Website: contoureducation.com.au | Phone: 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ½
Transformations [2.4]
Homework Solutions

Homework Outline:

Compulsory Questions	Pg 2 – Pg 19
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Section A: Compulsory Questions

Sub-Section [2.4.1]: Applying x' and y' Notation to Find Transformed Points, Find Interpretation of Transformations and Altered Order of Transformations

Question 1

Consider the following transformations on the plane:

- S a dilation by a factor 2 from the x-axis.
- T a translation 3 units to the right and 2 units down.
- W a reflection in the x-axis.

Find the image, (x', y'), of the point (x, y) and the transformation:

a. S

$$(x', y') = (x, 2y)$$

b. T

$$(x', y') = (x + 3, y - 2)$$

c. S then T.

$$(x, 2y) \mapsto (x + 3, 2y - 2) = (x', y')$$

d. T then W then S.

$$(x+3, y-2) \mapsto (x+3, 2-y) \mapsto (2x+6, 2-y) = (x', y')$$

Question 2

A transformation T is applied to points on the plane such that the image is given by $(x', y') = (2x+4, -y+2)$.

a. Describe T in words where dilations and reflections occur before translations.

- A dilation by factor 2 from the y-axis
- A reflection in the x-axis
- A translation 4 units to the right
- A translation 2 units up

b. Describe T in words where translations occur before reflections and dilations.

$(x', y') = (2(x+2), -(y-2))$

- A translation 2 units to the right
- A translation 2 units down
- A reflection in the x-axis
- A dilation by factor 2 from the y-axis.

Question 3

The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is described by the following sequence of transformations.

- A dilation by factor 2 from the x-axis.
- A dilation by factor 3 from the y-axis.
- A reflection in the x-axis.
- A translation 2 units left.
- A translation 6 units down.

a. Let (x', y') be the image of (x, y) under T. Find (x', y').

$$(x, y) \mapsto (3x, 2y) \mapsto (3x, -2y) \mapsto (3x - 2, -2y - 6) = (x', y')$$

b. Describe in words, the transformations T, in the order of translations, reflections, and dilations.

$$(x', y') = \left(3\left(x - \frac{2}{3}\right), -2(y + 3)\right).$$

- A translation 2/3 units left
- A translation 3 units up
- A reflection in the x-axis
- A dilation by factor 3 from the y-axis
- A dilation by factor 2 from the x-axis.

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Sub-Section [2.4.2]: Find Transformed Functions

Question 4

a. Find the rule for the image of $f(x) = x^2$ under the transformations:

- A dilation by factor 2 from the x-axis.
- A translation 1 unit up.

$$y = 2x^2 + 1$$

b. Find the rule for the image of $f(x) = \sqrt{x}$ under the transformations:

- A dilation by factor 4 from the y-axis.
- A translation 1 unit down.

$x' = 4x$ and $y' = y - 1$. The image is

$$y = \sqrt{\frac{1}{4}x} - 1 = \frac{\sqrt{x}}{2} - 1$$

c. Find the rule for the image of $f(x) = \frac{1}{x}$ under the transformations:

- A dilation by factor 2 from the x-axis.
- A translation 1 unit up and 3 units to the left.

$\frac{1}{x} \mapsto \frac{2}{x} \mapsto \frac{2}{x+3} + 1.$

The image has rule $y = \frac{2}{x+3} + 1$

Question 5

a. Find the rule for the image of $f(x) = 2x^2 + 4$ under the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (2x + 1, -y + 2)$.

$x' = 2x + 1 \implies x = \frac{x' - 1}{2}.$

$y' = -y + 2 \implies y = 2 - y'.$

Thus for the image we have

$$2 - y = f\left(\frac{x-1}{2}\right)$$

$$y = 2 - \left(2\left(\frac{x-1}{2}\right)^2 + 4\right)$$

$$y = 2 - \frac{1}{2}(x-1)^2 - 4$$

$$y = -2 - \frac{1}{2}(x-1)^2$$

b. Find the rule for the image of $f(x) = \frac{3}{x-3}$ under the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x,y) = (x+1, -2y+2)$.

$x' = x + 1 \implies x = x' - 1$ and $y' = -2y + 2 \implies y = \frac{2 - y'}{2}$.
So for the image we have

$$\frac{2-y}{2} = \frac{3}{x-4}$$
$$y = 2 - \frac{6}{x-4}$$

c. Find the rule for the image of $f(x) = \sqrt{2x-4} + 3$ under the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x,y) = (-2x+1, 2y+3)$.

$x' = -2x + 1 \implies x = \frac{1 - x'}{2}$ and $y' = 2y + 3 \implies y = \frac{y' - 3}{2}$.
So for the image we have

$$\frac{y-3}{2} = \sqrt{1 - x - 4} + 3$$
$$y = 2\sqrt{-3 - x} + 9$$

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Question 6

a. Find the rule for the image of $f(x) = 2(x-1)^2 + 3$ under the transformations:

- A dilation by factor 2 from the x-axis.
- A translation 3 units to the left.
- A translation 1 unit up.
- A reflection in the x-axis.

Sequentially applying the transformations yields

$$f(x) \mapsto 4(x-1)^2 + 6 \mapsto 4(x+2)^2 + 6 \mapsto 4(x+2)^2 + 7 \mapsto -4(x+2)^2 - 7$$

so the image is $y = -4(x+2)^2 - 7$

b. Find the rule for the image of $f(x) = \frac{1}{x-1}$ under the transformations:

- A dilation by factor 4 from the x-axis.
- A dilation by factor $\frac{1}{2}$ from the y-axis.
- A reflection in the x-axis.
- A translation 2 units right.

Sequentially applying the transformations yields

$$f(x) \mapsto \frac{4}{x-1} \mapsto \frac{4}{2x-1} \mapsto -\frac{4}{2x-1} \mapsto -\frac{4}{2x-5}$$

so the image is $y = -\dfrac{4}{2x-5}$

c. Find the rule for the image of $f(x) = \sqrt{2x+6} - 4$ under the transformations:

- A translation 1 unit up.
- A translation 4 units to the right.
- A reflection in the y-axis.
- A dilation by factor 2 from the x-axis.

Sequentially applying the transformations yields

$$f(x) \mapsto \sqrt{2x+6} - 3 \mapsto \sqrt{2x-2} - 3 \mapsto \sqrt{-2-2x} - 3 \mapsto 2\sqrt{-2-2x} - 6$$

so the image is $y = 2\sqrt{-2-2x} - 6$

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Sub-Section [2.4.3]: Find Transformations from Transformed Function

Question 7

a. Let $f(x) = x^2$ and $g(x) = 4x^2 + 1$.

Describe a sequence of transformations that maps the graph of f onto the graph of g.

$f(x) \mapsto 4x^2 \mapsto 4x^2 + 1 = g(x)$.
Therefore transformations are

- A dilation by factor 4 from the x-axis (or $1/2$ from the y-axis)
- A translation 1 unit up

b. Let $f(x) = \sqrt{x}$ and $g(x) = 2\sqrt{x+1} - 3$.

Describe a sequence of transformations that maps the graph of f onto the graph of g.

$f(x) \mapsto 2\sqrt{x} \mapsto 2\sqrt{x+1} \mapsto 2\sqrt{x+1} - 3$.
Therefore transformations are

- A dilation by factor 2 from the x-axis
- A translation 1 left
- A translation 3 units down

c. Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{3}{x+2}$.

Describe a sequence of transformations that maps the graph of f onto the graph of g.

$f(x) \mapsto \frac{3}{x} \mapsto \frac{3}{x+2}$.
Therefore transformations are

- A dilation by factor 3 from the x-axis
- A translation 2 units left.

Question 8

a. Let $f(x) = x^2$ and $g(x) = 4(x-2)^2 + 3$.

Describe a sequence of transformations that maps the graph of f onto the graph of g.

$f(x) \mapsto 4x^2 \mapsto 4(x-2)^2 \mapsto 4(x-2)^2 + 3$.
Therefore transformations are

- A dilation by factor 4 from the x-axis
- A translation 2 units right
- A translation 3 units up

b. Let $f(x) = \sqrt{2x}$ and $g(x) = 2\sqrt{4x-2}$.

Describe a sequence of transformations that maps the graph of f onto the graph of g.

$f(x) \mapsto \sqrt{4x} \mapsto \sqrt{4(x-1/2)} \mapsto 2\sqrt{4x-2}$.
Therefore transformations are

- A dilation by factor $1/2$ from the y-axis
- A translation $1/2$ units to the right
- A dilation by factor 2 from the x-axis.

c. Let $f(x) = \frac{6}{x-1}$ and $g(x) = \frac{3}{x+2} + 1$.

Describe a sequence of transformations that maps the graph of f onto the graph of g.

$f(x) \mapsto \dfrac{3}{x-1} \mapsto \dfrac{3}{x+2} \mapsto \dfrac{3}{x+2} + 1.$
Therefore transformations are

- A dilation by factor $1/2$ from the x-axis
- A translation 3 units to the left
- A translation 1 unit up

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Question 9

a. Let $f(x) = 2\sqrt{x+1}$ and $g(x) = 5\sqrt{5-3x} + 4$.

Describe a sequence of transformations that maps the graph of f onto the graph of g.

$f(x) \mapsto 5\sqrt{x+1} \mapsto 5\sqrt{x+5} \mapsto 5\sqrt{5-3x} \mapsto 5\sqrt{5-3x}+4.$
Therefore transformations are

- A dilation by factor $5/2$ from the x-axis
- A translation 4 units to the left
- A dilation by factor $1/3$ from the y-axis
- A reflection in the y-axis
- A translation 4 units up

b. Let $f(x) = 2(x-3)^2 + 2$ and $g(x) = x^2 + 4x + 7$.

Describe a sequence of transformations that maps the graph of f onto the graph of g.

Note that $g(x) = (x+2)^2 + 3$. Then
$f(x) \mapsto (x-3)^2 + 1 \mapsto (x+2)^2 + 1 \mapsto (x+2) + 3$.
Therefore transformations are

- A dilation by factor $1/2$ from the x-axis
- A translation 5 units left
- A translation 2 units up

c. Let $f(x) = \frac{3}{x^2} + 1$ and $g(x) = -\frac{6}{(2x-3)^2} + 4$.

Describe a sequence of transformations that maps the graph of f onto the graph of g.

$x = 2x' - 3 \implies x' = \frac{x+3}{2}$

$$f(x) \mapsto \frac{3}{(2x)^2} + 1 \mapsto \frac{3}{(2(x-3/2))^2} + 1 \mapsto -\frac{3}{(2x-3)^2} - 1 \mapsto -\frac{6}{(2x-3)^2} - 2 \mapsto -\frac{6}{(2x-3)^2} + 4$$

Therefore transformations are

- A dilation by factor $1/2$ from the y-axis
- A translation $3/2$ units right
- A reflection in the x-axis
- A dilation by factor 2 from the x-axis
- A translation 6 units up.

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Sub-Section: The 'Final Boss'

Question 10

Consider the function $f(x) = x^2 - 4x + 7$ and the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (3x - 6, -2y + 2)$.

a. Use words to describe the transformation T with:

i. Dilations and reflections before translations.

- A dilation by factor 3 from the y-axis
- A dilation by factor 2 from the x-axis
- A reflection in the x-axis
- A translation 6 units left and 2 units up.

ii. Translations before reflections and dilations.

Note that $T(x, y) = (3(x - 2), -2(y - 1))$. Thus

- A translation 2 units left and 1 unit down
- A reflection in the x-axis
- A dilation by factor 3 from the y-axis
- A dilation by factor 2 from the x-axis.

b. Write $f(x)$ in turning point form.

$$f(x) = (x-2)^2 + 3$$

c. Find the rule for the image of $f(x)$ under T.

Sequentially apply the transformations in the order from **part a.ii**

$$f(x) \mapsto x^2 + 3 \mapsto x^2 + 2 \mapsto -x^2 - 2 \mapsto -\frac{1}{9}x^2 - 2 \mapsto -\frac{2}{9}x^2 - 4$$

the rule for the image is $y = -\frac{2}{9}x^2 - 4$

d. Determine a sequence of transformations that map $f(x)$ to $g(x) = 2x^2 - 16x + 28$.

Note that $g(x) = 2(x-4)^2 - 4$. Then the sequence

$$g(x) \mapsto (x-4)^2 - 2 \mapsto (x-2)^2 - 2 \mapsto (x-2)^2 + 3 = f(x)$$

is according to the transformations

- A dilation by factor $1/2$ from the x-axis
- A translation 2 units to the left
- A translation 5 units up.

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Section B: Supplementary Questions

Sub-Section [2.4.1]: Applying x' and y' Notation to Find Transformed Points, Find Interpretation of Transformations and Altered Order of Transformation

Question 11

Find the coordinates of the image point for the following:

a. The point $(2, 3)$ undergoes a dilation by a factor of 6 from the y-axis, a reflection in the x-axis, followed by a translation 1 unit up.

$x' = 6x, y' = -y + 1$
$(x', y') = (6(2), -3 + 1)$
$(x', y') = (12, -2)$

b. The point $(1,5)$ undergoes a translation 2 units left, a dilation by a factor of $\frac{1}{4}$ from the y-axis, a translation 3 units up, followed by a reflection in the x-axis.

$x' = \frac{1}{4}(x - 2), y' = -(y + 3)$
$(x', y') = \left(\frac{1}{4}(1 - 2), -(5 + 3)\right)$
$(x', y') = \left(-\frac{1}{4}, -8\right)$

c. The point $(-4,2)$ is dilated by a factor of 3 from the x-axis, translated 1 unit right, reflected in the x-axis, reflected in the y-axis, dilated by a factor of 2 from the y-axis, and then translated 5 units down.

$x' = -2(x + 1), y' = -3y - 5$
$(x', y') = (-2(-4 + 1), -3(2) - 5)$
$(x', y') = (6, -11)$

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Question 12

Consider the sequence of transformations:

- A dilation by a factor of $\frac{1}{2}$ from the y-axis.
- A reflection in the x-axis.
- A dilation by a factor of 6 from the x-axis.
- A translation 4 units down.
- A translation 1 unit right.
- A translation 9 units up.

a. Rewrite the transformations in the order of a dilation, a translation, a dilation, a reflection, and then a translation.

A dilation by a factor of $\frac{1}{2}$ from the y-axis
A translation 1 unit right
A dilation by a factor of 6 from the x-axis
A reflection in the x-axis
A translation 5 units up

b. Express the transformations as a sequence of two translations, followed by two dilations and a reflection.

$x' = \frac{1}{2}x + 1, y' = -6y + 5$

$x' = \frac{1}{2}(x + 2), y' = -6\left(y - \frac{5}{6}\right)$

A translation 2 units right

A translation $\frac{5}{6}$ units left

A dilation by a factor of $\frac{1}{2}$ from the y-axis

A dilation by a factor of 6 from the x-axis

A reflection in the x-axis

c. Express the transformations in the order of a dilation, a translation, a dilation, a translation, and then a reflection.

$x' = \frac{1}{2}(x + 2), y' = -(6y - 5)$

A dilation by a factor of $\frac{1}{2}$ from the y-axis

A translation 2 units right

A dilation by a factor of 6 from the x-axis

A translation 5 units down

A reflection in the x-axis

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Question 13

The transformation T is defined as $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (5 - 2x, 6y + 1)$.

a. Evaluate $T(-3,8)$.

$T(-3,8) = (5 - 2(3), 6(8) + 1)$
$T(-3,8) = (11,49)$

b. Find the pre-image of $(7, -35)$ under the transformation T.

$T(x, y) = (7, -35)$
$(5 - 2x, 6y + 1) = (7, -35)$
$x = -1, y = -6$
$(-1, -6)$

c. Express T as a sequence of two translations, two dilations, and a reflection.

$x' = -2\left(x - \frac{5}{2}\right), y' = 6\left(y + \frac{1}{6}\right)$
Translation $\frac{5}{2}$ units left
Translation $\frac{1}{6}$ units up
Dilation by a factor of 2 from the y-axis
Dilation by a factor of 6 from the x-axis
Reflection in the y-axis

d. Identify a sequence of transformations that maps the point $(-3,8)$ to the image of $(-3,8)$ under T and also maps the point $(1,-2)$ to the point $(23,-1)$.

$x' = ax + b, y' = cy + d$

$(-3,8) \to (11,49)$: $11 = a(-3) + b, 8 = c(49) + d$

$(1,-2) \to (23,-1)$: $23 = a(1) + b, -1 = c(-2) + d$

Solve equations simultaneously: $a = 3,\ b = 20,\ c = 5,\ d = 9$

$x' = 3x + 20, y' = 5y + 9$

A dilation by a factor of 3 from the y-axis

A translation 20 units right

A dilation by a factor of 5 from the x-axis

A translation 9 units up

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Question 14

a. Consider the transformation T described by:

- A translation 2 units left.
- A dilation by a factor of 3 from the x-axis.
- A dilation by a factor of $\frac{1}{4}$ from the y-axis.
- A reflection in the x-axis.
- A translation 1 unit up.
- A reflection in the line $y = x$.
- A translation 4 units right.

i. Apply T to the point $(5,2)$.

$x' = -3y + 1 + 4, y' = \frac{1}{4}(x - 2)$

$(x', y') = \left(-3(2) + 1 + 4, \frac{1}{4}(5 - 2)\right)$

$(x', y') = \left(-1, \frac{3}{4}\right)$

ii. Express T as a sequence of 2 dilations followed by 2 reflections, and then 2 translations.

A reflection in $y = x$ will swap x and y values

$x' = -3y + 5, y' = \frac{1}{4}x - \frac{1}{2}$

A dilation by a factor of 3 from the x-axis

A dilation by a factor of $\frac{1}{4}$ from the y-axis

A reflection in $y = x$

A reflection in the y-axis

A translation 5 units right

A translation $\frac{1}{2}$ units left

b. Consider the transformation S described by:

- A dilation by a factor of 2 from the x-axis.
- A reflection in the y-axis.
- A dilation by a factor of $\frac{1}{3}$ from the y-axis.
- A reflection in the line $y = 4$.
- A translation 5 units down.
- A translation 1 unit right.

i. S can also be defined $S : \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (ax + b, cy + d)$. Find the values of a, b, c, and d.

A reflection in the line $y = 4$ can be treated as a translation 4 units down, followed by a reflection in the x axis and then a translation 4 units up

$x' = -\frac{1}{3}x + 1, y' = -(2y - 4) + 4 - 5$

$x' = -\frac{1}{3}x + 1, y' = -2y + 3$

$a = -\frac{1}{3}, b = 1, c = -2, d = 3$

ii. Hence, evaluate $S(-2,4)$.

$S(-2,4) = \left(-\frac{1}{3}(-2) + 1, -2(4) + 3\right)$

$S(-2,4) = \left(\frac{5}{3}, -5\right)$

c. A point (x, y) undergoes the transformations T followed by S. Find the image point.

$$T(x, y) = \left(-3y + 5, \frac{1}{4}x - \frac{1}{2}\right)$$
$$S\left(-3y + 5, \frac{1}{4}x - \frac{1}{2}\right) = \left(-\frac{1}{3}(-3y + 5) + 1, -2\left(\frac{1}{4}x - \frac{1}{2}\right) + 3\right)$$
$$S\left(-3y + 5, \frac{1}{4}x - \frac{1}{2}\right) = \left(y - \frac{2}{3}, -\frac{1}{2}x + 4\right)$$
$$(x', y') = \left(y - \frac{2}{3}, -\frac{1}{2}x + 4\right)$$

d. Given that the image point from **part c.**, is $(-4, 6)$, find the pre-image.

$$(-4,6) = \left(y - \frac{2}{3}, -\frac{1}{2}x + 4\right)$$
$$-4 = y - \frac{2}{3}, 6 = -\frac{1}{2}x + 4$$
$$y = -\frac{10}{3}, x = -4$$
$$\left(-4, -\frac{10}{3}\right)$$

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Sub-Section [2.4.2]: Find Transformed Functions

Question 15

Find the resultant function when:

a. $y = x^2$ is dilated by a factor of 2 from the y-axis, reflected in the x-axis, translated 3 units up, and translated 1 unit left.

$x' = 2x - 1, y' = -y + 3$

$x = \frac{1}{2}(x' + 1), y = -(y' - 3)$

$-(y' - 3) = \left(\frac{1}{2}(x' + 1)\right)^2$

$y' = -\frac{1}{4}(x' + 1)^2 + 3$

b. $y = \frac{1}{x}$ is reflected in the y-axis, translated 3 units up, dilated by a factor of 2 from the x-axis, dilated by a factor of $\frac{1}{4}$ from the y-axis, and translated 2 units right.

$x' = -\frac{1}{4}x + 2, y' = 2(y + 3)$

$x = -4(x' - 2), y = \frac{1}{2}(y' - 6)$

$\frac{1}{2}(y' - 6) = \frac{1}{-4(x' - 2)}$

$y' = -\frac{1}{2(x-2)} + 6$

c. $y = \sqrt{x}$ is translated 3 units down, translated 5 units right, reflected in the y-axis, dilated by a factor of 3 from the x-axis, dilated by a factor of 2 from the y-axis, and reflected in the x-axis.

$$x' = -2(x+5),\ y' = -3(y-3)$$
$$x = -\frac{1}{2}x' - 5,\ y = -\frac{1}{3}(y'-9)$$
$$-\frac{1}{3}(y'-9) = \sqrt{-\frac{1}{2}x' - 5}$$
$$y' = -3\sqrt{-\frac{1}{2}x' - 5} + 9$$

Question 16

Find the resultant function when:

a. $y = -2(x+5)^2 + 1$ is dilated by a factor of $\frac{1}{3}$ from the x-axis, translated 4 units right, translated 1 unit down, reflected in the y-axis, and dilated by a factor of 2 from the y-axis.

$$x' = -2(x+4),\ y' = \frac{1}{3}y - 1$$
$$x = -\frac{1}{2}x' - 4,\ y = 3(y'+1)$$
$$3(y'+1) = -2\left(-\frac{1}{2}x' - 4 + 5\right)^2 + 1$$
$$y' = -\frac{1}{6}(x'-2)^2 - \frac{2}{3}$$

b. $y = \frac{2}{(5-x)^2} + 7$ is reflected in the x-axis, translated 2 units up, dilated by a factor of 3 from the y-axis, reflected in the y-axis, translated 4 units right, and dilated by a factor of $\frac{1}{4}$ from the x-axis.

$x' = -3x + 4,\ y' = \frac{1}{4}(-y + 2)$

$x = -\frac{1}{3}(x' - 4),\ y = -4\left(y' - \frac{1}{2}\right)$

$-4\left(y' - \frac{1}{2}\right) = \frac{2}{5-\left(-\frac{1}{3}x'-4\right)^2} + 7$

$y' = -\frac{9}{90-2(x'+12)^2} - \frac{5}{4}$

c. $y = 4 - 2(x + 1)^3$ is translated 4 units right, dilated by a factor of 3 from the x-axis, reflected in the y-axis, translated 5 units up, reflected in the x-axis, and dilated by a factor of 2 from the y-axis.

$x' = -2(x + 4),\ y' = -(3y + 5)$

$x = -\frac{1}{2}x' - 4,\ y = -\frac{y'+5}{3}$

$-\frac{y'+5}{3} = 4 - 2\left(-\frac{1}{2}x' - 4 + 1\right)^3$

$y' = -\frac{3}{8}(x' + 6)^3 - 17$

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Question 17

Find the resultant function when:

a. $(x-2)^2 + (y+5)^2 = 9$ is dilated by a factor of 3 from the y-axis, reflected in the x-axis, translated 4 units up, translated 1 unit left, and dilated by a factor of 3 from the x-axis.

$x' = 3x - 1, y' = 3(-y + 4)$

$x = \frac{1}{3}x' + \frac{1}{3}, y = -\frac{1}{3}y' + 4$

$\left(\frac{1}{3}x' + \frac{1}{3} - 2\right)^2 + \left(-\frac{1}{3}y' + 4 + 5\right)^2 = 9$

$(x-5)^2 + (y-27)^2 = 81$

b. $y = 2x^2 + 3x - 6$ is reflected in the y-axis, dilated by a factor of 4 from the x-axis, translated 5 units down, translated 1 unit right, dilated by a factor of $\frac{1}{2}$ from the y-axis.

$x' = \frac{1}{2}(-x + 1), y' = 4y - 5$

$x = -2x' + 1, y = \frac{y'+5}{4}$

$\frac{y'+5}{4} = 2(-2x' + 1)^2 + 3(2x' + 1) - 6$

$y' = 32x' - 8x' - 9$

c. $x = -\sqrt{-y^2 + 6y + 15} + 4$ is translated 2 units down, dilated by a factor of 3 from the x-axis, reflected in the y-axis, translated 5 units left, dilated by a factor of $\frac{1}{4}$ from the y-axis, translated 5 units up, reflected in the y-axis, and dilated by a factor of 2 from the x-axis.

$$x' = -\frac{1}{4}(-x - 5), y' = 2(3(y - 2) + 5)$$

$$x = 4\left(x' - \frac{5}{4}\right), y = \frac{1}{3}y' - \frac{2}{3}$$

$$4\left(x' - \frac{5}{4}\right) = -\sqrt{-\left(\frac{1}{3}y' - \frac{2}{3}\right)^2 + 6\left(\frac{1}{3}y' - \frac{2}{3}\right) + 15} + 4$$

$$x' = -\frac{1}{12}\sqrt{y'^2 + 14y + 103} + \frac{9}{4}$$

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Question 18

a. When the graph $y = 6 - 2(x+1)^2$ undergoes the transformation T, described as:

- A translation 4 units right.
- A dilation by a factor of 2 from the y-axis.
- A translation 4 units down.
- A reflection in the y-axis.
- A dilation by a factor of 3 from the x-axis.

It is mapped onto an equation $y = a(x-h)^2 + k$, where $a, h, k \in \mathbb{R}$.

Find the values of a, h, and k.

$x' = -2(x+4),\ y' = 3(y-4)$

$x = -\frac{1}{2}x' - 4,\ y = \frac{y'+12}{3}$

$\frac{y'+12}{3} = 6 - 2\left(-\frac{1}{2}x' - 4 + 1\right)^2$

$y' = -\frac{3}{2}(x+6)^2 + 6$

$a = -\frac{3}{2}, b = -6, k = 6$

b. The graph $y = 6 - 2(x+1)^2$ can also be mapped to the same equation from **part a.**, by a sequence of 2 dilations, a reflection, and 2 translations. Describe this sequence of transformations.

$x' = -2x - 8,\ y' = 3y - 12$
A dilation by a factor of 2 from the y-axis.
A dilation by a factor of 3 from the x-axis.
A reflection in the y-axis.
A translation 8 units left.
A translation 12 units down.

c. Find the pre-image, that when undergoes the transformation T, results in the equation $y = 6 - 2(x+1)^2$.

$x' = -2x - 8,\ y' = 3(y - 4)$
$3(y-4) = 6 - 2(-2x-8)^2$
$y = -\frac{8}{3}(x+4)^2 + 6$

d. The graph of $y = 6 - 2(x+1)^2$ undergoes the transformation T, followed by a dilation by a factor of 2 from the x-axis, a reflection in the line $x = 6$, a reflection in the line $y = x$, and a translation 1 unit up. Find the image equation.

$x' = 2y, y' = -(x-6) + 6 + 1$

$y = \frac{1}{2}x', x = -y' + 13$

$\frac{1}{2}x' = -\frac{3}{2}(-y' + 13 + 6)^2 + 6$

$x' = -3(y' - 19)^2 + 6$

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Sub-Section [2.4.3]: Find Transformations from Transformed Function (Reverse Engineering)

Question 19

Find the sequence of transformations that map:

a. $y = x^2$ to $y = -3(x+1)^2 + 7$.

$\frac{y'-7}{-3} = (x'+1)^2$

$y = \frac{y'-7}{-3}, x = x' + 1$

$y' = -3y + 7, x' = x - 1$

Dilation by a factor of 3 from the x-axis

Reflection in the x-axis

Translation 7 units up

Translation 1 unit down

b. $y = \frac{1}{x}$ to $y = \frac{3}{5-2x} + 6$.

$\frac{y'-6}{3} = \frac{1}{5-2x'}$

$y = \frac{y'-6}{3}, x = 5 - 2x'$

$y' = 3y + 6, x' = -\frac{1}{2}x + \frac{5}{2}$

Dilation by a factor of 3 from the x-axis

Translation 6 units up

Dilation by a factor of $\frac{1}{2}$ from the y-axis

Reflection in the y-axis

Translation $\frac{5}{2}$ units right

c. $y = \sqrt{x}$ to $y = 1 - \frac{\sqrt{4-3x}}{2}$.

$-2(y' - 1) = \sqrt{4 - 3x'}$
$y = -2(y' - 1), x = 4 - 3x'$
$y' = -\frac{1}{2}y + 1, x' = -\frac{1}{3}x + \frac{4}{3}$
Dilation by a factor of $\frac{1}{2}$ from the x-axis
Reflection in the x-axis
Translation 1 unit up
Dilation by a factor of $\frac{1}{3}$ from the y-axis
Reflection in the y-axis
Translation $\frac{4}{3}$ units right

Question 20

Find the sequence of transformations that map:

a. $y = 4(x + 8)^3 - 5$ to $y = 5 - 2(6x - 1)^3$.

$\frac{y+5}{4} = (x + 8)^3$
$\frac{y'-5}{-2} = (6x' - 1)^3$
$\frac{y+5}{4} = \frac{y'-5}{-2}, x + 8 = 6x' - 1$
$y' = -\frac{1}{2}y + \frac{5}{2}, x' = \frac{1}{6}x + \frac{3}{2}$
Dilation by a factor of $\frac{1}{2}$ from the x-axis
Reflection in the x-axis
Translation $\frac{5}{2}$ units up
Dilation by a factor of $\frac{1}{6}$ from the y-axis
Translation $\frac{3}{2}$ units right

b. $y = 3\sqrt{16-(x+1)^2}+5$ to $y = 1-2\sqrt{16-(3x+5)^2}$.

$\frac{y-5}{3} = \sqrt{16-(x+1)^2}$

$\frac{y'-1}{-2} = \sqrt{16-(3x'+5)^2}$

$\frac{y-5}{3} = \frac{y'-1}{-2}$, $x+1 = 3x'+5$

$y' = -\frac{2}{3}y + \frac{8}{3}$, $x' = \frac{1}{3}x - \frac{4}{3}$

Dilation by a factor of $\frac{2}{3}$ from the x-axis

Reflection in the x-axis

Translation $\frac{8}{3}$ units up

Dilation by a factor of $\frac{1}{3}$ from the y-axis

Translation $\frac{4}{3}$ units left

c. $y = \frac{3}{(4-2x)^2}+7$ to $y = -\frac{6}{(x+1)^2}+5$.

$\frac{y-7}{3} = \frac{1}{(4-2x)^2}$

$\frac{y'-5}{-6} = \frac{1}{(x'+1)^2}$

$\frac{y-7}{3} = \frac{y'-5}{6}$, $4-2x = x'+1$

$y' = 2y-9$, $x' = -2x+3$

Dilation by a factor of 2 from the x-axis

Translation 9 units left

Dilation by a factor of 2 from the y-axis

Reflection in the y-axis

Translation 3 units right

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Question 21

a. The function $y = -2(3(x-1))^4 + 5$ undergoes a sequence of 2 transformations, a reflection, and 2 dilations to become the graph $y = 6(2-x)^4 - 1$.

$\frac{y-5}{-2} = (3(x-1))^4$

$\frac{y'+1}{6} = (x'-2)^4$

$\frac{y-5}{-2} = \frac{y'+1}{6}, 3(x-1) = x'-2$

$y' = -3\left(y - \frac{14}{3}\right), x' = 3(x - \frac{1}{3})$

Translation $\frac{14}{3}$ units down

Translation $\frac{1}{3}$ units left

Reflection in the x-axis

Dilation by a factor of 3 from the x-axis

Dilation by a factor of 3 from the y-axis

b. The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x,y) = (ax+b, y+c)$ maps the equation $y = 11 + 5(x+3)^2$ onto the equation $y = 20(x-6)^2 + 9$. Find the values of a, b, and c.

$\frac{y-11}{5} = (x+3)^2$

$\frac{y'-9}{5} = (2(x'-6))^2$

$\frac{y-11}{5} = \frac{y'-9}{5}, x+3 = 2(x'-6)$

$y' = y - 2, x' = \frac{1}{2}x + \frac{15}{2}$

$a = \frac{1}{2}, b = \frac{15}{2}, c = -2$

c. The graph $y = \frac{\sqrt{6x-4}}{3} + 2$ is mapped onto $y = 5 - 2\sqrt{-1-x}$ by a sequence of 2 dilations and 2 reflections, followed by a translation.

$\frac{3(y-2)}{2} = \sqrt{\frac{3}{2}x - 1}$

$\frac{y'-5}{-2} = \sqrt{-x' - 1}$

$\frac{3(y-2)}{2} = \frac{y'-5}{-2}, \frac{2}{3}x - 1 = -x' - 1$

$y' = -3y + 11, x' = -\frac{2}{3}x$

Dilation by a factor of 3 from the x-axis
Dilation by a factor of $\frac{2}{3}$ from the y-axis
Reflection in the x-axis
Reflection in the y-axis
Translation 11 units up

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Question 22

Find the sequence of transformations that map:

a. $y = x^2 - 4x + 6$ onto $y = -2x^2 + 10x - 7$.

$y = (x-2)^2 + 2$
$y - 2 = (x-2)^2$
$y' = -2\left(x' - \frac{5}{2}\right)^2 + \frac{11}{2}$
$\frac{y' - \frac{39}{2}}{-2} = \left(x' + \frac{5}{2}\right)^2$
$y - 2 = \frac{y' - \frac{39}{2}}{-2}, x - 2 = x' + \frac{5}{2}$
$y' = -2y + \frac{47}{2}, x' = x - \frac{1}{2}$
Dilation by a factor of 2 from the x-axis
Reflection in the x-axis
Translation $\frac{47}{2}$ units up
Translation $\frac{1}{2}$ units left

b. $y = 2\sqrt{(x+4)^2 + 1} - 5$ onto $y = 3 - \sqrt{(2x-6)^2 + 9}$.

$\frac{y+5}{2} = \sqrt{(x+4)^2 + 1}$
$y' = 3 - 3\sqrt{\left(\frac{2}{3}x' - 2\right)^2 + 1}$
$\frac{y'-3}{-3} = \sqrt{\left(\frac{2}{3}x' - 2\right)^2 + 1}$
$\frac{y+5}{2} = \frac{y'-3}{-3}, x + 4 = \frac{2}{3}x' - 2$
$y' = -\frac{3}{2}y - \frac{9}{2}, x' = \frac{3}{2}x + 9$
Dilation by a factor of $\frac{3}{2}$ from the x-axis
Reflection in the x-axis
Translation $\frac{9}{2}$ units down
Dilation by a factor of $\frac{3}{2}$ from the y-axis
Translation 9 units right

c. $f : [-4, \infty) \to \mathbb{R}, f(x) = -x^2 - 8x + 9$ onto $g : (-\infty, 5] \to \mathbb{R}, g(x) = 2x^2 - 20x + 13$.

$y = -(x+4)^2 + 25$

$-(y - 25) = (x+4)^2$

$y' = 2(-(x' - 5))^2 - 37$

$\frac{y'+37}{2} = (-(x'-5))^2$

$-(y-25) = \frac{y'+37}{2}, x + 4 = -(x' - 5)$

$y' = -2y + 13, x' = -x + 1$

Dilation by a factor of 2 from the x-axis

Reflection in the x-axis

Translation 13 units up

Reflection in the y-axis

Translation 1 units right

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