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VCE Mathematical Methods ½
Transformations [2.4]
Workbook

Outline:

Introduction to Transformations Pg 2-8
- Image and Pre-Image
- Dilation
- Reflection
- Translation

Transformation of Points Pg 9-20
- Basic Transformation of Points
- The Order of Transformations
- Interpreting the Transformation of Points

Transformation of Functions Pg 21-28
- Applying Transformations to Functions
- Finding the Applied Transformations

Learning Objectives:

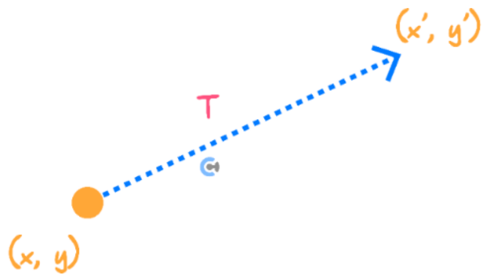
- ☐ **MM12 [2.4.1]** - Applying x' and y' Notation to Find Transformed Points, Find the Interpretation of Transformations and Altered Order of Transformations
- ☐ **MM12 [2.4.2]** - Find Transformed Functions
- ☐ **MM12 [2.4.3]** - Find Transformations From Transformed Function (Reverse Engineering)

Section A: Introduction to Transformations

Sub-Section: Image and Pre-Image

What do we call an original coordinate and a transformed coordinate?

Image and Pre-Image



- The original coordinate is called the Pre-image.
- The transformed coordinate is called the Image.

Pre-Image: (x, y)

Image: (x', y')

Question 1

It is known that $(1,4)$ transformed into $(3,5)$. State the value of x' and y'.

$\therefore$ $x' = 3$

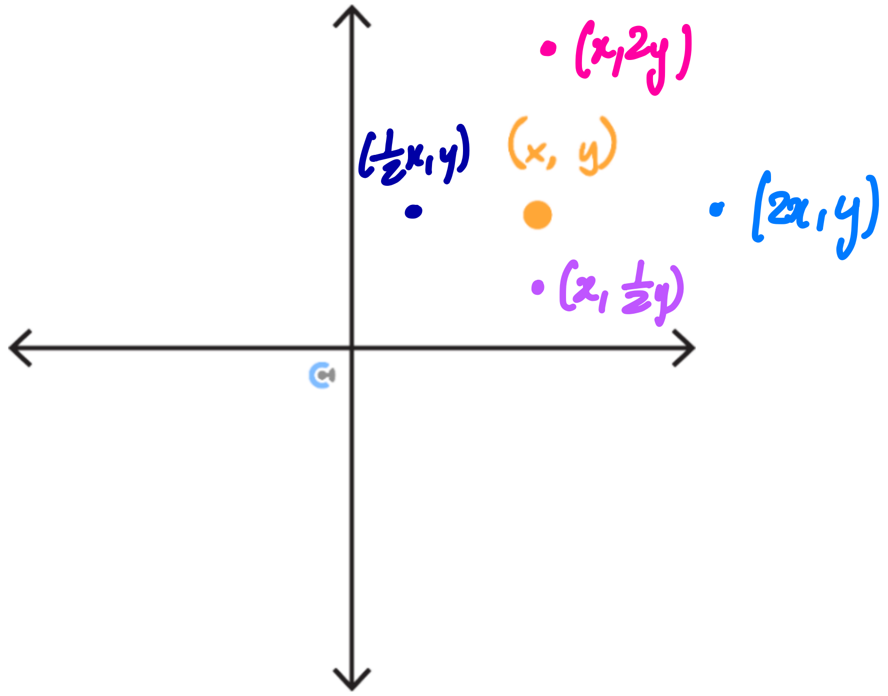
$y' = 5$

NOTE: The x' and y' notation will be used quite heavily!

Sub-Section: Dilation

Exploration: Dilation

- Consider the point below:



- Let's plot the coordinates:

  - $P1$: Dilation by a factor 2 from the x-axis. $(x, 2y)$
  - $P2$: Dilation by a factor $\frac{1}{2}$ from the x-axis. $(x, \frac{1}{2}y)$
  - $P3$: Dilation by a factor 2 from the y-axis. $(2x, y)$
  - $P4$: Dilation by a factor $\frac{1}{2}$ from the y-axis. $(\frac{1}{2}x, y)$

Dilation

Dilation by a factor a from the x-axis: $y' = ay$

Dilation by a factor b from the y-axis: $x' = bx$

Question 2 Walkthrough.

Find the image (x', y') after applying the following transformations to (x, y).

Dilation by factor 2 from the x-axis. $(x, 2y)$

Dilation by factor $\frac{1}{3}$ from the y-axis. $\left(\frac{1}{3}x, 2y\right)$

Question 3

Find the image (x', y') after applying the following transformations to (x, y).

Dilation by factor $\frac{1}{2}$ from the x-axis. $(x, \frac{1}{2}y)$

Dilation by factor 4 from the y-axis. $(4x, \frac{1}{2}y)$

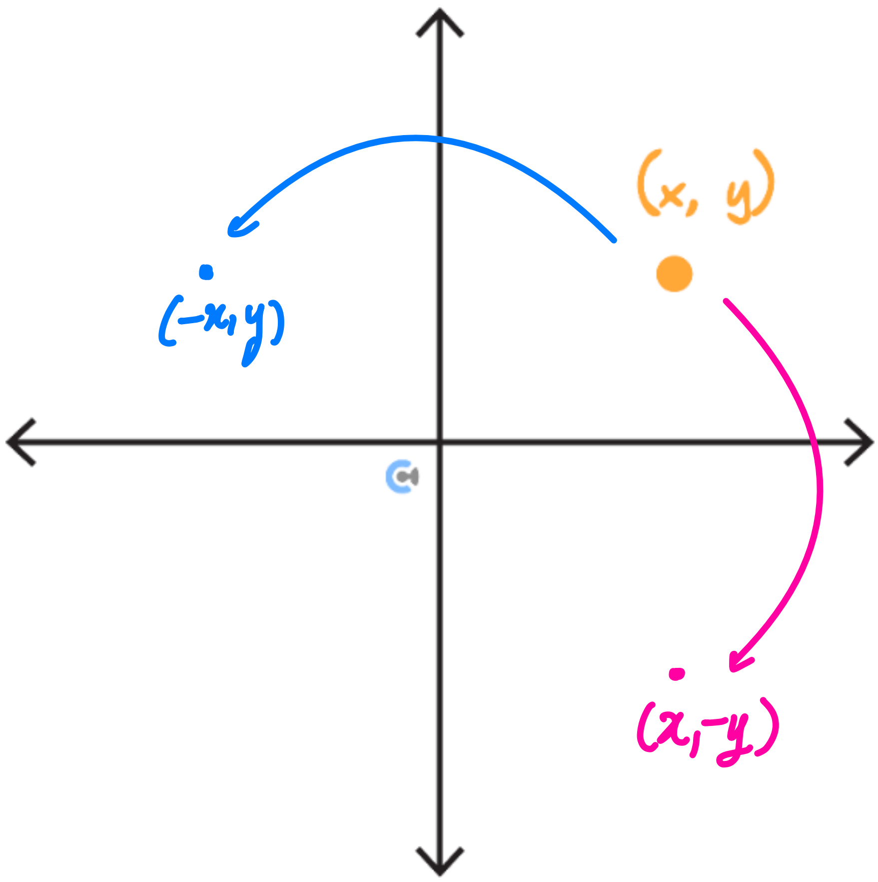
NOTE: We are applying the transformations on (x, y) not (x', y').

Space for Personal Notes

Sub-Section: Reflection

Exploration: Reflection

- Consider the point below:



- Let's plot the coordinates:
  - $P1$: Reflection in the x-axis. (x, -y)
  - $P2$: Reflection in the y-axis. (-x, y)

Reflection

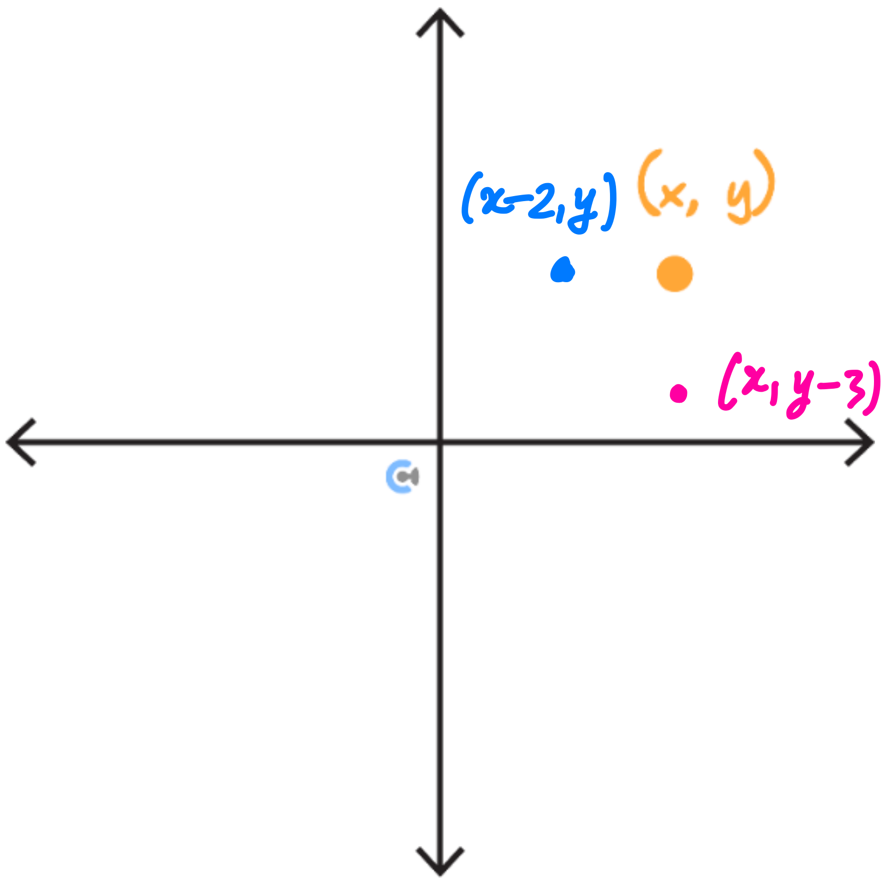
Reflection in the x-axis: $y' = -y$

Reflection in the y-axis: $x' = -x$

Sub-Section: Translation

Exploration: Translation

- Consider the point below:



- Let's plot the coordinates (ignore the scale):
  - $P1$: Translation by 2 units in the negative direction of the x-axis. (x-2, y)
  - $P2$: Translation by 3 units in the negative direction of the y-axis. (x, y-3)

Translation

Translation by c units in the positive direction of the x-axis: $x' = x + c$

Translation by d units in the positive direction of the y-axis: $y' = y + d$

Question 4

Find the image (x', y') after applying the following transformations to (x, y).

Translation by 3 units in the positive direction of the x-axis. $(x+3, y)$

Translation by 2 units in the negative direction of the y-axis. $(x+3, y-2)$

Key Takeaways

- The transformed point is called the image and is denoted by (x', y').
- The dilation factor is multiplied by the original coordinates.
- Reflection makes the original coordinates the negative of their original values.
- Translation adds a unit to the original coordinates.

Section B: Transformation of Points

Sub-Section: Basic Transformation of Points

Let's try to apply all types of transformations to a point!

Question 5 Walkthrough.

Find the image (x', y') after applying the following transformations to (x, y).

Dilation by a factor 2 from the x-axis. $(x, 2y)$

Dilation by a factor 4 from the y-axis. $(4x, 2y)$

Reflection in the x-axis. $(4x, -2y)$

Translation by 2 units in the negative direction of the x-axis. $(4x-2, -2y)$

Translation by 3 units in the positive direction of the y-axis. $(4x-2, -2y+3)$

Question 6

Find the image (x', y') after applying the following transformations to (x, y).

Translation by 4 units in the positive direction of the x-axis. $(x+4, y)$

Translation by 3 units in the negative direction of the y-axis. $(x+4, y-3)$

Dilation by a factor of $\frac{1}{5}$ from the x-axis. $(x+4, \frac{1}{5}(y-3))$

Dilation by a factor of 2 from the y-axis. $(2(x+4), \frac{1}{5}(y-3))$

Reflection in the x-axis.

$(2(x+4), -\frac{1}{5}(y-3))$ //

NOTE: Order Matters.

Question 7 Extension.

Find the image (x', y') after applying the following transformations to (x, y).

Translation by a units in the negative direction of the x-axis. $(x-a, y)$

Translation by b units in the positive direction of the y-axis. $(x-a, y+b)$

Dilation by a factor c from the x-axis. $(x-a, c(y+b))$

Dilation by a factor $\frac{3}{d}$ from the y-axis. $\left(\frac{3}{d}(x-a), c(y+b)\right)$

Reflection in the x-axis. $\left(\frac{3}{d}(x-a), -c(y+b)\right)$

Space for Personal Notes

Sub-Section: The Order of Transformations

Discussion: From the previous question, what happens when the translation is applied first?

Brackets $\Rightarrow$ inside first

$x' = 2x + 4$
D 1st T 2nd

$x' = 2(x+2)$
D 2nd T 1st

What is the order of transformations the same as?

The Order of Transformation

Order = BODMAS Order

Question 8 Walkthrough.

1. Dil 3 from y
2. 6 right

Consider the point (x, y) which was transformed into a point $(3x + 6, y)$ by the transformation T.

Jennifer thinks the transformation was:

$3(x+2)$

1. 2 right
2. Dil 3 from y

"Translation 6 units in the positive direction of the x-axis and dilation by a factor of 3 from the y-axis."

Meanwhile, David thinks the transformation was:

"Dilation by a factor of 3 from the y-axis and translation 6 units in the positive direction of the x-axis."

Who is correct? And why?

1. 5 left
2. Dil 2 from y

Question 9

Consider the point (x, y) was transformed into a point $(2(x-5), y)$ by the transformation T.

Mary thinks the transformation was:

> "Translation 5 units in the negative direction of the x-axis and dilation by a factor of 2 from the y-axis."

Meanwhile, Sam thinks the transformation was:

> "Dilation by a factor of 2 from the y-axis and translation 5 units in the negative direction of the x-axis."

Who is correct? And why?

1. Dil 2a from y
2. 6a right

Question 10 Extension.

Consider the point (x, y) was transformed into a point $(2ax + 6a, y)$ by the transformation T.

$2a(x+3)$

1. 3 right
2. Dil 2a from y

Jennifer thinks the transformation was:

"A translation by 3 units in the positive direction of the x-axis, followed by a dilation by a factor $2a$ from the y-axis."

Meanwhile, David thinks the transformation was:

"A dilation by a factor $2a$ from the y-axis, followed by a translation by $3a$ units in the positive direction of the x-axis."

Who is correct? And why?

Discussion: If the order is the same as the **BODMAS** order, how do we change the order of transformations?

Translation First:

Factorised Form

$x' = 2(x+2)$

Dilation First:

Expanded Form

$x' = 2x+4$

Question 11 Walkthrough.

$(x, \frac{1}{2}y)$ $(x, \frac{1}{2}y+3)$

The series of transformations, "a dilation by a factor $\frac{1}{2}$ from the x-axis and a translation by 3 units up" yields the same result as the series of transformations, "a translation by a units up and a dilation by a factor b from the x-axis." Find the values of a and b.

$(x, y+a)$ $(x, b(y+a))$

$$\frac{1}{2}y+3 = b(y+a)$$

$$\frac{1}{2}(y+6) = b(y+a)$$

$$\therefore a=6, b=\frac{1}{2}$$

Question 12

$(4x, y)$ $(-4x, y)$

The series of transformations, "a dilation by a factor 4 from the y-axis, a reflection in the y-axis and a translation by 8 units left" yields the same result as the series of transformations, "a translation by c units right, a reflection in the y-axis and a dilation by a factor d from the y-axis." Find the values of c and d.

$(x+c, y)$ $(-(x+c), y)$

$(-d(x+c), y)$

$(-4x-8, y)$

$-4x-8 = -d(x+c)$

$-4(x+2) = -d(x+c)$

$\therefore c=2, d=4$

Question 13 Extension.

$(2x, y)$ $(-2x, y)$

The series of transformations, "a dilation by a factor 2 from the y-axis, a reflection in the y-axis, a dilation by a factor 2 from the x-axis, a translation by 4 units left and a translation by 6 units down", yields the same result as the series of transformations, "a translation by c units right, a reflection in the y-axis, a dilation by a factor d from the y-axis, a translation k units down, and a dilation by a factor m from the x-axis." Find the values of c, d, k and m.

$(-2x, 2y) \Rightarrow (-2x-4, 2y) \Rightarrow (-2x-4, 2y-6)$

$(x+c, y) \Rightarrow (-(x+c), y) \Rightarrow (-d(x+c), y) \Rightarrow (-d(x+c), y-k)$

$\Downarrow$

$(-d(x+c), m(y-k))$

$-2x-4 = -d(x+c)$

$-2(x+2) = -d(x+c)$

$\therefore c=2, d=2$

$2y-6 = m(y-k)$

$2(y-3) = m(y-k)$

$\therefore k=3, m=2$

NOTE: Dilation factors don't change!

Space for Personal Notes

Sub-Section: Interpreting the Transformation of Points

Active Recall: Order of Transformation

Order = BODMAS Order

Question 14 Walkthrough.

Dilate by factor __ from y/x axis

Consider the transformation which maps:

$$x' = 2x + 4$$

$$y' = -3(y - 1)$$

Translate __ units L/R/U/D

Say it fully

a. State the transformation in DRT (Dilation, Reflection, Translation) order.

1. Dil 2 from y
2. Dil 3 from x
3. Reflection in x
4. 4 right
5. 3 up

Expanded:

$x' = 2x+4$

$y' = -3y+3$

b. State the transformation in the translation first order.

1. 2 right
2. 1 down
3. Reflection in x
4. Dil 2 from y
5. Dil 3 from x

Factorised:

$x' = 2(x+2)$

$y' = -3(y-1)$

NOTE: Expanding or factorising changes the order of transformation.

Question 15

Consider the transformation which maps:

$$x' = 3x + 6$$

$$y' = -2(y + 2)$$

a. State the transformation in DRT (Dilation, Reflection, Translation) order.

1. Dil 3 from y
2. Dil 2 from x
3. Reflection in x
4. 6 right
5. 4 down

$x' = 3x + 6$

$y' = -2y - 4$

b. State the transformation in the translation first order.

1. 2 right
2. 2 up
3. Reflection in x
4. Dil 3 from y
5. Dil 2 from x

$x' = 3(x + 2)$

$y' = -2(y + 2)$

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Discussion: Could the order of x and y transformations change?

Yes!

Key Takeaways

- ☑ Transformations should be interpreted when x' and y' are isolated.
- ☑ The order of transformation follows the BODMAS order.
- ☑ To change the order of transformations, we either factorise or expand.

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Section C: Transformation of Functions

Sub-Section: Applying Transformations to Functions

Let's now work with Functions!

Transformation of Functions

- The aim is to get rid of the old variables, x and y, and have the new variables, x' and y', instead.

NEW FUNCTION $f\left(\frac{x'-4}{2}\right)$

$$y = f(x) \rightarrow y' = f(x')$$

Make OLD var subject

- Steps:

1. Transform the points.
2. Make x and y the subjects.
3. Substitute them into the function.

$x = \frac{x'-4}{2}$

$x' = 2x+4$

1. Dil 2 from y
2. 4 right

Question 16 Walkthrough.

Apply the transformations given below to $y = x^2$.

Reflect in the y-axis. $(-x, y)$

Translate 1 unit to the right. $(-x+1, y)$

Dilate by a factor of 2 from the y-axis. $(2(-x+1), y)$

① Find x', y'

$x' = -2x+2$

$y' = y$

② Make OLD x/y the subject

$x = \frac{x'-2}{-2}$

$y = y'$

③ REPLACE OLD w/ NEW

$y = x^2$

$\Downarrow \quad \Downarrow$

$y' = \left(\frac{x'-2}{-2}\right)^2$

③.⑤ Remove dashes

④ Make y subject

$\therefore y = \left(\frac{x-2}{-2}\right)^2$

Your turn!

Active Recall: Transformation of Functions

- The aim is to get rid of the old variables, x and y, and have the new variables, x' and y', instead.

$$y = f(x) \rightarrow y' = f(x')$$

- Steps:

1. Transform the points.
2. Make x and y the subject.
3. Replace them into the function.

Space for Personal Notes

Question 17

Apply the following transformations to the functions given:

a. $f(x) = x^2$

Dilation by factor 3 from the x-axis. $(x, 3y)$

Reflect in the y-axis. $(-x, 3y)$

Translate 3 units to the left. $(-x-3, 3y)$

Dilate by a factor of 5 from the y-axis. $(5(-x-3), 3y)$

$x' = -5x - 15$

$y' = 3y$

$x = \frac{x'+15}{-5}$

$y = \frac{y'}{3}$

$y = x^2$

$\frac{y'}{3} = \left(\frac{x'+15}{-5}\right)^2$

$\therefore y = 3\left(\frac{x+15}{-5}\right)^2$

b. $f(x) = \sqrt{x}$

Dilate by a factor of $\frac{1}{4}$ from the y-axis. $(\frac{1}{4}x, y)$

Dilate by a factor of 3 from the x-axis. $(\frac{1}{4}x, 3y)$

Translate 4 units to the left. $(\frac{1}{4}x - 4, 3y)$

Translate 1 unit up. $(\frac{1}{4}x - 4, 3y + 1)$

Reflect in the y-axis. $(-(\frac{1}{4}x - 4), 3y + 1)$

$x' = -\frac{1}{4}x + 4$

$y' = 3y + 1$

$x = -4(x' - 4)$

$y = \frac{y'-1}{3}$

$y = \sqrt{x}$

$\frac{y'-1}{3} = \sqrt{-4(x'-4)}$

$\therefore y = 3\sqrt{-4x+16} + 1$

$y = 6\sqrt{4-x} + 1$

Question 18 Extension.

Apply the following transformations to $y = 2^x$.

Translation by 2 units to the right. $(x+2, y)$

Reflection in the y-axis. $(-(x+2), y)$

Dilation by a factor 3 from the y-axis. $(-3(x+2), y)$

Translation by 3 units up. $(-3(x+2), y+3)$

A dilation by a factor 2 from the x-axis. $(-3(x+2), 2(y+3))$

A reflection in the x-axis. $(-3(x+2), -2(y+3))$

$\therefore x' = -3x-6 \Rightarrow x = \frac{x'+6}{-3}$

$\therefore y' = -2y-6 \Rightarrow y = \frac{y'+6}{-2}$

$y = 2^x$

$\Downarrow$

$\Rightarrow \frac{y'+6}{-2} = 2^{\frac{x'+6}{-3}}$

$\therefore y = -2 \cdot 2^{\frac{x+6}{-3}} - 6$

$y = -2^{\frac{x+3}{-3}} - 6$

Sub-Section: Finding the Applied Transformations

Now let's go backwards!

Reverse Engineering

➤ Steps:

1. Add the dashes (') back to the transformed function.
2. Make $f(\ )$ the subject.
3. Equate the LHS of the original and transformed functions to the RHS of the original and transformed functions.
4. Make x' and y' the subjects and interpret the transformations.

Question 19 Walkthrough.

Find the transformations required for $y = x^2$ to be transformed to $y = 3\left(\frac{x+3}{2}\right)^2 + 5$.

① Put ' back in

$y = x^2$

$y' = 3\left(\frac{x'+3}{2}\right)^2 + 5$

② Move y transformations left

$\frac{y'-5}{3} = \left(\frac{x'+3}{2}\right)^2$

③ Equate OLD w/ NEW x & y

$y = \frac{y'-5}{3}$ $\quad x = \frac{x'+3}{2}$

$y' = 3y+5$ $\quad x' = 2x-3$

④ Make x', y' the subject & READ

1. Dil 3 from x
2. 5 up
3. Dil 2 from y
4. 3 left

Your turn!

Active Recall: Steps for reverse engineering

- **Steps**:

1. Add the dashes (′) back to the transformed function.
2. Make $f(\ )$ the Subject.
3. Equate the LHS of the original and transformed functions to the RHS of the original and transformed functions.
4. Make x' & y' the subjects and interpret the transformations.

Space for Personal Notes

Question 20

State a series of transformations (in order) that allow $f(x)$ to be transformed into $g(x)$.

a. $f(x) = 2(x+1)^2 + 3$ and $g(x) = 6(x-4)^2 - 3$.

$y = 2(x+1)^2 + 3$ $\quad$ $y' = 6(x'-4)^2 - 3$

$\frac{y-3}{2} = (x+1)^2$ $\quad$ $\frac{y'+3}{6} = (x'-4)^2$

$\frac{y-3}{2} = \frac{y'+3}{6}$ $\quad$ $x+1 = x'-4$

$\therefore y' = 3y - 12$ $\quad$ $x' = x+5$

1. Dil 3 from x
2. 12 down

1. 5 right

b. $f(x) = 3(x-1)^2$ and $g(x) = \frac{1}{2}(2x+3)^2 + 1$.

$y = 3(x-1)^2$ $\quad$ $y' = \frac{1}{2}(2x'+3)^2 + 1$

$\frac{y}{3} = (x-1)^2$ $\quad$ $2(y'-1) = (2x'+3)^2$

$\frac{y}{3} = 2y' - 2$ $\quad$ $x-1 = 2x'+3$

$y' = \frac{1}{6}y + 1$ $\quad$ $x' = \frac{1}{2}x - 2$

1. Dil $\frac{1}{6}$ from x
2. 1 up

1. Dil $\frac{1}{2}$ from y
2. 2 left

Question 21 Extension.

Find a sequence of transformations required for $y = 2(x-3)^2 + 4$ to be transformed to $y = -x^2 - 4x - 9$.

$y = -(x^2 + 4x) - 9$ CTS

$= -((x+2)^2 - 4) - 9$

$= -(x+2)^2 - 5$

$y = 2(x-3)^2 + 4$

$y' = -(x'+2)^2 - 5$

$\frac{y-4}{2} = (x-3)^2$

$-(y'+5) = (x'+2)^2$

$\frac{y-4}{2} = -(y'+5)$

$x - 3 = x' + 2$

$\therefore y' = -\frac{1}{2}y + 2 - 5$

$\therefore x' = x - 5$

$y' = -\frac{1}{2}y - 3$

1. 5 left

1. Dil $\frac{1}{2}$ from x
2. Reflection in x
3. 3 down

Key Takeaways

- We transform the coordinates first, then transform the function.
- To transform the function, replace its old variables with the new ones.
- To find the transformations, simply equate LHS with RHS after separating the transformations of x and y.

Contour Checklist

☐ Learning Objective: [2.4.1] – Applying x' and y' Notation to Find Transformed Points, Find the Interpretation of Transformations and Altered Order of Transformations

Key Takeaways

- ☐ The transformed point is called the image and is denoted by (x', y').
- ☐ The dilation factor is multiplied to the original coordinate.
- ☐ Reflection makes the original coordinates the negative of their original values.
- ☐ Translation adds a unit to the original coordinate.
- ☐ Transformations should be interpreted when x' & y' are isolated.
- ☐ The order of transformation follows the BODMAS order.
- ☐ To change the order of transformations, we either factorise or expand.

☐ Learning Objective: [2.4.2] – Find Transformed Functions

Key Takeaways

- ☐ To transform the function, replace its old variables with the new one.

Learning Objective: [2.4.3] - Find Transformations From Transformed Function (Reverse Engineering)

Key Takeaways

- To find the transformations, simply equate the LHS & RHS after separating the transformations of x and y.

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