9A + 3 = 0$

$2A^2 - 3A + 1 = 0$

$(A-1)(2A-1) = 0$

$\boxed{A = 1, \frac{1}{2}}$

$\sin(2x) = 1$

$2x = \frac{\pi}{2}$

$x = \frac{\pi}{4}$

$\sin(2x) = \frac{1}{2}$

$2x = \frac{\pi}{6}, \frac{5\pi}{6}$

$x = \frac{\pi}{12}, \frac{5\pi}{12}$

No point add/subtracting period

$x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}$

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Question 18 Walkthrough.

Solve the following equations for x over the domains specified.

$$2\tan(2x+\pi)+2\sqrt{3}=0 \text{ for } x \in [0, 2\pi]$$

Discussion: Why do we need to find one angle only for tangents?

Question 19

Solve the following equations for x over the domains specified.

$$\sqrt{3}\tan\left(x-\frac{\pi}{3}\right)-1=0 \text{ for } x \in (0, 3\pi)$$

$\tan(x-\frac{\pi}{3})=\frac{1}{\sqrt{3}}$

① Solutions for 1 period:

$x-\frac{\pi}{3}=\frac{\pi}{6}$

$x=\frac{\pi}{2}$

② Add /subtract period, period $=\pi$

$x=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

Calculator Commands: Particular Solutions on Technology

Mathematica	TI-Nspire	Casio Classpad
Solve $[TrigEquation\ \&\&\ Domain, x]$	Solve $(trigequation, x)\vert domain$	Solve $(trigequation, x)\vert domain$

Question 20 Tech-Active.

Solve the following equations for x over the domains specified.

a. $\sqrt{3}\tan\left(x-\frac{\pi}{3}\right)-1=0$ for $x\in(0,3\pi)$.

```
In[1]:= Solve[Sqrt[3] * Tan[x - Pi / 3] - 1 == 0 && 0 < x < 3 Pi, x]
        解く   平方根    正接     円周率                   円周率
```

Out[1]= $\left\{\left\{x\to\frac{\pi}{2}\right\},\left\{x\to\frac{3\pi}{2}\right\},\left\{x\to\frac{5\pi}{2}\right\}\right\}$

```
In[2]:= Solve[2 Cos[2 x - Pi / 2] + 1 == 0 && 0 ≤ x ≤ 2 Pi, x]
        解く    余弦      円周率                 円周率
```

Out[2]= $\left\{\left\{x\to\frac{7\pi}{12}\right\},\left\{x\to\frac{11\pi}{12}\right\},\left\{x\to\frac{19\pi}{12}\right\},\left\{x\to\frac{23\pi}{12}\right\}\right\}$

b. $2\cos\left(2x-\frac{\pi}{2}\right)+1=0$ for $x\in[0,2\pi]$.

1.1 *Doc RAD

$\text{solve}\left(\sqrt{3}\cdot\tan\left(x-\frac{\pi}{3}\right)-1=0,x\right)\vert 0<x<3\cdot\pi$

$x=1.5708$ or $x=4.71239$ or $x=7.85398$

$\text{solve}\left(2\cdot\cos\left(2\cdot x-\frac{\pi}{2}\right)+1=0,x\right)\vert 0\le x\le 2\cdot\pi$

$x=\frac{7\cdot\pi}{12}$ or $x=\frac{11\cdot\pi}{12}$ or $x=\frac{19\cdot\pi}{12}$ or $x=\frac{23\cdot\pi}{12}$

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Contour Check

Learning Objective: [4.1.1] - Evaluate exact values for sine, cosine, and tangent

Key Takeaways

- Radians and Degrees

$$1^c = \left(\frac{180}{\pi}\right)^o$$

$$1^o = \left(\frac{\pi}{180}\right)^c$$

$$180^o = \pi^c$$

- The Exact Values Table:

x	$0\ (0^o)$	$\frac{\pi}{6}\ (30^o)$	$\frac{\pi}{4}\ (45^o)$	$\frac{\pi}{3}\ (60^o)$	$\frac{\pi}{2}\ (90^o)$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Learning Objective: [4.1.2] - Applying Pythagorean identity to evaluate trigonometric functions

Key Takeaways

- Pythagorean Identity:

$$\sin^2(\theta) + \cos^2(\theta) = \underline{\quad 1 \quad}$$

Learning Objective: [4.1.3] – Apply supplementary and complementary relationship to evaluate trigonometric functions

Key Takeaways

- Supplementary Relationships:
- Look at the quadrant to find the correct sign.
  - Second Quadrant $(\pi - \theta)$
    - Reflection around y-axis.

$$\cos(\pi - \theta) = \underline{\quad -\cos\theta \quad}$$

$$\sin(\pi - \theta) = \underline{\quad +\sin\theta \quad}$$

$$\tan(\pi - \theta) = \underline{\quad -\tan\theta \quad}$$

- **Third Quadrant** $(\boldsymbol{\pi} + \boldsymbol{\theta})$

  - Reflection around x- and y-axis.

$$\cos(\pi + \theta) = \underline{\ -\cos\theta\ }$$

$$\sin(\pi + \theta) = \underline{\ -\sin\theta\ }$$

$$\tan(\pi + \theta) = \underline{\ +\tan\theta\ }$$

- **Fourth Quadrant** $(-\boldsymbol{\theta})$

  - Reflection around x-axis.

$$\cos(-\theta) = \underline{\ +\cos\theta\ }$$

$$\sin(-\theta) = \underline{\ -\sin\theta\ }$$

$$\tan(-\theta) = \underline{\ -\tan\theta\ }$$

- **Steps**

  1. Equate to $\cos$ / $\sin$ / $\tan(\theta)$.
  2. Determine the sign $(\pm)$ by considering the quadrant.

- Complementary Relationships:
  - First Quadrant $\left(\frac{\pi}{2}-\theta\right)$

$$\cos\left(\frac{\pi}{2}-\theta\right) = +\sin\theta$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = +\cos\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right) = +\frac{1}{\tan\theta}$$

  - Second Quadrant $\left(\frac{\pi}{2}+\theta\right)$

$$\sin\left(\frac{\pi}{2}+\theta\right) = +\cos\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$

$$\tan\left(\frac{\pi}{2}+\theta\right) = -\frac{1}{\tan\theta}$$

  - Third Quadrant $\left(\frac{3\pi}{2}-\theta\right)$

$$\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta$$

$$\cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta$$

$$\tan\left(\frac{3\pi}{2}-\theta\right) = +\frac{1}{\tan\theta}$$

- Fourth Quadrant $\left(\frac{3\pi}{2}+\theta\right)$

$$\sin\left(\frac{3\pi}{2}+\theta\right) = \underline{-\cos\theta}$$

$$\cos\left(\frac{3\pi}{2}+\theta\right) = \underline{+\sin\theta}$$

$$\tan\left(\frac{3\pi}{2}+\theta\right) = \underline{-\frac{1}{\tan\theta}}$$

- Steps:

1. Note complementary relationship by identifying a vertical angle $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
2. Equate to the opposite trigonometric function $\cos$ / $\sin$ / $\frac{1}{\tan(\theta)}$.
3. Determine the sign $(\pm)$ by considering the quadrant.

Learning Objective: [4.1.4] - Solve particular solution for trigonometric functions

Key Takeaways

- Period

$$\text{Period of } \sin(nx) \text{ and } \cos(nx) \text{ functions} = \frac{2\pi}{n}$$

$$\text{Period of } \tan(nx) \text{ functions} = \frac{\pi}{n}$$

$$\text{where } n = \text{coefficient of } x \text{ and } n > 0$$

- Particular Solutions:
  - Solving trigonometric equations **for finite solutions.**
  - Steps:
    1. Make the trigonometric function the subject.
    2. Find the necessary angles for one period.
    3. Solve for x by equating the necessary angles to the inside of the trigonometric functions.
    4. Add and subtract the period to find all other solutions in the domain.