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VCE Mathematical Methods ½
Combinations & Permutations [3.3]
Workbook

Outline:

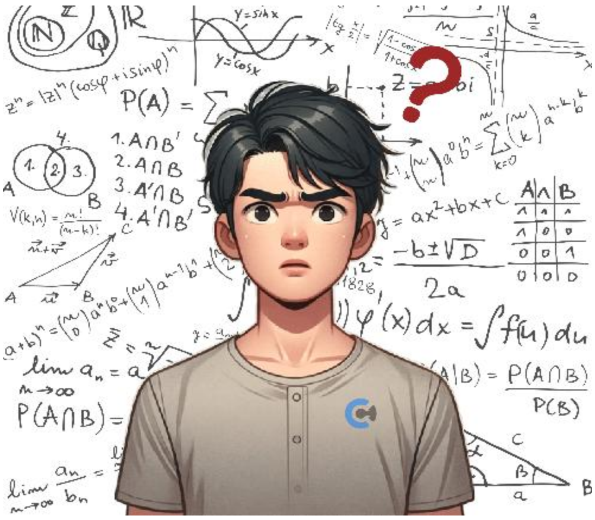
Learning Objectives:

- ☐ MM12 [3.3.1] - Find Number of Permutations and Combinations
- ☐ MM12 [3.3.2] - Find Number of Permutations and Combinations with Restrictions/Composite
- ☐ MM12 [3.3.3] - Find Probabilities Using Counting Methods

Section A: Introduction to Counting Methods

Sub-Section: Addition Principle

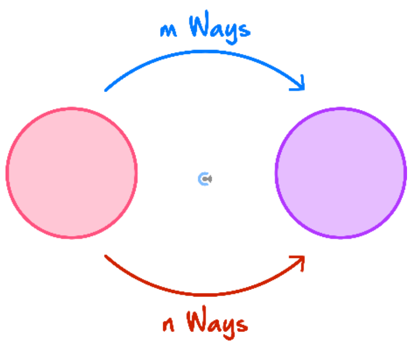
Context: Addition Principle



- Sam is choosing between his 4 pants and 3 shorts to go on his date with Emily.
- How many different options does he have? [7/12]
- We [added/multiplied] the options as Sam will choose one option [or/and] the other.

alternatives

Addition Principle



- Associated with the use of the word "OR."

Total Possibilities $= m + n$

Question 1

A restaurant offers three vegan dishes or four vegetarian dishes.

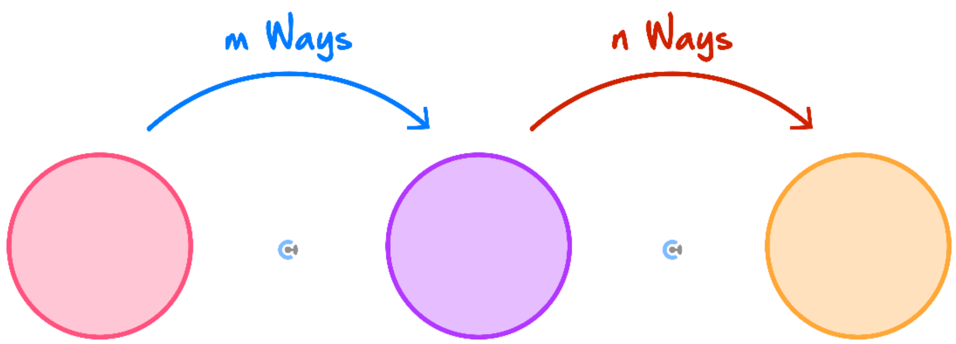
How many selections of one main meal does a customer have?

$3+4 = 7$ options

Space for Personal Notes

Sub-Section: Multiplication Principle

The Multiplication Principle → same time



- Associated with the use of the word "AND."

Total Possibilities $= m \times n$

Question 2

Emily has two different pants, five different tops and three different pairs of shoes.

How many different choices does she have for a complete outfit?

$2 \times 5 \times 3 = 30$ outfits

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Section B: Arrangements (Permutations)

Sub-Section: Introduction to Arrangements

What are arrangements?

Arrangements

- **Definition:** It is a study of a number of ways to order things.

Discussion: How many ways can you arrange letters a, b, c?

- abc
- acb
- bac
- bca
- cab
- cba

→ 6 ways

$\boxed{3 \mid 2 \mid 1} = 3 \times 2 \times 1$

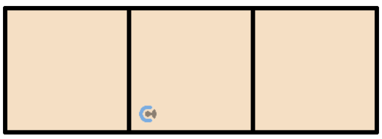
$= 3!$

factorial

Is there a way to visualise the number of arrangements?

Box Diagram for Arrangements

- **Definition:** We can use it to write down a number of arrangements for each position represented by each box.



Question 3 Walkthrough.

a, b, c

3 different letters are ordered for a 3 letter word.

How many different words can you get?

$\boxed{3 \mid 2 \mid 1} = 3 \cdot 2 \cdot 1 \quad (3!)$

$= 6$ words

Question 4

A family of 4 sits next to each other and is interested in a number of ways they could be seated.

How many different ways can the family of 4 sit in their 4 seats?

$\boxed{4 \mid 3 \mid 2 \mid 1} = 24$ ways

$4!$

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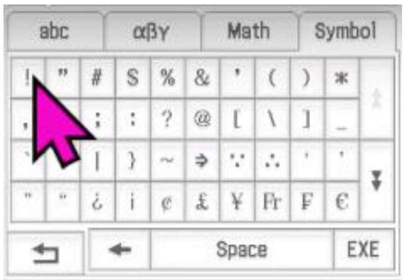
Calculator Commands: Factorial on Technology

- **Mathematica**
  - Exclamation Mark

  $x!$

- **TI-Nspire**
  - Menu 51

- **Casio Classpad**

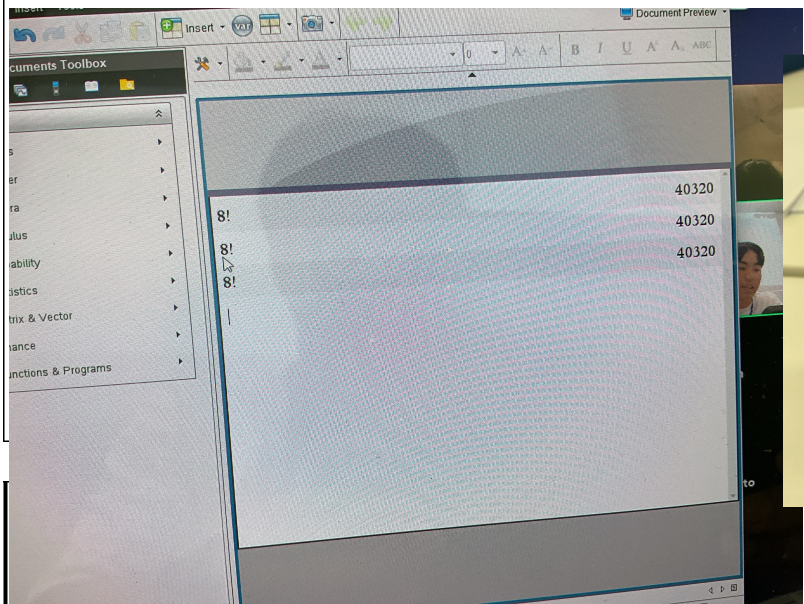


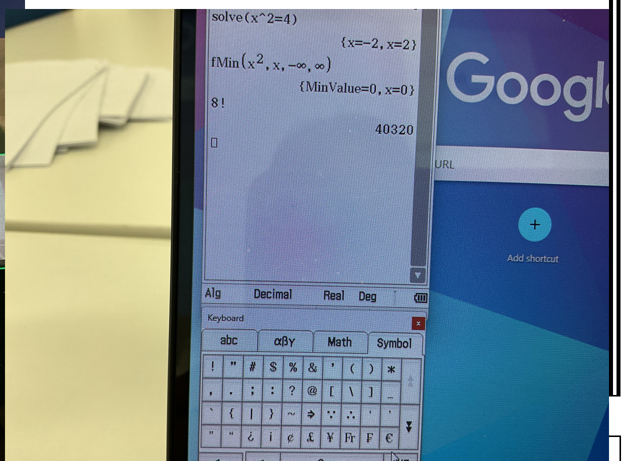
Question 5 Tech-Active.

A school is organising a photo session for 8 students. They are to stand in a single row for a group picture.

In how many different ways can the 8 students be arranged in a row?

8!





Sub-Section: General Arrangement

What if we do not have enough spots for everything?

Exploration: General Arrangement

- Previously, we considered the case where **everything was ordered.**

CUE: How many ways can 5 people sit in 5 seats?

5	4	3	2	1

- What would happen if we don't have enough seats for everyone?

CUE: How many ways can 5 people sit in 2 seats?

5	4

$\div(3\times2\times1)$

- In summary:

$^5P_2$

$$\boxed{5\ \ 4} = \frac{\boxed{5\ \ 4\ \ 3\ \ 2\ \ 1}}{\boxed{3\ \ 2\ \ 1}} = \frac{5!}{3!}$$

NOT ordered

- Let's generalise this for n people with only r many seats! $(n > r)$.
  - The numerator represents a number of ways where we seat everyone.

    total

    It is given by [$n!$, $r!$, $(n-r)!$].

    n = total

    r = order

  - The denominator represents a number of arrangements we missed out due to lack of a seat.

    How many people aren't sitting? [n, r, $(n-r)$]

    Hence, the denominator is given by [$n!$, $r!$, $(n-r)!$]

$$\text{Number of ways for } n \text{ people to sit in } r \text{ seats} = \boxed{\frac{n!}{(n-r)!}} \quad \frac{\text{total}}{\text{not ordered}}$$

  - We call this ${}^nP_r$! Or Permutations!

Definition

<u>Arrangement</u> = Permutations

- Generally:

$$\text{Ways to arrange/order } n \text{ many things for } r \text{ spots} = \frac{n!}{(n-r)!}$$

- We call this ${}^nP_r$.

$${}^nP_r = \_\_\_\_\_\_\_\_$$

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Question 6 Walkthrough.

James is trying to make a three-digit number by using the numbers $1, 2, 3, 4, 5, 6, 7$ without repeating them.

How many different numbers can James have?

$$^{7}P_{3} = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \times 6 \times 5 \times \cancel{4!}}{\cancel{4!}}$$

$$= 7 \times 30$$

$$= 210 \text{ numbers}$$

NOTE: We did **not** arrange all 7 numbers we had, as the number was three-digit.

Question 7

The teacher decides to pick 2 students from 10 student class and appoints them as a class captain and a vice-captain.

How many different ways could the teacher do this?

$$^{10}P_{2} = \frac{10!}{8!} = 10 \times 9$$

$$= 90 \text{ ways}$$

TIP: Only multiply the numbers after all the common factors are cancelled in the fraction.

Calculator Commands: Arrangements on Technology

Mathematica

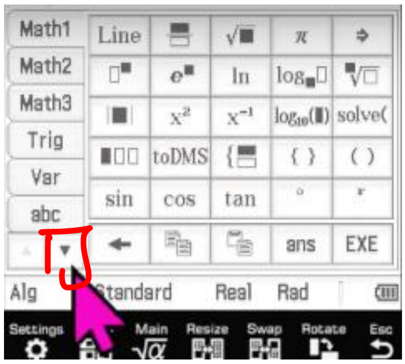
- Just use factorial. Rip

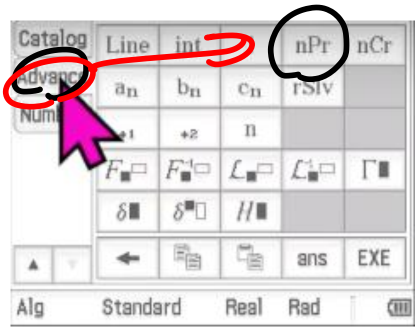
TI-Nspire

- Menu 52

  $^nP_r(n,r)$

Casio Classpad





$^nP_r(n,r)$

Question 8 Tech-Active.

A painter is to paint the five circles of the Olympic flag. He cannot remember the colours to use for any of the circles, but he knows they should all be different. He has eight colours of paint available.

In how many ways can he paint the circles on the flag?

$$^8P_5 = \left(\frac{8!}{3!}\right) = 6720$$

Sub-Section: Composite Arrangements ⇒ Grouping

Nested

Discussion: How can we arrange a, b, c, d if a and b needs to be next to each other?

c | ab | d

1 2 3

2!

3 group 3!

$3! \times 2!$

group within

Composite Arrangements

- **Definition:** Occurs when an arrangement happens within another arrangement.
- Steps:
  - Consider each group as one object and find the arrangements.
  - Consider the arrangements within the groups and multiply.

Question 9 Walkthrough.

Consider a family of 4 which consists of a mum, dad, son, and daughter.

If it is known that the mum must sit next to the daughter and the dad must sit next to the son, how many different ways could they be sitting?

groups: m.d f.s

2! 2! 2!

$2! \times 2! \times 2! = 8$ ways

Question 10

Pranit wants to install seating plans for his rowdy class of 10 students. Fearing the backlash of the students, he lets them choose one friend to sit next to each other.

How many different seating plans could Pranit come up with?

group:

5!

2! 2! 2! 2! 2!

$$5! \times (2!)^5 = 3840$$

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Question 11 Tech-Active.

There are 12 animals in the animal farm; 4 dogs, 4 cats and 4 hamsters. Nayuta decides to label them with numbers ranging from 1-12. He decides to finish labelling all animals in the specific type of species before labelling animals from different species.

a. How many different ways could the animals be numbered?

groups $\boxed{4!}$ $\boxed{4!}$ $\boxed{4!}$

→ $3!$

within

$3! \times 4! \times 4! \times 4!$
$= 82944$

b. If Nayuta decided to label the animals without considering their species, are there more ways to number them compared to **part a.** or less?

$12! = 479001600$

$\Rightarrow$ MORE

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Sub-Section: Arrangements with Restrictions

Discussion: What do we have to consider when making a 4-digit number with $0, 1, 2, 3$?

1, 2, 3

NOT 0

Arrangements with Restrictions

- **Definition:** The general principle to deal with restrictions is to:
  - Use the boxes.
  - Fill in the number of options for the slot that has the restriction first.

Question 12 Walkthrough.

How many different odd numbers between 1000 - 9999 exist, which only have the digits $1, 2, 3, 4, 5, 6, 7, 8$ given each digit can only be used once?

8

end with odd

$\underbrace{\Box\ \Box\ \Box}_{^7P_3}\ \boxed{4}$

$$= {}^7P_3 \times 4$$
$$= \frac{7!}{4!} \times 4$$
$$= (7 \cdot 6 \cdot 5) \times 4$$
$$= 840 \text{ ways}$$

TIP: Consider the number of options for the last digit. And do this before considering the rest!

Question 13

If no digit can be used more than once, find how many numbers that can be formed from the digits $2, 3, 4, 5, 6, 7, 8$ that are:

7

a. Three-digit numbers.

$$^7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4!}}{\cancel{4!}}$$

$= 210$ numbers

b. Even three-digit numbers.

$$\underbrace{\square\ \square}_{^6P_2}\ \boxed{4}$$

$$^6P_2 \times 4$$

$$= \frac{6!}{4!} \times 4$$

$$= 6\cdot 5\cdot 4 \quad = 120 \text{ numbers}$$

c. Three-digit numbers greater than 700.

7 or 8

$$\boxed{2}\ \underbrace{\square\ \square}_{^6P_2}$$

$$= 2 \times {}^6P_2$$

$$= 2 \times \frac{6!}{4!}$$

$= 60$ numbers

TIP: Target the digit with restrictions first.

Question 14 Tech-Active.

Four boys and two girls sit in a line on stools in front of a counter. Find the number of ways in which they can arrange themselves:

a. If one of the boys, Raj, insists on being on (either) one of the ends.

Raj | everyone else

1! 5!

groups: 2!

within

$2! \times (1!) \times (5!)$

$= 240$ ways

b. If the girls want to be at opposite ends of the line.

2 | 4! | 1

$= 2 \times 4! \times 1$

$= 48$ ways

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Section C: Selections (Combinations)

order doesn't matter

Sub-Section: Introduction to Selections

Discussion: Two different permutations AB and BA. Are they considered to be different selections?

1 selection

Question 15

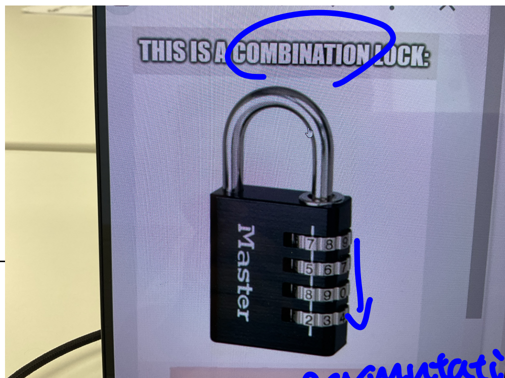
Consider letters A, B, C.

a. State the number of ways we can arrange two letters from A, B, C.

$3! = 6$ ways

b. State the number of ways we can select two letters from A, B, C.

3 ways



permutation lock

Okay, let's try generalising this!

Exploration: Selections

- Consider the following example.

Number of selections when we select two alphabets from A, B, C?

- We can first solve for the number of arrangements.

$$\text{Number of Arrangements} = {}^3P_2 = \frac{3!}{(3-2)!}$$

How do we find number of selections?

- Let's consider the number of arrangements with the same selection.

$EG: AB$ and BA

- How many different arrangements with same selection of lettters? 2

$$\text{Number of Selections} = \frac{\textit{Number of Arrangements}}{\text{no. of arrangements for the chosen elements}}$$

$$= \frac{\textit{Number of Arragenements}}{r!}$$

- Let's generalise this for n people with only r many selected! $(n > r)$
  - The numerator represents number of ways where we arrange n people in r spots.

    It is given by [${}^nP_r$, $n!$, $r!$].

  - The denominator represents number of arrangements in the same selections.

    It is given by [$n!$, $r!$, $(n-r)!$].

Number of Selections $= \dfrac{{}^nP_r}{r!} = \dfrac{n!}{r!(n-r)!}$

^nP_r

- We call this ^nC_r! Or combinations!

Definition

<u>Selection</u>

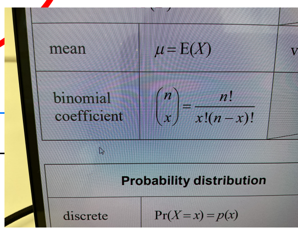
- Generally:

$$\text{Ways to select } r \text{ things from } n \text{ many things} = \frac{{}^nP_r}{r!}$$

- We call this ^nC_r,

$$^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

- Where $r =$ number of selection spots.



mean	$\mu = E(X)$
binomial coefficient	$\binom{n}{x} = \frac{n!}{x!(n-x)!}$
Probability distribution	
discrete	$\Pr(X=x)=p(x)$

Question 16 Walkthrough.

A leadership team of three people is to be chosen from a group of eleven students. How many different teams are possible?

$n = 11$

$$^{11}C_3 = \frac{n!}{r!(n-r)!} = \frac{11!}{3!\,8!}$$

$r = 3$

$= 165$

Discussion: Why do we divide by $r!$ again?

order doesn't matter

Question 17

How many ways are there to choose exactly two pets from a store with 8 dogs and 12 cats?

$$^{20}C_2 = \frac{20!}{18!\,2!} = \frac{20 \times 19 \times \cancel{18!}}{\cancel{18!}\,2!}$$

$$= \frac{20 \times 19}{2}$$

$$= 190 \text{ ways}$$

NOTE: We can treat dogs and cats the same although dogs are better

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Calculator Commands: Combinations of Technology

Mathematica

- Binomial $[n, r]$

TI-Nspire

- Menu 53

${}^{n}C_{r}\ (n, r)$

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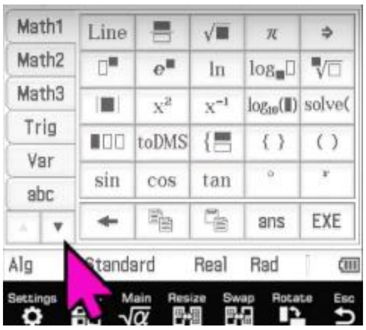

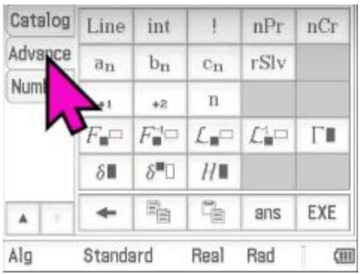

${}^{n}C_{r}\ (n, r)$

Question 18 Tech-Active.

A team of three boys and three girls is to be chosen from a group of nine boys and six girls. How many different teams are possible?

${}^{9}C_{3} \times {}^{6}C_{3}$

$= 1680$

Section D: Probability with Counting Method

Sub-Section: Probability with Arrangements

Discussion: How would you find the probability of getting an even number from arrangements of $1, 3, 4, 5$?

$$\frac{\text{no. even arrangements}}{\text{total}}$$

$$\frac{\text{desired}}{\text{total}}$$

Probability with Arrangements

$$\mathbf{Pr} = \frac{n\,(Wanted\ Arrangements)}{n\,(Total\ Arrangements)}$$

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Question 19 Walkthrough.

Three-letter 'words' are to be made by arranging the letters of the word METHODS. We cannot use the same letter twice:

a. How many three-letter words can be made from the letters METHODS?

order ✓

7 letters

$$^7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \cdot 6 \cdot 5$$

$= 210$ ways

b. How many three-letter words start with the vowel?

$$\boxed{2}\ \boxed{6}\ \boxed{5} = 60 \text{ words}$$

c. Hence, what is the probability that the word begins with a vowel?

$$\frac{\text{desired}}{\text{total}} = \frac{\text{begins with vowel}}{\text{total}} = \frac{60}{210} = \frac{2}{7}$$

NOTE: We get a different word depending on their arrangements. Hence, we use arrangements rather than selections!

ALSO NOTE: In combinations and permutations 'word' is used to mean an arrangement of letters, it does not have to be an actual English word.

Question 20

Four-digit number is to be made by arranging the numbers $1, 2, 3, 4$. We cannot use the same number twice.

a. How many four-digit numbers can be made in general?

$4! = 24$

b. How many four-digit even numbers can be made?

$$\underbrace{\square\,\square\,\square}_{3!}\,\boxed{2} = 3! \times 2 = 12$$

c. Hence, what is the probability that the four-digit number is an even number?

$$\frac{12}{24} = \frac{1}{2}$$

NOTE: That was an obvious answer but make sure to follow these steps for harder questions!

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Sub-Section: Probability with Selections

Discussion: How would you find the probability of selecting A and B out of 4 letters A, B, C and D?

Probability with Selections

$$\mathbf{Pr} = \frac{n\,(\boldsymbol{Wanted\ Selections})}{n\,(\boldsymbol{Total\ Selections})}$$

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Question 21 ~~Walkthrough~~.

We decide to make a team of 4 people from the pool of 4 males and 4 females.

a. How many different selections of three women and one man is possible?

$$^4C_3 \times {}^4C_1 = \frac{4!}{3!\,1!} \times \frac{4!}{(1,3)} = 16$$

b. How many different teams of 4 can we make from the selection?

$$^8C_4 = 70$$

c. What is the probability that a team of 4 would consist of three women and one man?

$$\frac{16}{70} = \frac{8}{35}$$

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Question 22

desired
Total

What is the probability that a team of four chosen at random from a group of eight friends, four males and four females, would consist of three women and one man?

$$\frac{8}{35}$$

TIP: Find the number of selections for three women and one man first.

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Question 23 Tech-Active.

We decide to select a team of 11 from the pool of 5 bowlers and 12 non-bowlers.

a. How many different possible teams can be selected?

$^{17}C_{11} = 12376$

$\frac{17!}{11!\,6!}$

b. In how many ways can the team include exactly 4 bowlers?

4B 7N. B

$^5C_4 \times {}^{12}C_7 = 3960$

c. Hence, what is the probability of having a team of 4 bowlers?

$$\frac{3960}{12376} = \frac{495}{1547}$$

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Contour Check

Learning Objective: [3.3.1] – Find number of permutations and combinations

Key Takeaways

- Box Diagram for Arrangements
  - **Definition:** We can use it to write down number of arrangement for each position represented by each box.
- Arrangement
  - Generally:

Ways to arrange/order n many things for r spots =

  - We call this ^nP_r.

$$^nP_r = \frac{n!}{(n-r)!}$$

- Selection
  - Generally:

Ways to select r things from n many things $= \frac{^nP_r}{r!}$

  - We call this ^nC_r.

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

  - Where r = number of selection spots.

Learning Objective: [3.3.2] – Find number of permutations and combinations with restrictions/composite

Key Takeaways

- **Composite Arrangements** → grouping
  - **Definition:** Occurs when an arrangement happens within another arrangement.
  - Steps:
    - Consider each group as one object and find the arrangements.
    - Consider the permutation within the groups and multiply.
- **Arrangements with Restrictions**
  - **Definition:** The general principle to deal with restrictions is to:
    - Use a box diagram.
    - Fill in the number of options for the slot that has the restriction first.

Learning Objective: [3.3.3] - Find probabilities using counting methods

Key Takeaways

- Probability with Arrangements

$$\mathrm{Pr} = \frac{n\,(Wanted\ Arrangements)}{n(\text{total})}$$

- Probability with Selections

$$\mathrm{Pr} = \frac{n\,(Wanted\ Selections)}{n(\text{total})}$$

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