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VCE Mathematical Methods ½
Functions & Relations II [2.2]
Workbook

Outline:

Domain and Range Pg 02-16 ➤ Set Notation ➤ Interval Notation ➤ Maximal Domain ➤ Range ➤ Functional Notation **Hybrid (Piecewise) Functions** Pg 17-21	**Inverse Functions** Pg 22-36 ➤ Basics of Inverses ➤ Swapping x and y ➤ Symmetry Around $y = x$ ➤ Validity of Inverse Function ➤ Intersection Between Inverses

Section A: Domain and Range

Sub-Section: Set Notation

Let's have a look at set notations!

Set Operators

- Intersection: "AND".

$$A \cap B = \textit{What values are in set } A \textit{ AND in set } B.$$

- Union: "OR".

$$A \cup B = \textit{What values are in set } A \textit{ OR in set } B.$$

- Set difference: "Except".

$$A \setminus B = \textit{What values are in set } A \textit{ except those also in set } B.$$

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Question 1

For the sets given below, find:

$$A = \{0, 2, 3, 5, 6, 11\} \text{ and } B = \{0, 1, 2, 3, 5, 7, 9, 10\}$$

a. $A \cap B =$

$A = \{0, 2, 3, 5, 6, 11\}$ $B = \{0, 1, 2, 3, 5, 7, 9, 10\}$

$A \cap B = \{0, 2, 3, 5\}$

b. $A \cup B =$

$A = \{0, 2, 3, 5, 6, 11\}$ $B = \{0, 1, 2, 3, 5, 7, 9, 10\}$

$A \cup B = \{0, 1, 2, 3, 5, 6, 7, 9, 10, 11\}$

c. $A \backslash B =$

$A = \{0, 2, 3, 5, 6, 11\}$ $B = \{0, 1, 2, 3, 5, 7, 9, 10\}$

$A \backslash B = \{6, 11\}$

d. $B \backslash A =$

$A = \{0, 2, 3, 5, 6, 11\}$ $B = \{0, 1, 2, 3, 5, 7, 9, 10\}$

$B \backslash A = \{1, 7, 9, 10\}$

Sub-Section: Interval Notation

Now interval notation!

Interval Notation

- Parentheses (non-inclusive):

$$x \in (a, b) \Rightarrow a < x < b$$

- Square brackets [inclusive]:

$$x \in [a, b] \Rightarrow a \le x \le b$$

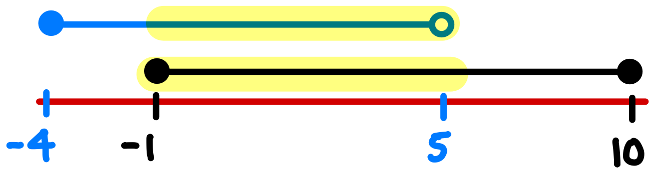
Question 2 Walkthrough.

Simplify the following set.

$$A = [-1, 10] \text{ and } B = [-4, 5)$$

a. Find $A \cap B$.

$= [-1, 5)$ //



b. Find $A \cup B$.

$= [-4, 10]$ //

NOTE: Use **number lines** to find the intersection and union of sets.

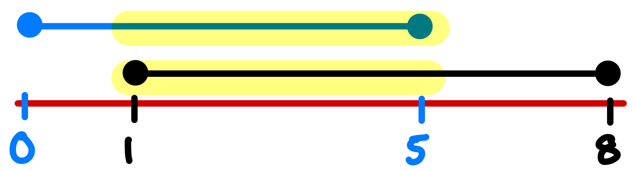
Now your turn!

Question 3

Find the following sets:

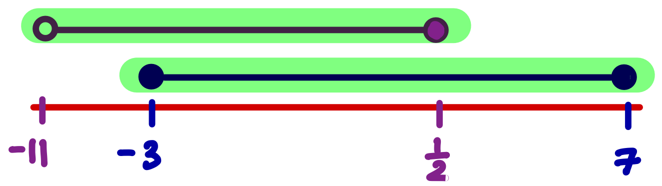
a. $[0,5] \cap [1,8]$

$= [1,5]$ //



b. $[-3,7] \cup \left(-11,\frac{1}{2}\right]$

$= (-11,7]$ //



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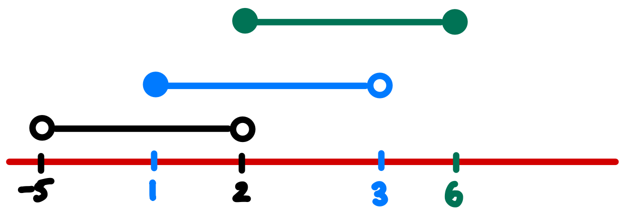
~~**Question 4 Extension:**~~

Find the following set.

$$[1, 3) \cap [2, 6] \cup (-5, 2)$$



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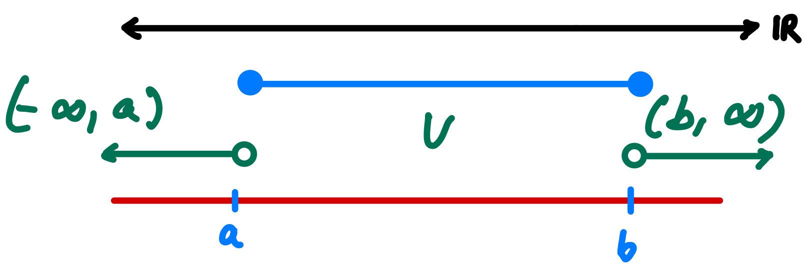
this one!

Discussion: What is $\mathbb{R}\backslash[a, b]$ equal to? Is it $(-\infty, a) \cup (b, \infty)$ or $(-\infty, a] \cup [b, \infty)$?



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Sub-Section: Maximal Domain

What is a maximal domain?

Maximal Domain

- The maximal domain is the largest possible domain for a rule without committing a mathematical crime.
- In Methods, we need to consider 3 important rules:

$$\sqrt{z}, \qquad z \geq 0$$

$$\log(z), \qquad z > 0$$

$$\frac{1}{z}, \qquad z \neq 0$$

NOTE: We will consider $\log$ in depth later throughout the year!

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Question 5 Walkthrough.

Find the maximal domain of each of the following functions.

a. $f(x) = \underbrace{\sqrt{x-3}}_{\geq 0}$

$x - 3 \geq 0$

$\therefore x \geq 3$

b. $h(x) = \log_2 \underbrace{(x+5)}_{>0}$

$x + 5 > 0$

$\therefore x > -5$

c. $h(x) = \frac{1}{\underbrace{x-4}_{\neq 0}}$ $\Rightarrow$ $x - 4 \neq 0$

$\therefore x \neq 4$ $\Rightarrow$ $\therefore x \in \mathbb{R} \setminus \{4\}$

Your turn!

Question 6

Find the maximal domain of the following functions.

a. $f(x) = \sqrt{-x-6} - 5$

≥ 0

$-x-6 \geq 0$

$-x \geq 6$

$x \leq -6$

b. $h(x) = -\log_2(x+10)$

> 0

$x+10 > 0$

$\boxed{\therefore x > -10}$

c. $\frac{1}{x^2-25}$

$\neq 0$

$x^2 - 25 \neq 0$

$x^2 \neq 25$

$\therefore x \neq \pm 5$

$\therefore x \in \mathbb{R} \setminus \{-5, 5\}$

Now harder ones!

Question 7

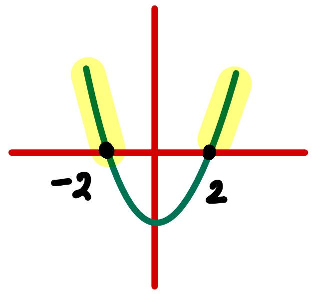
Find the maximal domain of the following functions.

a. $f(x) = \sqrt{\underbrace{x^2 - 4}_{\geq 0}} - 5$

$x^2 - 4 \geq 0$

$x^2 \geq 4$

$x \geq 2$ or $x \leq -2$

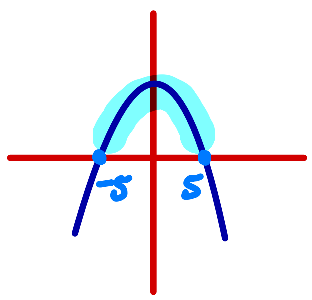


b. $h(x) = -\log_2 (\underbrace{25 - x^2}_{>0})$

$25 - x^2 > 0$

$x^2 < 25$

$-5 < x < 5$



NOTE: Always sketch the function when solving inequalities for many to one functions.

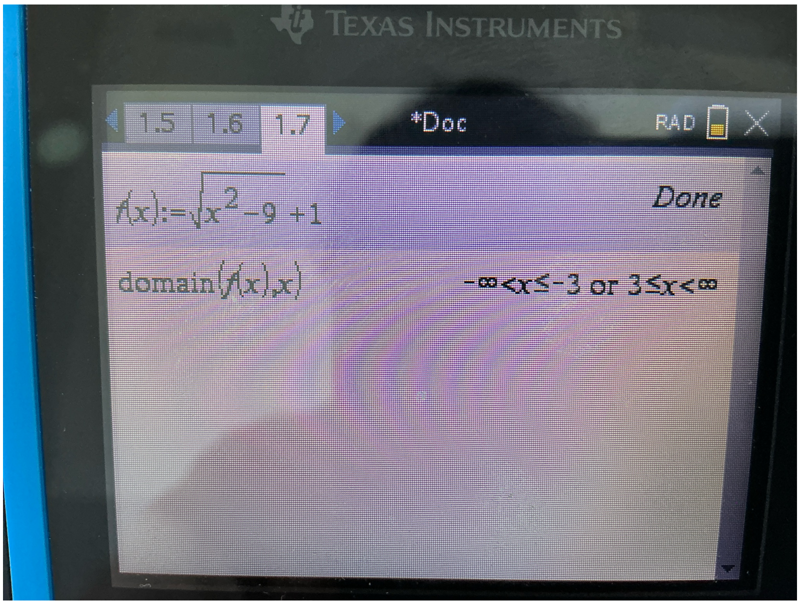
Calculator Commands

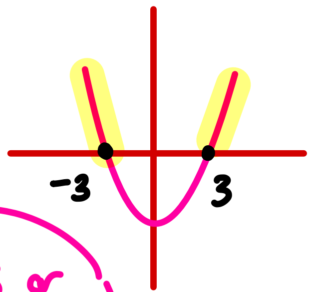
Mathematica	TI-Nspire	Casio Classpad
`FunctionDomain[func, x]`	Type up domain (or find it under the book button). domain(func,x)	Sketch the function and analyse.

Question 8 Tech Active.

Find the maximal domain of the following function.

$$f(x) = \sqrt{x^2 - 9} + 1$$





$x \leq -3$ or $x \geq 3$

Sub-Section: Range

Now the range!

Range

- The range is the possible values for the output of a function.

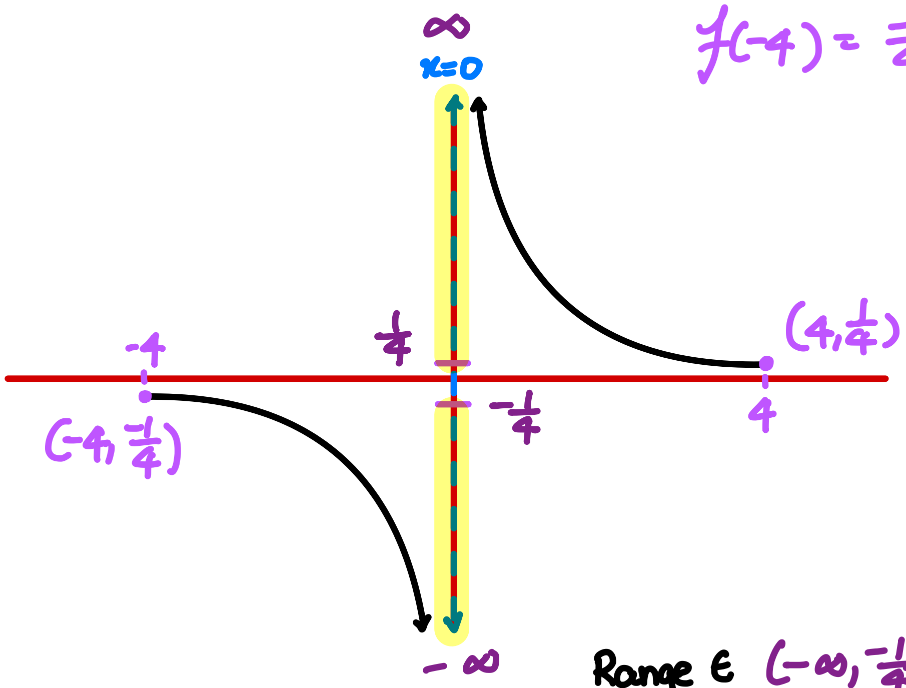
Question 9 Walkthrough.

Find the range of the following function:

$$f: [-4, 4]\backslash\{0\} \to \mathbb{R}, f(x) = \frac{1}{x}$$

$f(4) = \frac{1}{4}$

$f(-4) = \frac{-1}{4}$



Range $\in \left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$

TIP: Always **sketch** the function!

Question 10

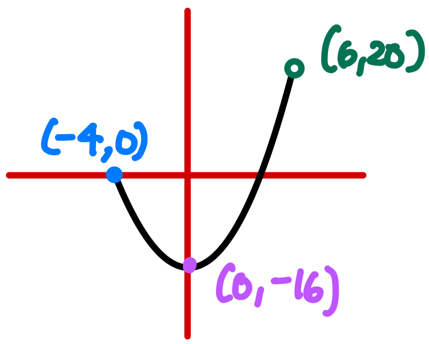
Find the range of the following functions.

a. $f: [-4, 6) \to R, f(x) = x^2 - 16$

$f(-4) = 16 - 16 = 0$

$f(6) = 36 - 16 = 20$

Range $\in [-16, 20)$



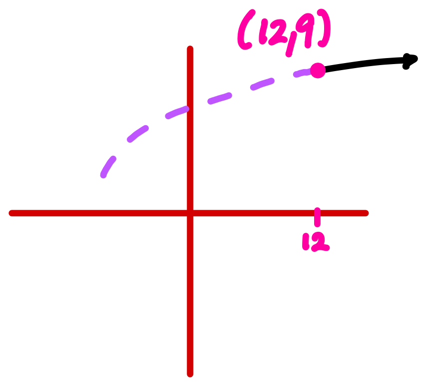
b. $f: [12, \infty) \to R, f(x) = 2\sqrt{x+4} + 1$

$f(12) = 2\sqrt{16} + 1$

$= 8 + 1$

$= 9$

Range $\in [9, \infty)$



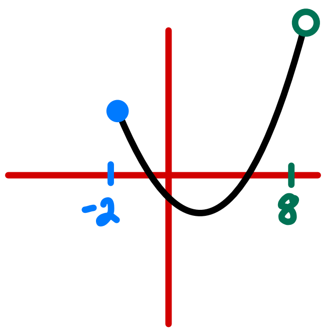
Question 11 Extension.

$(a+b)^2 = a^2 + 2ab + b^2$

Find the range of the following function.

$$f: [-2, 8) \to R, f(x) = \frac{1}{2}x^2 - 2x - 2$$

[-4,14)



Sub-Section: Functional Notation

How do we represent a function?

Functional Notation

Name:

$$f: Domain \rightarrow Codomain, f(x) = Rule$$

- Codomain is simply all the values the function works within.
- Codomain is **not** the same as range.

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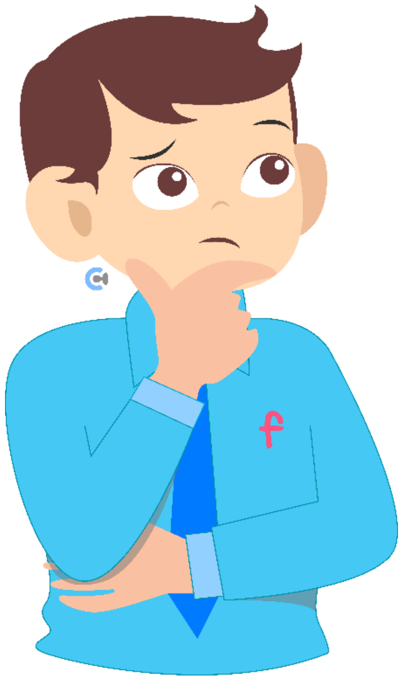
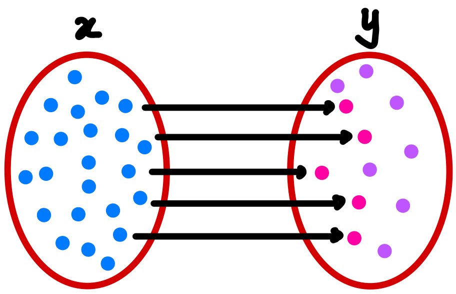
<u>Analogy</u>: **Functional notation is a "business card" for functions.**

- A function f wants to make a business card for themself.



- They decide to put their name, working hours, company associated and their role.

$$\textit{Name}: \textit{Working Hour} \rightarrow \textit{Company}, \textit{Role}$$



- Their name is simply f.
- Their working hours are their "domain".
- Their company is the "CoDomain".
- Their role is the rule!

$$f: \textit{Domain} \rightarrow \textit{Codomain}, f(x) = \textit{Rule}$$

• ⇒ Domain

• ⇒ Range

y ⇒ Codomain

- Now, does f have to make everything in their company?

No

- Hence, using this analogy, would his range (their output) be the same as the codomain (company)?

Range $\subseteq$ Codomain

Subset

"within"

Question 12

Consider the following function, written in functional notation:

$$f: [-1, 4] \to \mathbb{R}, f(x) = x + 5$$

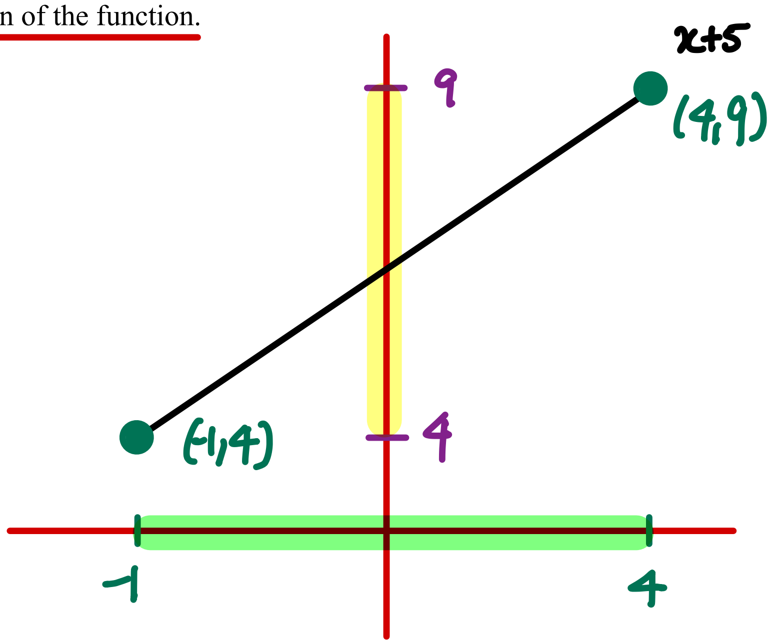
Identify the name, domain, range, and the equation of the function.

Name: f

Domain: $[-1, 4]$

Range: $[4, 9]$

Equation: $f(x) = x+5$



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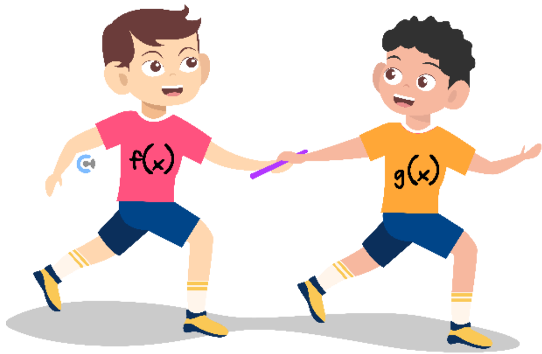
Section B: Hybrid (Piecewise) Functions

Analogy: Hybrid functions are like a relay race.

- Imagine the functions $f(x)$ and $g(x)$ participating in a relay race as part of the same team.



- $f(x)$ is running for $x < 4$ and $g(x)$ is running for $x \geq 4$.
- For $x = 5$ who do we look at?

$g(x)$

- This is how hybrid functions work!

Piecewise (Hybrid) Functions

- Series of functions.

$$h(x) = \begin{cases} f(x), & Domain_1 \\ g(x), & Domain_2 \end{cases}$$

- $Domain_1$ and $Domain_2$ represent the x values for which the two functions are defined.
- The two domains do not have to join!

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Question 13 Walkthrough.

Consider the hybrid function f.

$$f(x) = \begin{cases} x^2 - 5, & x \geq 0 \\ x + 4, & x < 0 \end{cases}$$

$x^2 - 5 = 0$

$x^2 = 5$

$x = \pm\sqrt{5} \approx 2.2$

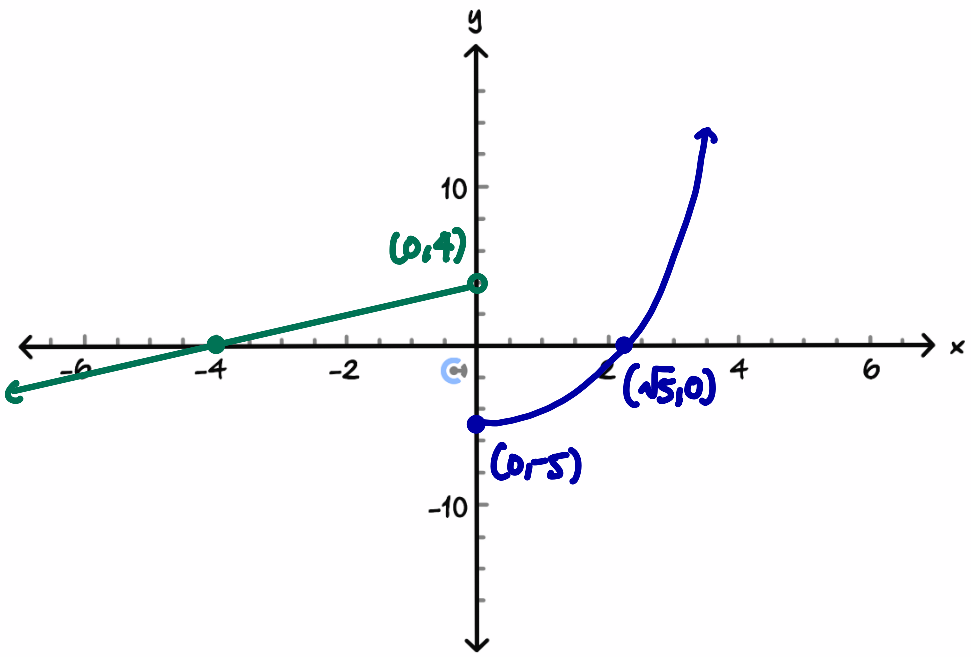
a. Find $f(-2)$.

$f(-2) = 2$

b. Find $f(5)$.

$f(5) = 5^2 - 5 = 25 - 5$
$= 20$

c. Graph $y = f(x)$.



Question 14

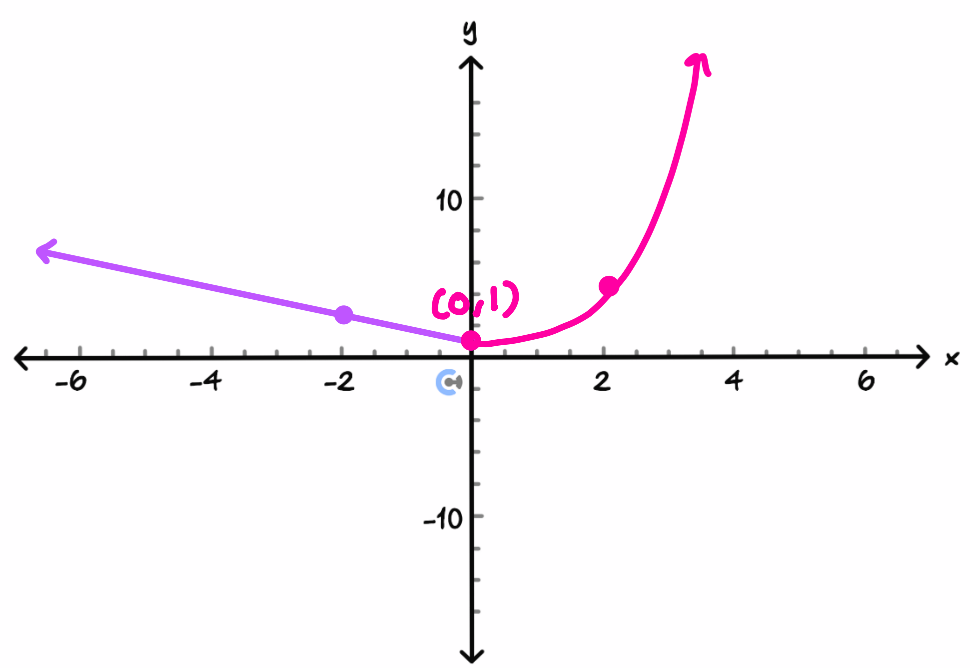
Consider the hybrid function g.

$f(2) = 5$

$f(-2) = 3$

$$g(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ 1 - x, & x < 0 \end{cases}$$

a. Graph $y = g(x)$.



b. Find the range of $g(x)$.

Range $\in [1, \infty)$

Question 15 (CAS Active)

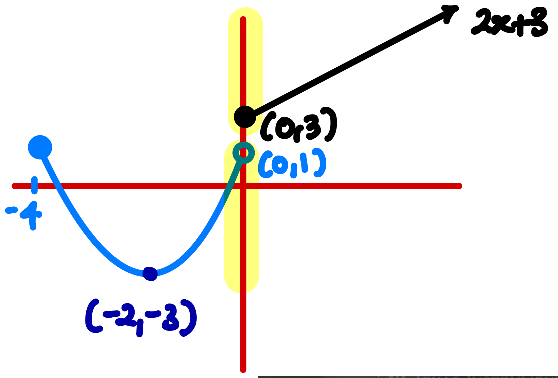
$g(-4) = 16-16+1$
$= 1$

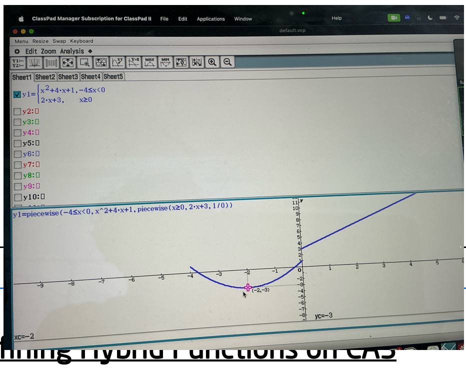
Consider the hybrid function g.

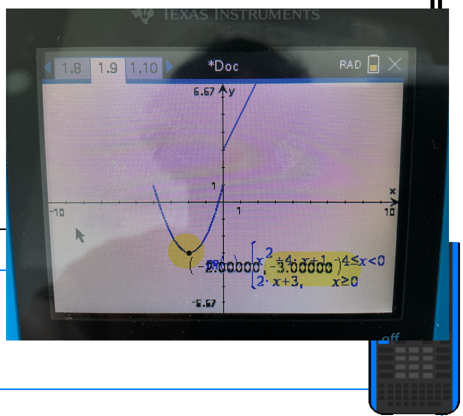
$$g(x) = \begin{cases} x^2 + 4x + 1, & -4 \le x < 0 \\ 2x + 3, & x \ge 0 \end{cases}$$

Find the range of $g(x)$.

Range $\in [-3, 1] \cup [3, \infty)$







Defining Hybrid Functions on CAS

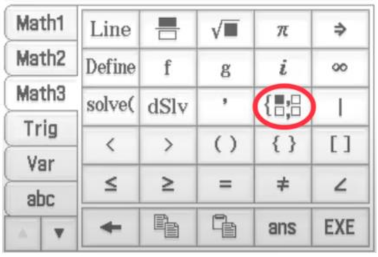
- **Mathematica**
  - "Esc PW" and Control Enter to create cells.

$$\begin{cases} \texttt{func1} & \texttt{dom1} \\ \texttt{func2} & \texttt{dom2} \end{cases}$$

- **TI-Nspire**

$$\begin{cases} func\,1, dom\,1 \\ func\,2, dom\,2 \end{cases}$$

- **Casio Classpad**



Question 16

Consider the hybrid function g.

$$g(x) = \begin{cases} x^3 + 6x - 5, & x < 1 \\ x + 4, & x \geq 1 \end{cases}$$

a. Evaluate $g(-2)$.

$g(-2) = -8 - 12 - 5$

$= -25$

b. Evaluate $g(3)$.

$g(3) = 7$

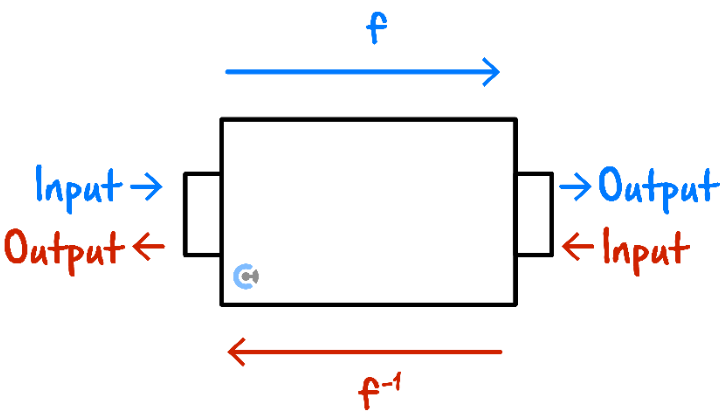
Section C: Inverse Functions

Sub-Section: Basics of Inverses

What does "inverse" mean?

Inverse Relation

- **Definition**: Inverse is a relation which does the opposite.



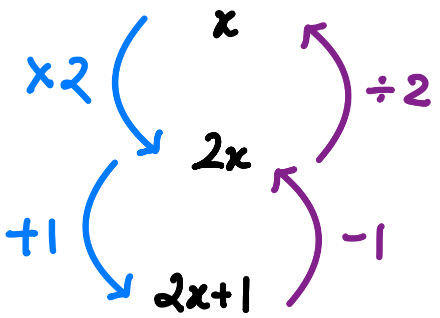
Discussion: What would be the inverse of $f(x) = x + 2$?

$f^{-1}(x) = x - 2$

Question 17

Find the inverse of $f(x) = 2x + 1$.

$f^{-1}(x) = \dfrac{x-1}{2}$

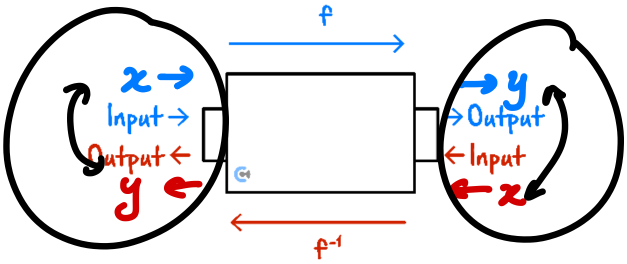


Sub-Section: Swapping x and y

Is there a better way of solving for an inverse relation?

Solving For an Inverse Relation

- Swap x and y.



Question 18

Find the inverse of $f(x) = 2x + 1$ by swapping x and y.

Let $y = f(x)$:

Swap x & y:

$x = 2y + 1$

$x - 1 = 2y$

$y = \frac{x-1}{2}$

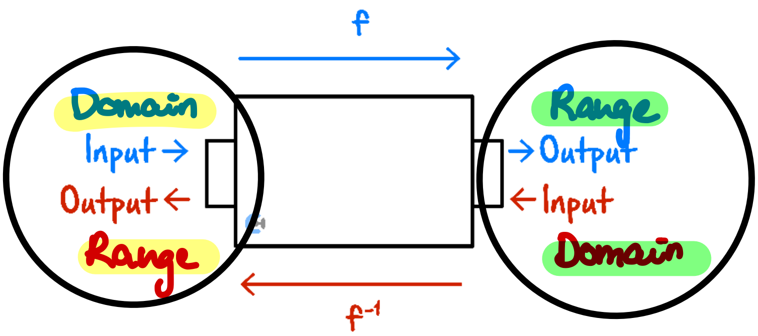
$\therefore f^{-1}(x) = \frac{x-1}{2}$

NOTE: $f(x) = y$.

Discussion: Hence, what would happen to the domain and range of the function when we find its inverse?

Swap Dom & Range

Domain and Range of Inverse Functions



$$Dom\ f^{-1} = \underline{\text{Ran } f}$$

$$Ran\ f^{-1} = \underline{\text{Dom } f}$$

Question 19 Walkthrough.

$x+2 \geq 0 \Rightarrow x \geq -2$

Consider the function $f(x) = \underbrace{\sqrt{x+2}}_{\geq 0} - 1$ defined for its maximal domain.

a. Find the rule for the inverse function.

Let $y = f(x)$:

Swap x & y:

$x = \sqrt{y+2} - 1$

$x + 1 = \sqrt{y+2}$

$(x+1)^2 = y+2$

$y = (x+1)^2 - 2$

$$\boxed{\therefore f^{-1}(x) = (x+1)^2 - 2}$$

b. State the domain and range of inverse function.

$\sqrt{x+2} - 1$

$[0, \infty) - 1$

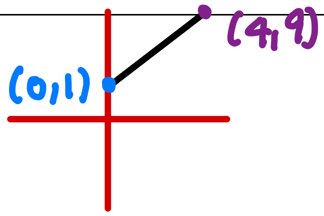
$= [-1, \infty)$

$Dom\ f^{-1} = Ran\ f = [-1, \infty)$

$Ran\ f^{-1} = Dom\ f = [-2, \infty)$

Question 20

Consider the function $f: [0, 4] \to R, f(x) = 2x + 1$.



a. Find the rule for the inverse function.

Let $y = f(x)$:

Swap x & y:

$x = 2y + 1$

$\therefore y = \frac{x-1}{2}$

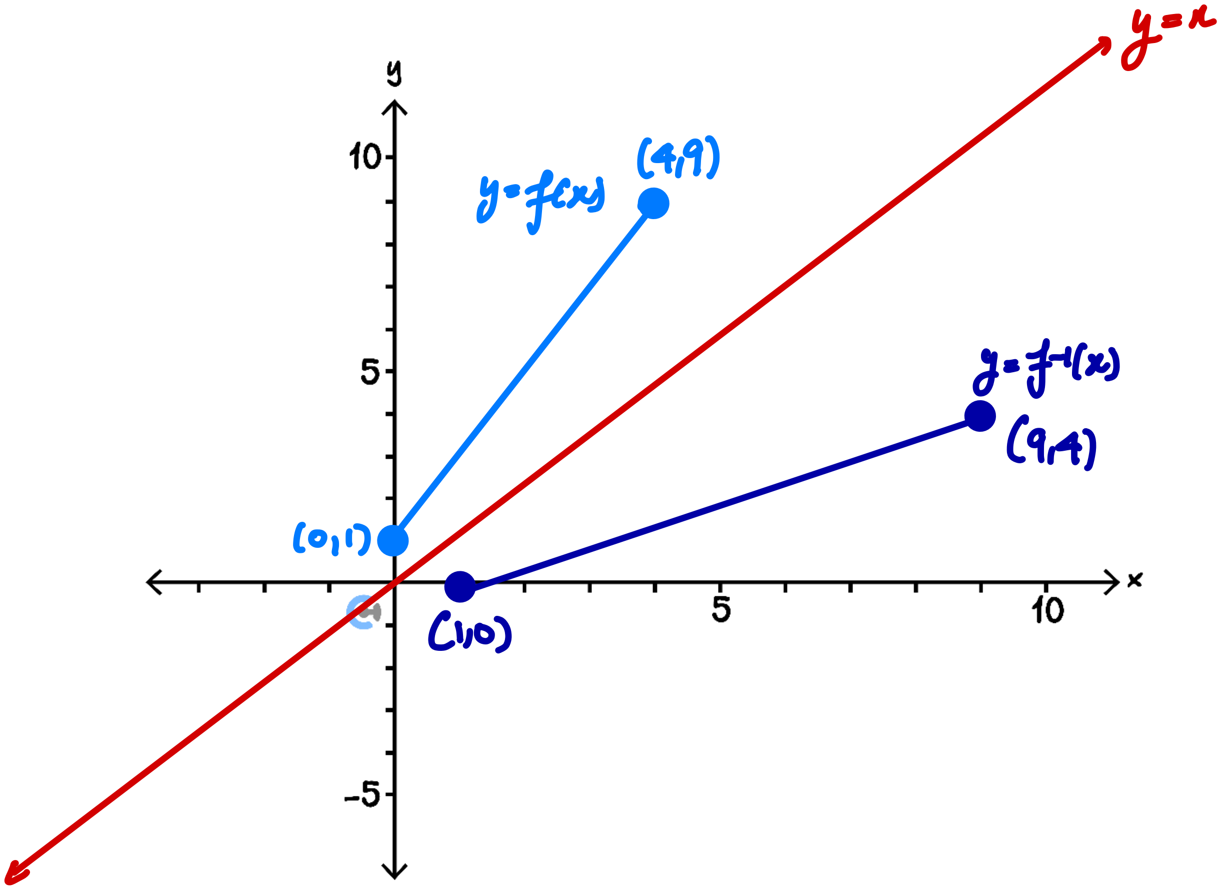
$\therefore f^{-1}(x) = \frac{x-1}{2}$

b. State the domain and range of inverse function.

$\text{Dom } f^{-1} = \text{Ran } f = [1, 9]$

$\text{Ran } f^{-1} = \text{Dom } f = [0, 4]$

c. Sketch the $f(x)$ and $f^{-1}(x)$ on the axis below.



Question 21 Extension.

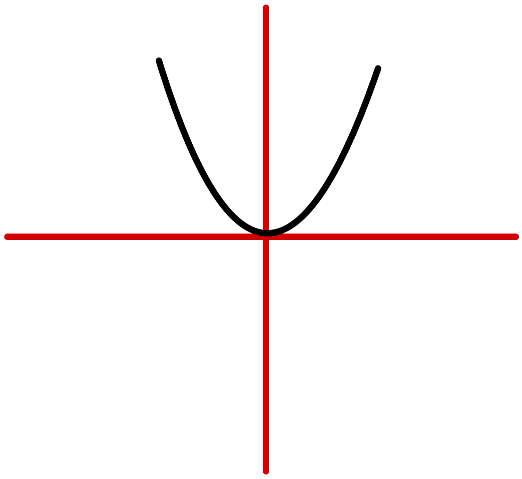
$\therefore f(x) =$

Consider the function $f: (-\infty, 2] \to R, f(x) = \frac{1}{2}x^2 - 2x + 4$.

a. Find the rule for the inverse function.

Let $y = f(x)$:

Swap x & y:



b. State the domain and range of inverse function.

$\text{Dom } f^{-1} = \text{Ran } f =$

$\text{Ran } f^{-1} = \text{Dom } f =$

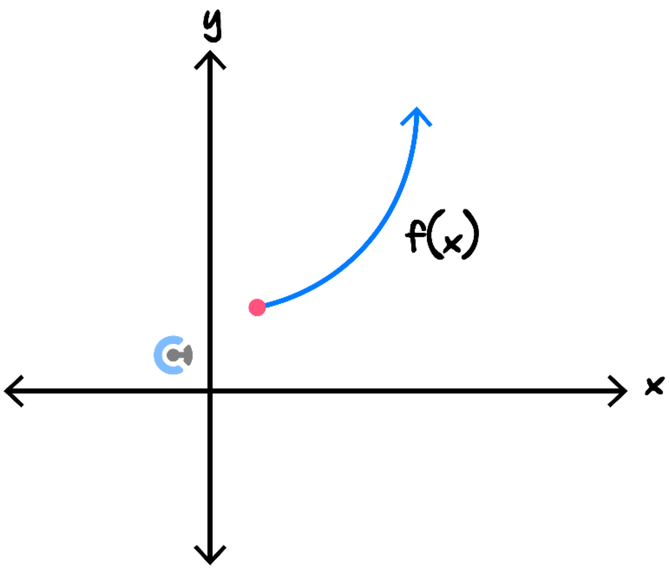
<u>Discussion:</u> In the previous question, which line were the two inverses symmetrical to?

<u>Sub-Section</u>: Symmetry Around $y = x$

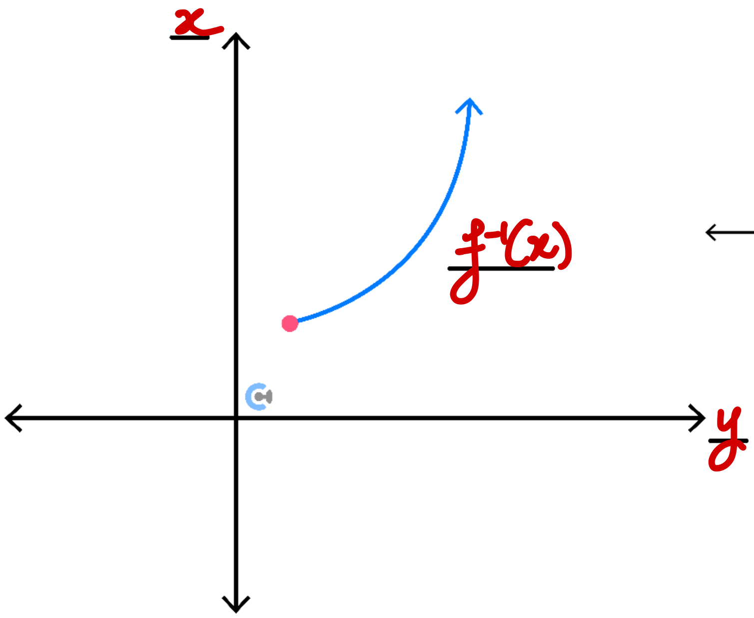
Why does this happen?

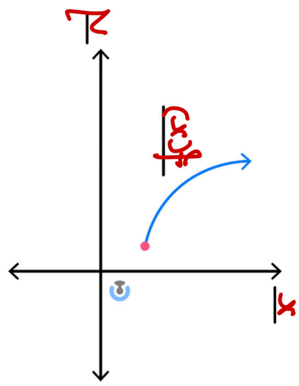
<u>Exploration</u>: Symmetry around $y = x$.

- Consider the following function:



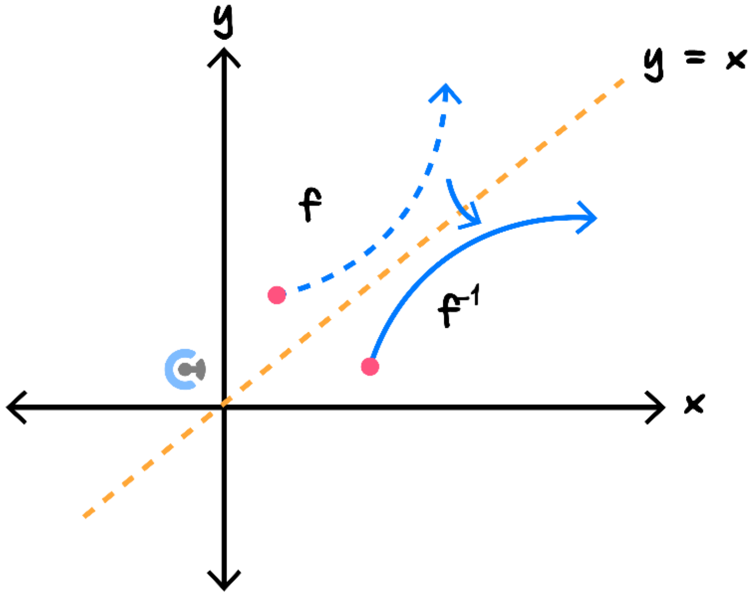
- What happens if you swap the x and y-axis on the label on our graph?



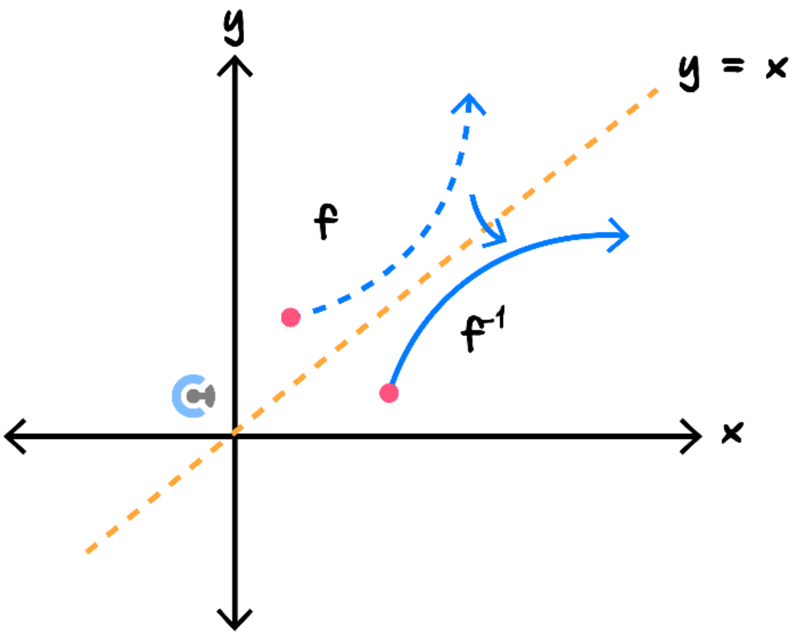


- Wait...do we want the x-axis to be the vertical one? [Yes/No]
- How should we reflect the graph so that the x and y-axis becomes horizontal and vertical again?

flip around y=x



<u>Symmetry of Inverse Functions</u>



- Inverse functions are always symmetrical around $y = x$.

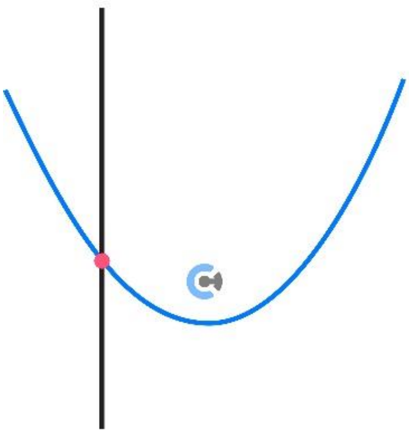
Sub-Section: Validity of Inverse Function

Does an inverse function always exist?

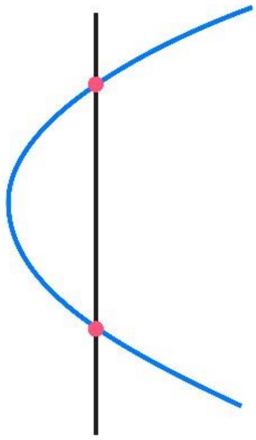
Discussion: If you find an inverse, can you guarantee that it is always a function?
Hence, is it always an inverse function?

ALWAYS inverse relation $\not\Rightarrow$ NOT ALWAYS an inverse function

REMINDER: Functions



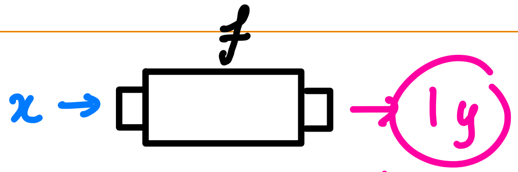
Passes : Function



Fails : Not function

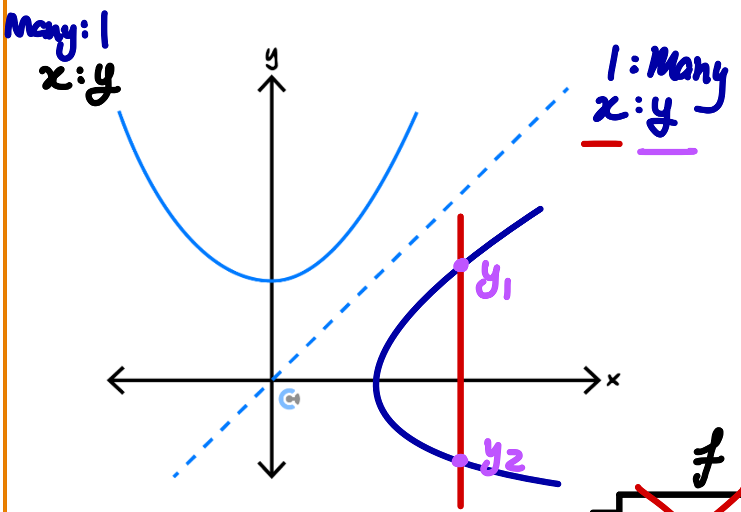
- Functions pass a vertical line test.

Exploration: Validity of inverse functions.

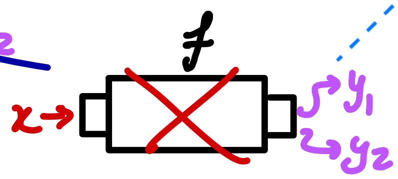


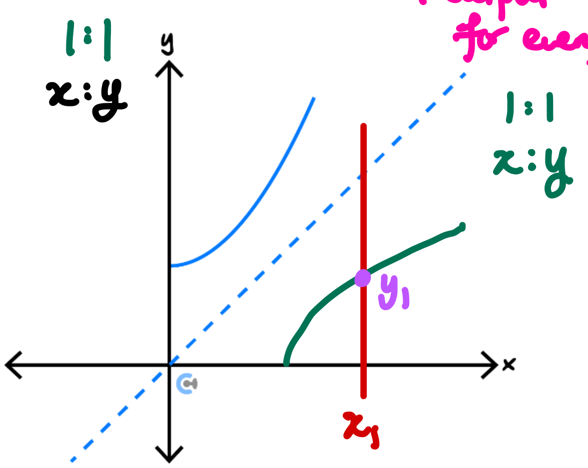
1 output for every input

- Consider the many to one and one to one functions.

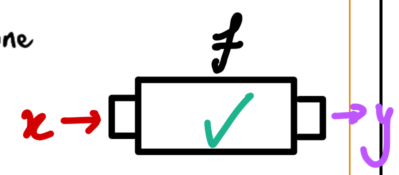


Many to one x_1





One to one

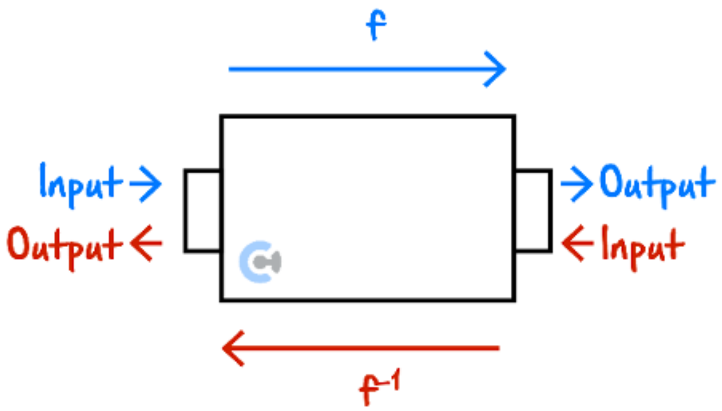


- Reflect them around $y = x$ and sketch the inverse! *(Label Above)*
- Which inverse is a function? (Passes through a vertical line test?)

  [neither] / [left] / [right] / [both]

- For an inverse **function** to exist, what must the original function be? [many to one] / [one to one]

Validity of Inverse Functions



- Requirement for Inverse Function:

f needs to be 1:1

Question 22 Walkthrough.

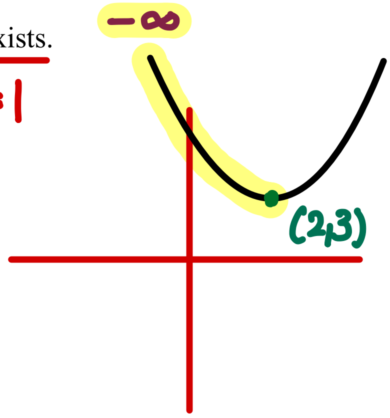
TP: $(2,3)$

Consider the function $f: (-\infty, a] \to \mathbb{R}, f(x) = (x-2)^2 + 3$.

a. Find the largest possible value of a such that the inverse function f^{-1} exists.

f must be 1:1

$\therefore a = 2$



b. Find the domain and range of the inverse function. (2 marks)

$\text{Dom } f^{-1} = \text{Ran } f = [3, \infty)$

$\text{Ran } f^{-1} = \text{Dom } f = (-\infty, 2]$

c. Find the rule for the inverse function. (2 marks)

Let $y = f(x)$:

Swap x & y:

$x = (y-2)^2 + 3$

$x - 3 = (y-2)^2$

$y - 2 = \pm\sqrt{x-3}$

$y = 2 \pm \sqrt{x-3}$

$\therefore f^{-1}(x) = 2 - \sqrt{x-3}$

TIP: Always try sketching the function to find the domain such that an inverse function can exist!

NOTE: You will need to complete the square when finding the inverse of quadratic functions!

Space for Personal Notes

Your turn!

Question 23

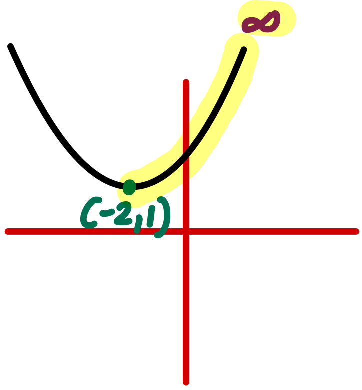
→ TP: $(-2,1)$

Consider the function $g: [b, \infty) \to \mathbb{R}, g(x) = (x+2)^2 + 1$.

a. Find the smallest possible value of b such that the inverse function g^{-1} exists.

g must be 1:1

$\therefore b = -2$



b. Find the domain and range of the inverse function. (2 marks)

$\text{Dom } g^{-1} = \text{Ran } g = [1, \infty)$

$\text{Ran } g^{-1} = \text{Dom } g = [-2, \infty)$

c. Find the rule for the inverse function. (2 marks)

Let $y = g(x)$:

Swap x & y:

$x = (y+2)^2 + 1$

$x - 1 = (y+2)^2$

$y + 2 = \pm\sqrt{x-1}$

$y = -2 \pm \sqrt{x-1}$

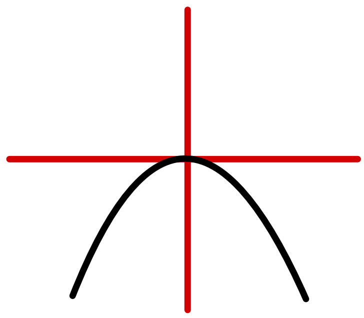
$\therefore g^{-1}(x) = -2 + \sqrt{x-1}$

Space for Personal Notes

Question 24 Extension.

Consider the function $g: (-\infty, b] \to \mathbb{R}, g(x) = -x^2 + 4x - 3$.

a. Find the largest possible value of b such that the inverse function g^{-1} exists.



b. Find the domain and range of the inverse function. (2 marks)

Dom g^{-1} = Ran g =

Ran g^{-1} = Dom g =

c. Find the rule for the inverse function. (2 marks)

Let $y = g(x)$:

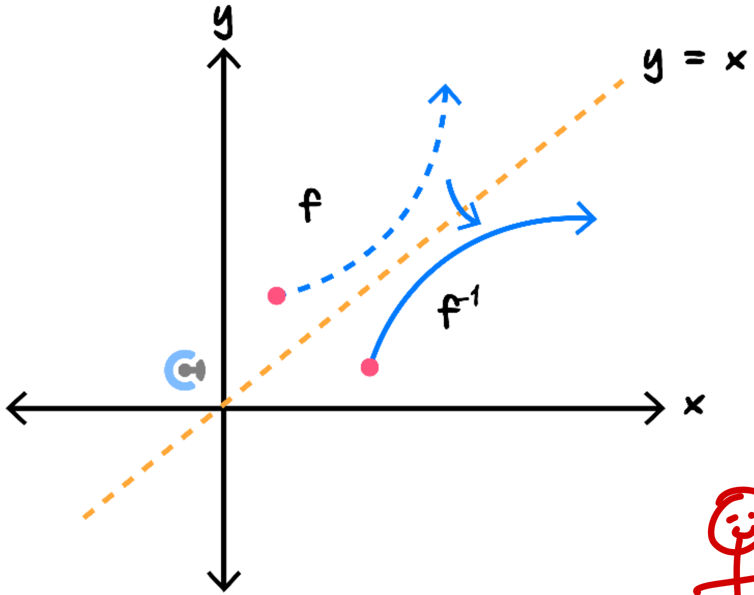
Swap x & y:

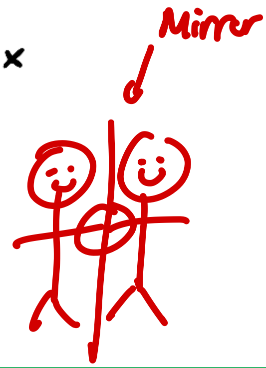
Space for Personal Notes

Sub-Section: Intersection Between Inverses

Where do inverses meet?

Active Recall: Symmetry around $y = x$.





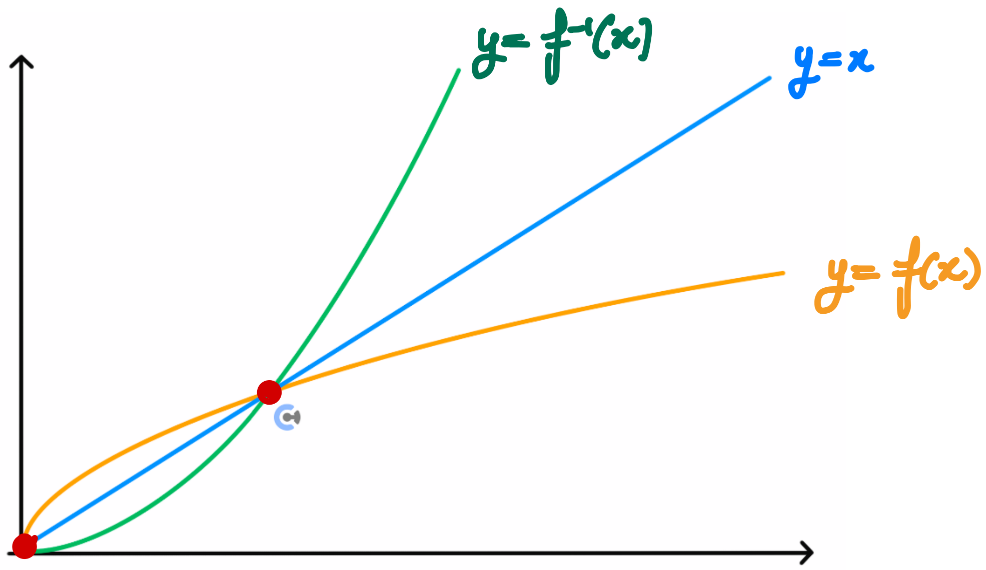
- Inverse functions are always symmetrical around $y = x$.

Discussion: Where could a function and its inverse meet?

on the mirror line $y=x$

<u>Exploration</u>: Intersections between a function and its inverse.

- Consider a function and its inverse below.



- Note the symmetry around $y = x$ for inverses!
- Circle the point where the two functions intersect.
- Where does this point also lie?

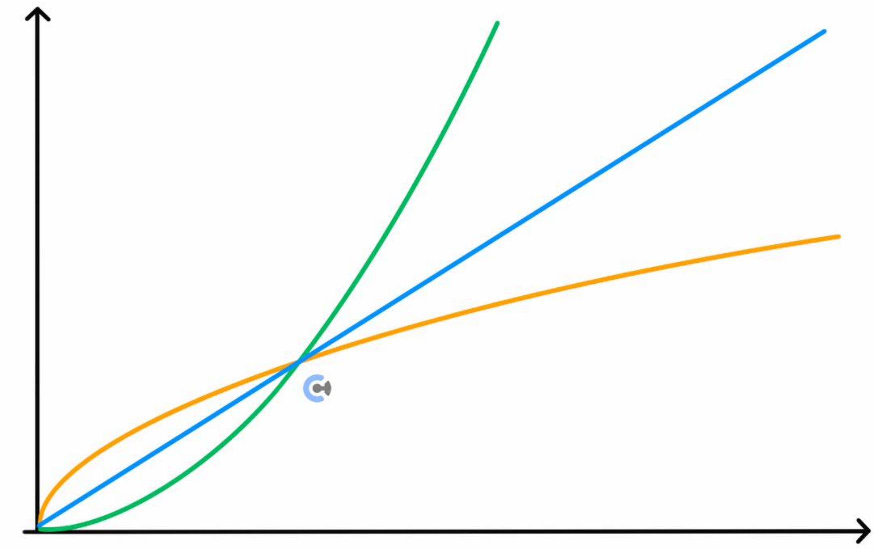
on $y=x$

<u>Discussion:</u> Hence, instead of solving $f(x) = f^{-1}(x)$, what can we solve instead of finding the point where a function and its inverse intersect?

$\therefore$ let $y=\underline{f(x)}$ intersect $y=\underline{x}$:

$\therefore f(x)=x$

Intersection Between a Function and Its Inverse



$$f(x) = x \quad \text{OR} \quad f^{-1}(x) = x$$

Question 25

Find the intersection (x, y) between $f: [0, \infty) \to R, f(x) = x^3$ and its inverse, without finding the inverse.

Let $f(x) = x$:

$x^3 = x$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x(x-1)(x+1) = 0$

$\therefore x = -1, \underline{0, 1}$ reject as $x \geq 0$

$\therefore$ IPs: $(0,0)$ & $(1,1)$

NOTE: We can always equate the function to x instead of the inverse function itself!

ALSO NOTE: This only works for an increasing function, however in VCAA, this is always the case. Something to note for SACS is that there COULD be intersections that are NOT on $y = x$.

Contour Checklist

Learning Objective: [2.2.1] – Find Domain and Range of Functions

Key Takeaways

Interval Notation:

- Parentheses (non-inclusive):

$$x \in (a, b) \Rightarrow a < x < b$$

- Square brackets [inclusive]:

$$x \in [a, b] \Rightarrow a \leq x \leq b$$

Maximal Domain:

- Inside of a log must be >0.
- Inside of a root must be ≥ 0.
- Denominator $\neq 0$.

Learning Objective: [2.2.2] – Sketch and Find the Domain and Range of Hybrid Functions

Key Takeaways

Piecewise (Hybrid) Functions:

- Series of functions.

$$h(x) = \begin{cases} f(x), & Domain_1 \\ g(x), & Domain_2 \end{cases}$$

- When we have an x intercept for one graph, sum graph intersects the other graph.
- $Domain_1$ and $Domain_2$ represent the x values for which the two functions are defined.
- The two domains do not have to join!

Learning Objective: [2.2.3] – Find the Rule, Domain, Range, and Intersections Between Inverse Functions

Key Takeaways

- f needs to be 1:1 for f^{-1} to exist.
- Domain of the inverse function equals to range of the original and vice versa.
- Symmetrical around $y=x$.
- For intersections of inverses, we can equate the function to x.