CONTOUR EDUCATION

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VCE Mathematical Methods ½
AOS 2 Revision [2.0]
Contour Check (Part 1) Solutions

Contour Check

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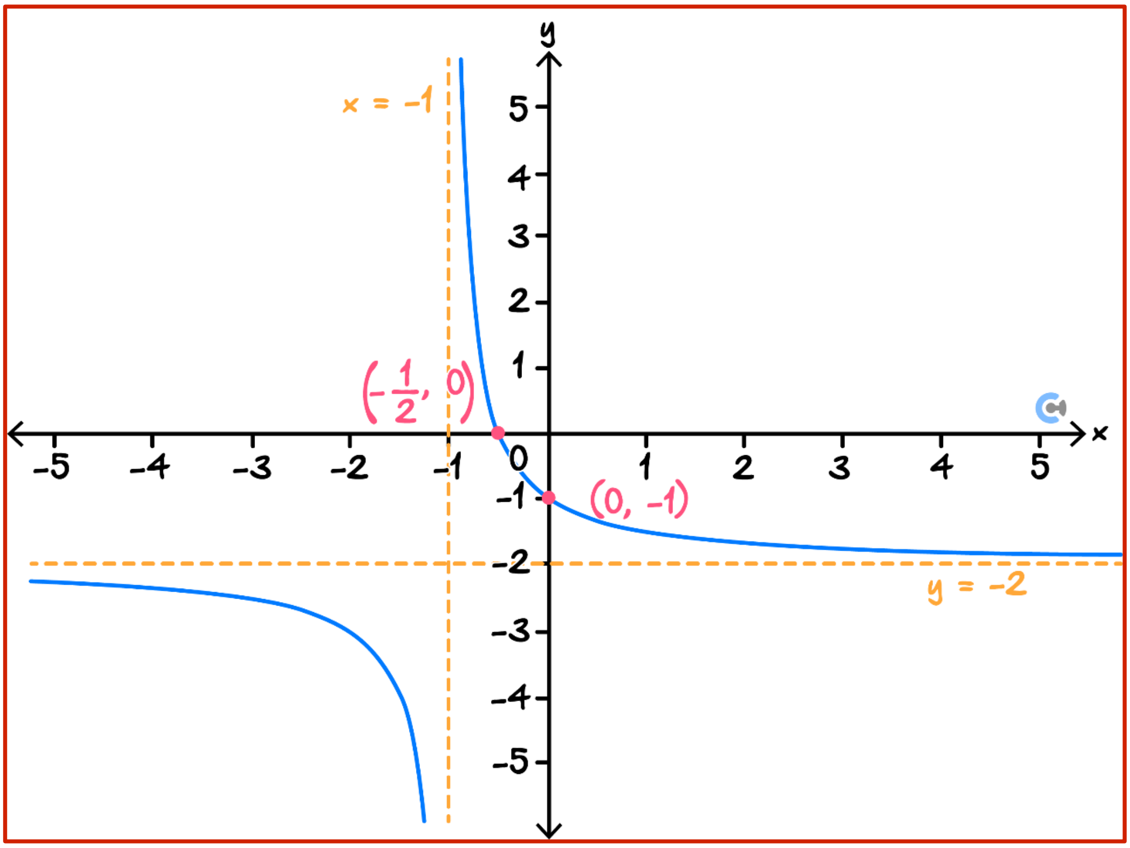
Section A: [2.1] - Functions & Relations I (Checkpoints)

Sub-Section [2.1.1]: Sketch and Find the Rule of Hyperbolas Functions

Question 1

Graph the following curve, labelling all intercepts and asymptotes with their equations.

$$y = \frac{1}{x+1} - 2$$



Solution Pending

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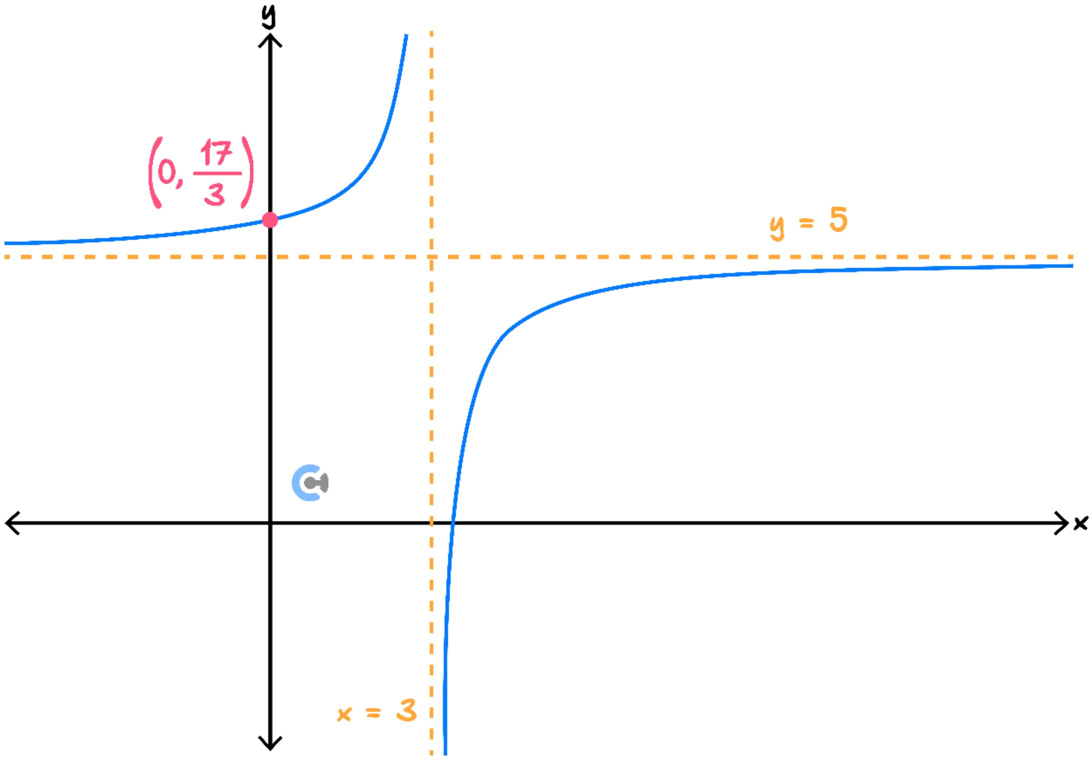
Question 2

Find the rule for the following graph, given it is of the form $y = \frac{a}{x-h} + k$.



Clear that $h = 3$ and $k = 5$.

Then, $\frac{17}{3} = \frac{a}{-3} + 5 \implies a = -2$

$$y = \frac{-2}{x-3} + 5$$

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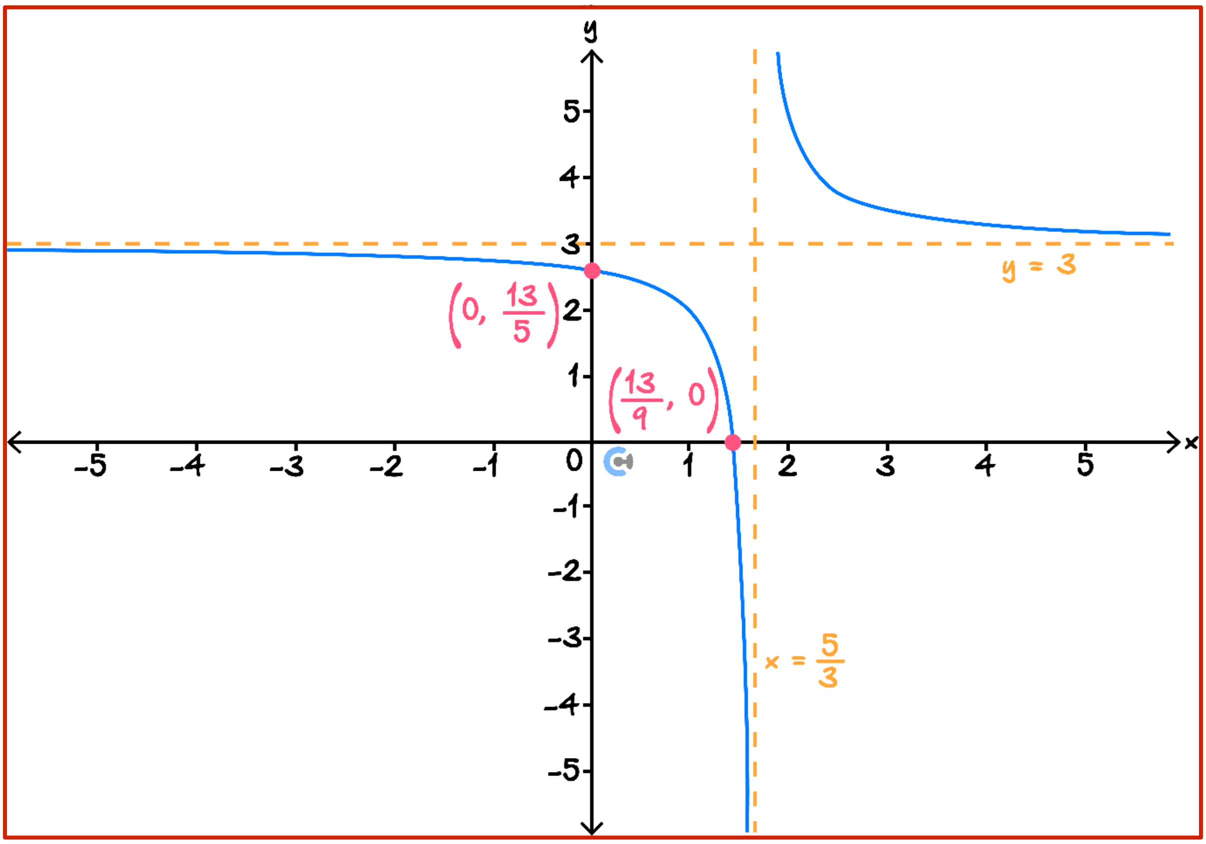
Question 3

Graph the following curve, labelling all intercepts and asymptotes with their equations.

$$y = 3 - \frac{2}{5 - 3x}$$



Solution Pending

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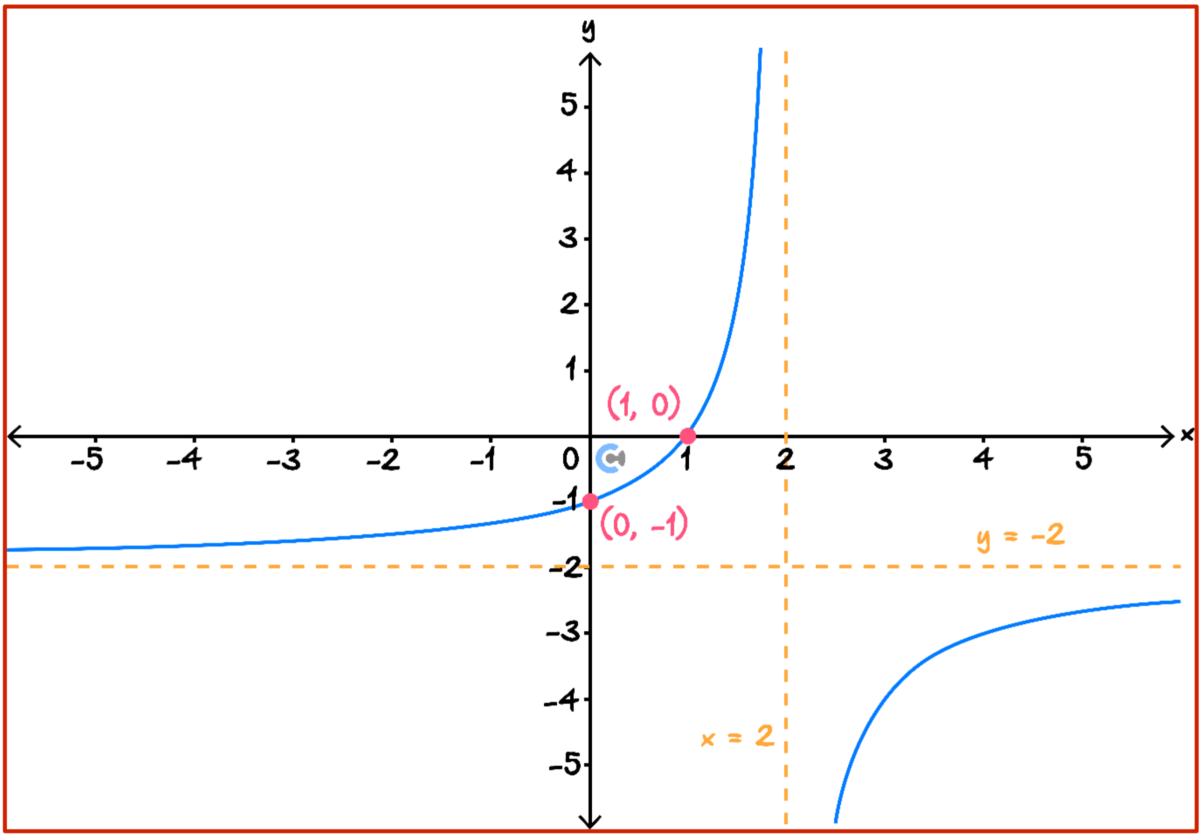
Question 4

Graph the following curve, labelling all intercepts and asymptotes with their equations.

$$y = \frac{2 - 2x}{x - 2}$$



Solution Pending

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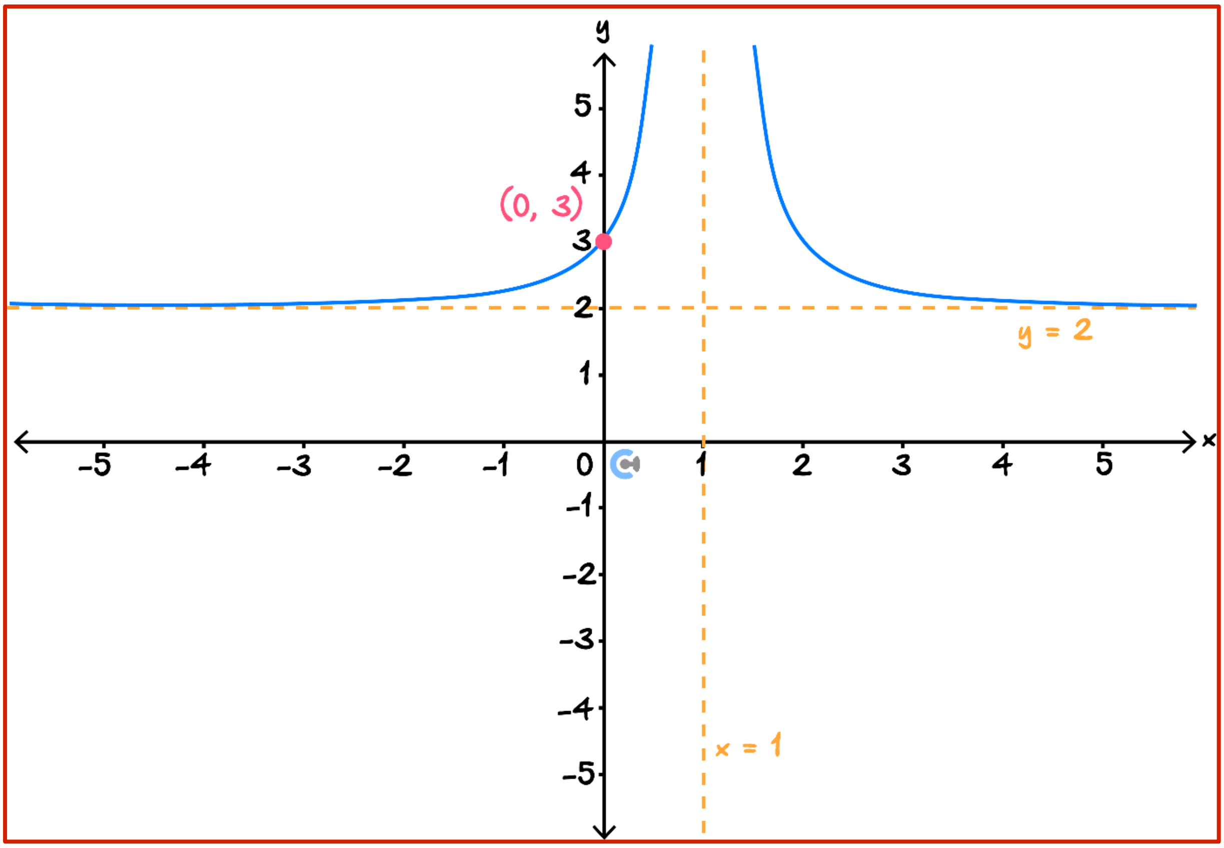
Sub-Section [2.1.2]: Sketch and Find the Rule of Truncus Functions

Question 5

Graph the following curve, labelling all intercepts and asymptotes with their equations.

$$y = \frac{1}{(x-1)^2} + 2$$



Solution Pending

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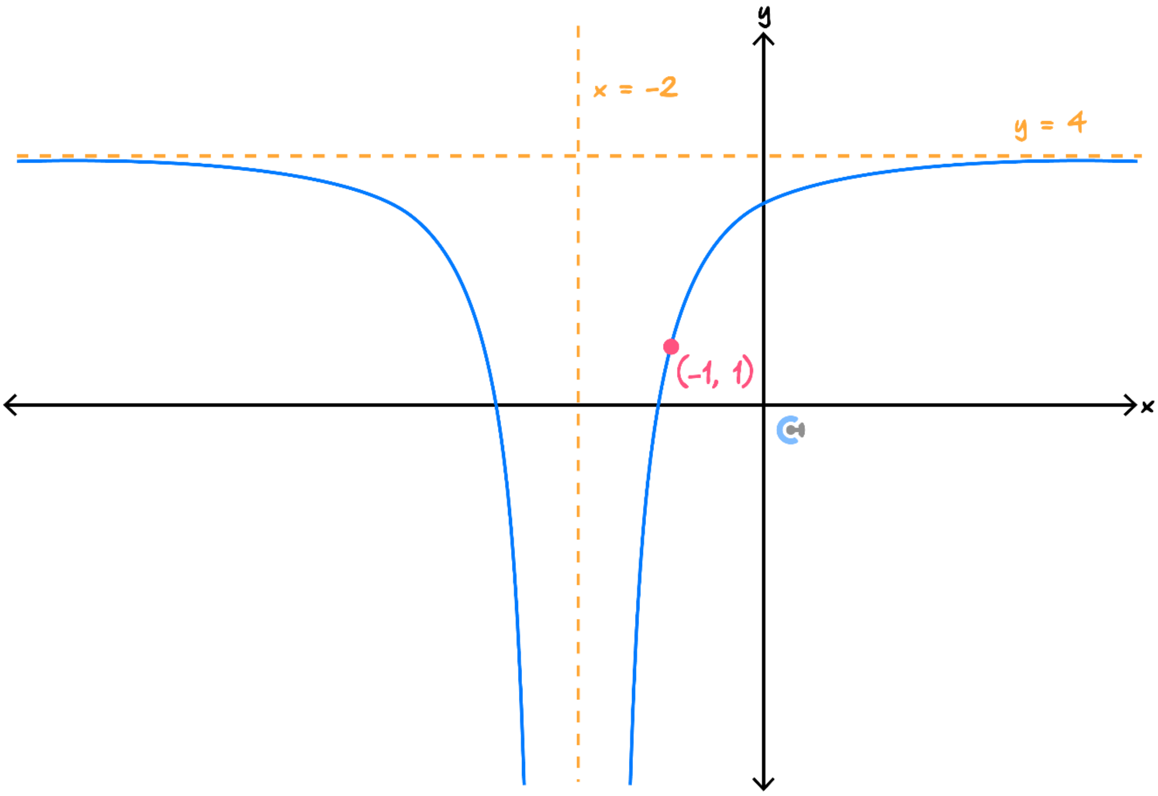
Question 6

Find the rule for the following graph, given it is of the form $y = \frac{a}{(x-h)^2} + k$.



Clear that $h = -2$ and $k = 4$.

Then, $1 = \frac{a}{(-1+2)^2} + 4 \implies a = -3$.

$$y = \frac{-3}{(x+2)^2} + 4$$

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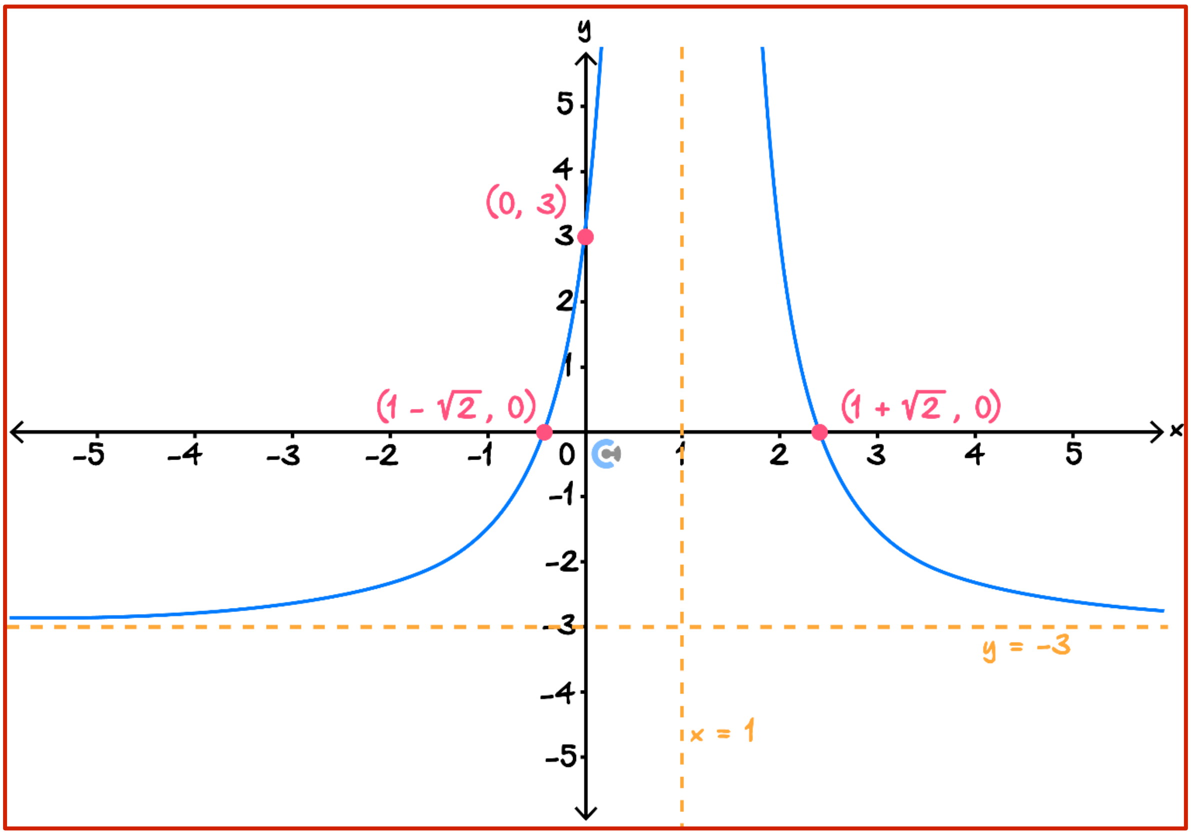
Question 7

Graph the following curve, labelling all intercepts and asymptotes with their equations.

$$y = \frac{6}{(1-x)^2} - 3$$



Solution Pending

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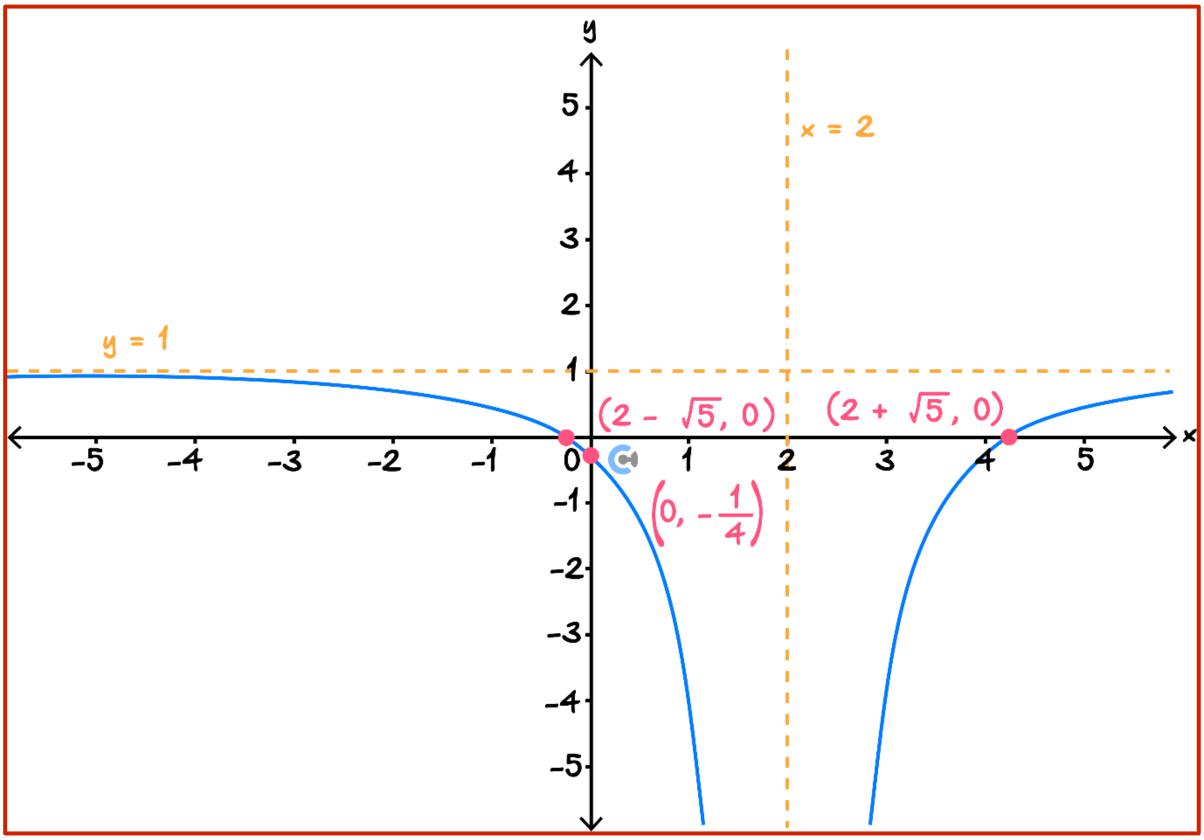
Question 8

Graph the following curve, labelling all intercepts and asymptotes with their equations.

$$y = \frac{x^2 - 4x - 1}{(x-2)^2}$$



Solution Pending

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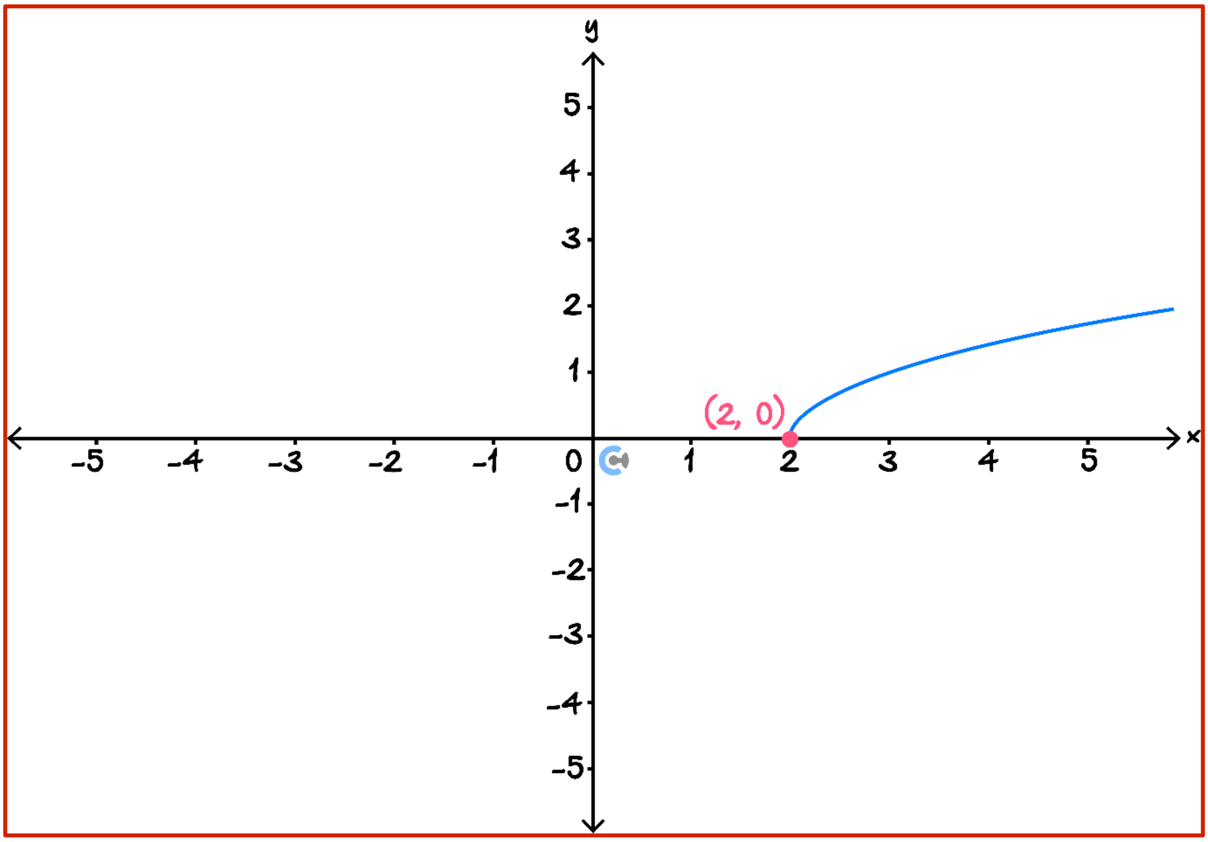
Sub-Section [2.1.3]: Sketch and Find the Rule of Root Functions

Question 9

Graph the following curve, labelling all intercepts and start points.

$$y = \sqrt{x - 2}$$



Solution Pending

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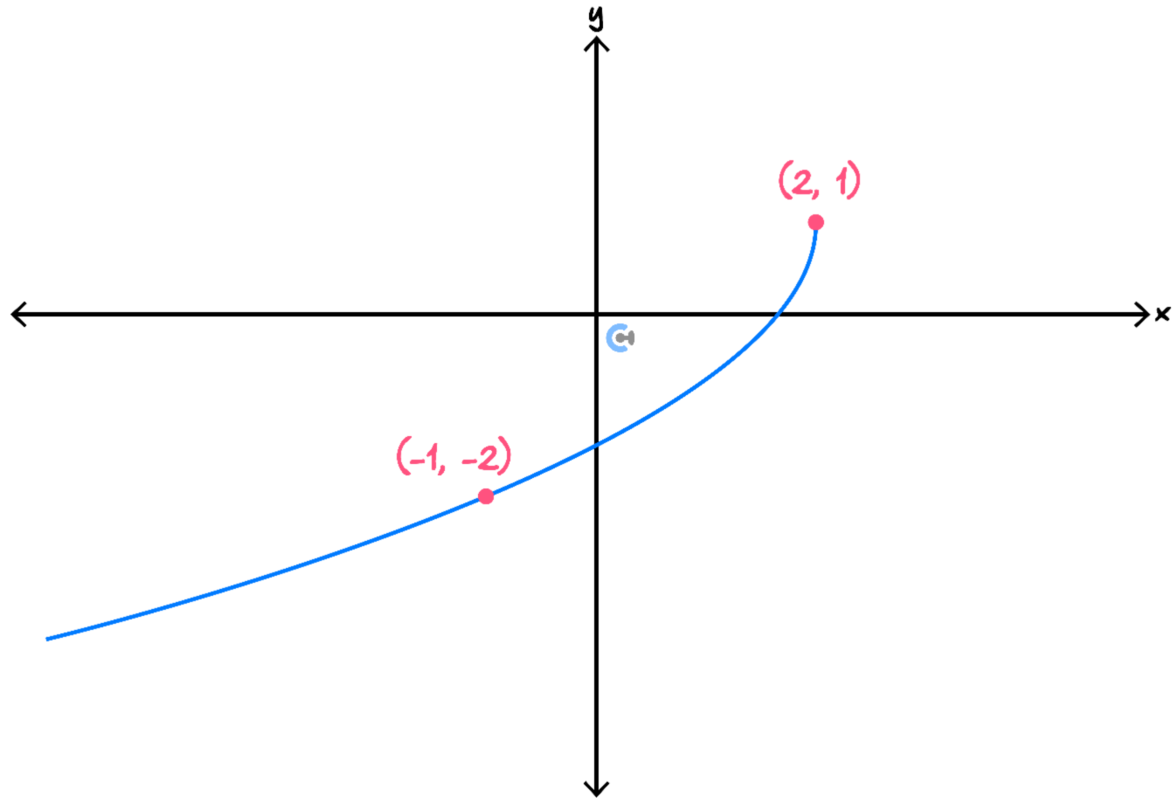
Question 10

Find the rule for the following graph, given it is of the form $y = a\sqrt{h-x} + k$.



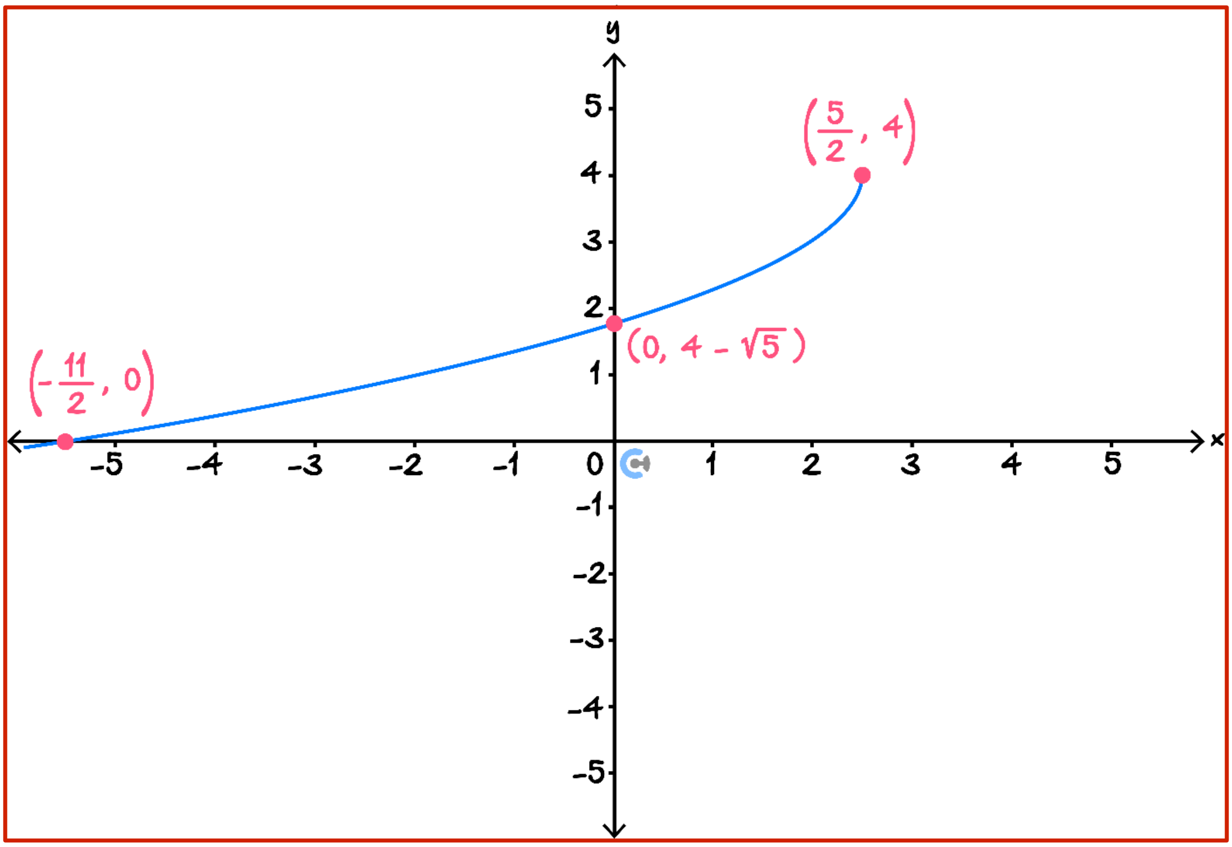
From the start point, $h = 2$ and $k = +1$.
Then, $-2 = a\sqrt{2-(-1)} + 1 \Longrightarrow a = -\sqrt{3}$.
$y = -\sqrt{3}\sqrt{2-x} + 1$

Space for Personal Notes

Question 11

Graph the following curve, labelling all intercepts and start points.

$$y = 4 - \sqrt{5 - 2x}$$



Solution Pending

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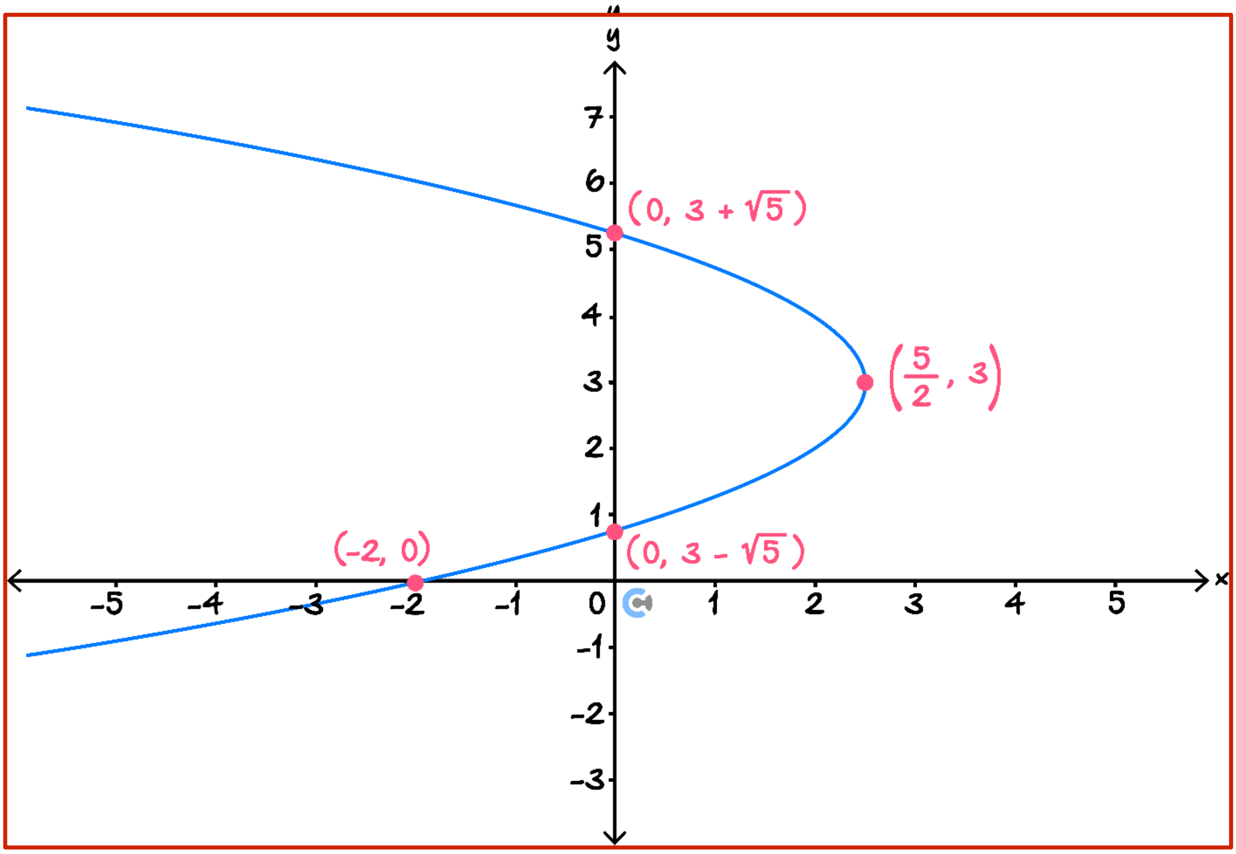
Question 12

Graph the following curve, labelling all intercepts and turning points.

$$(y-3)^2 = 5 - 2x$$



Solution Pending

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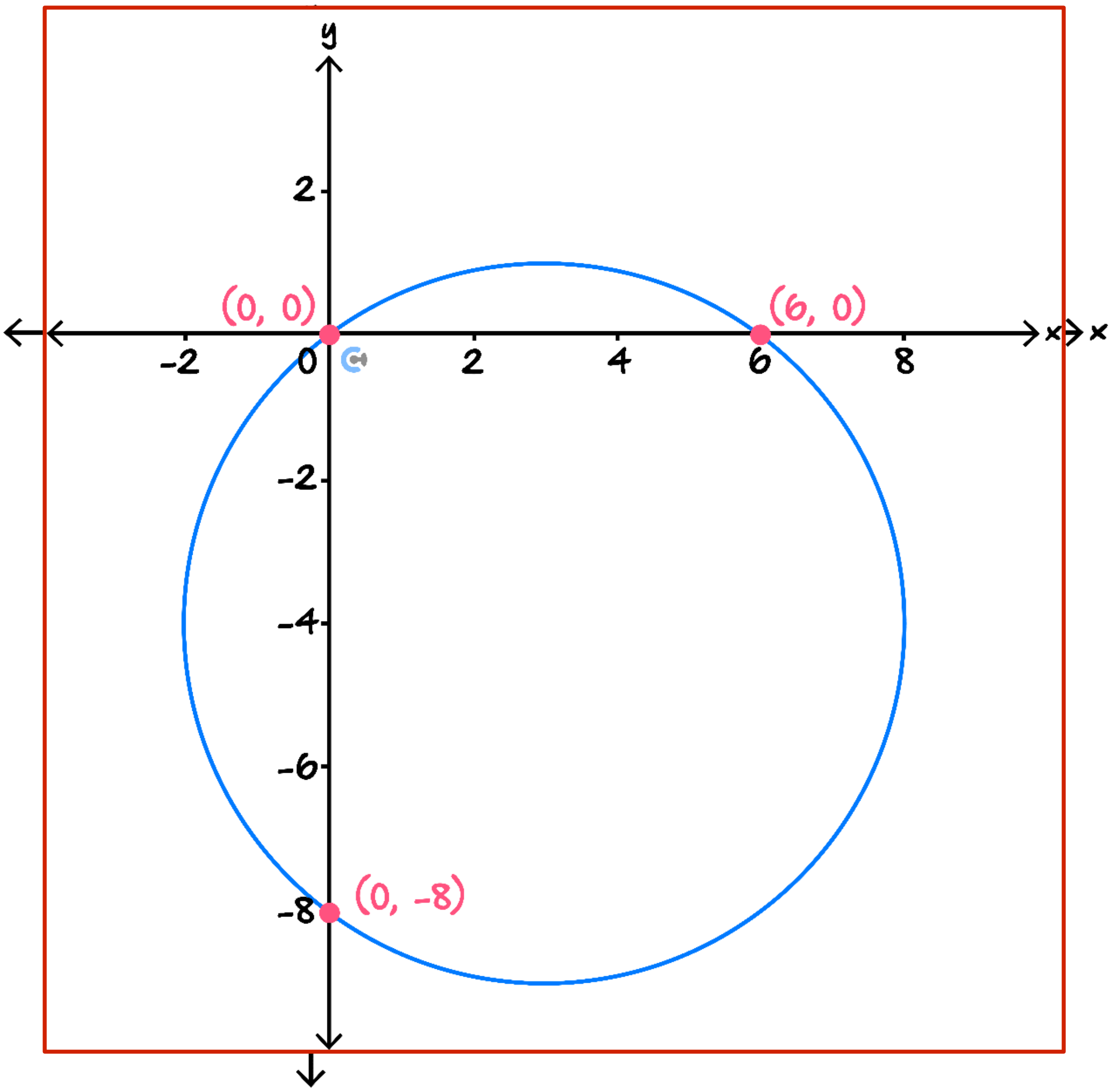
Sub-Section [2.1.4]: Sketch and Find the Rule of Semicircles and Circles

Question 13

Graph the following circle, label all intercepts.

$$(x-3)^2 + (y+4)^2 = 25$$

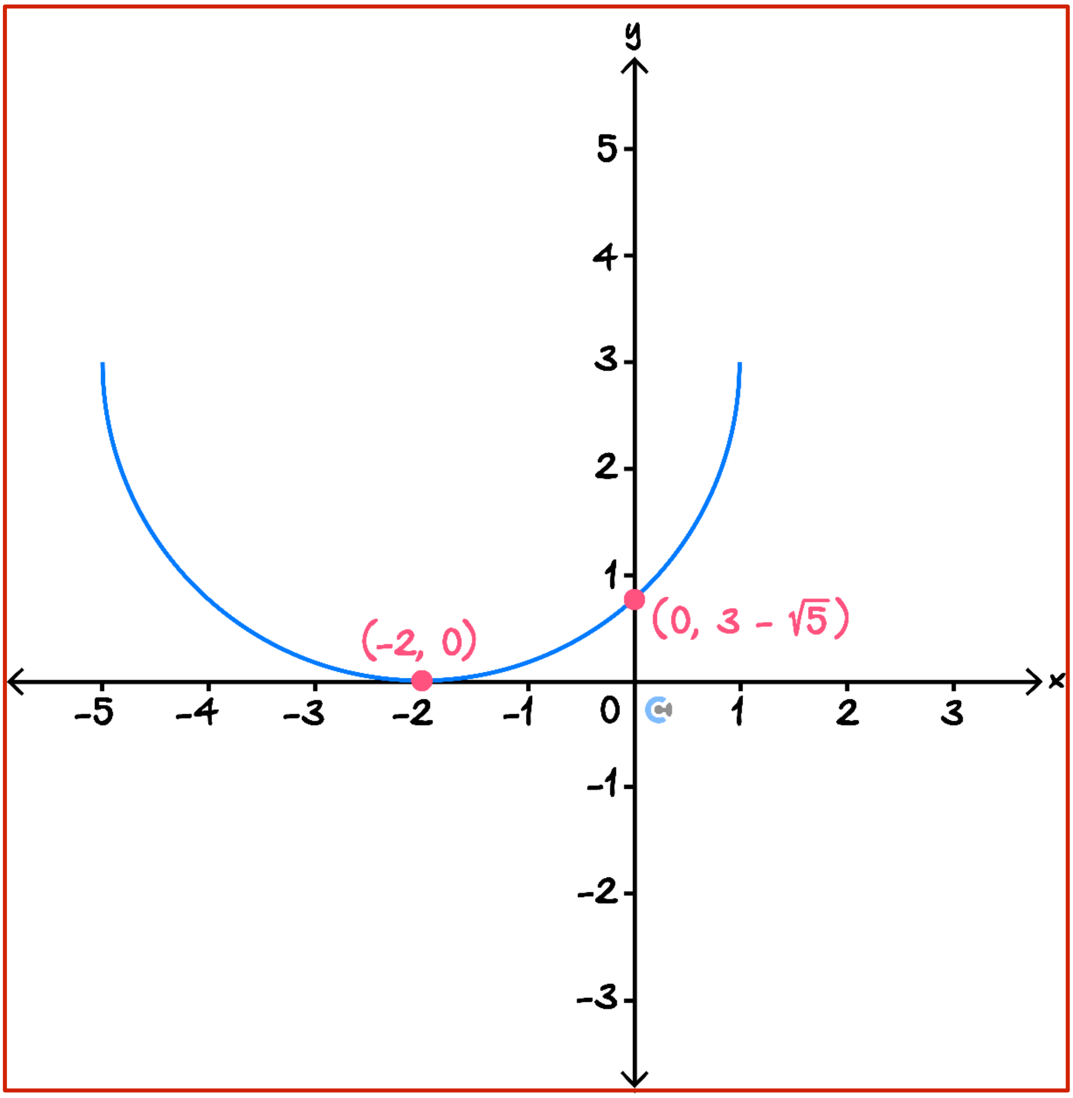


Solution Pending

Question 14

Graph the following semi-circle, label all intercepts.

$$y = 3 - \sqrt{9 - (x+2)^2}$$



Solution Pending

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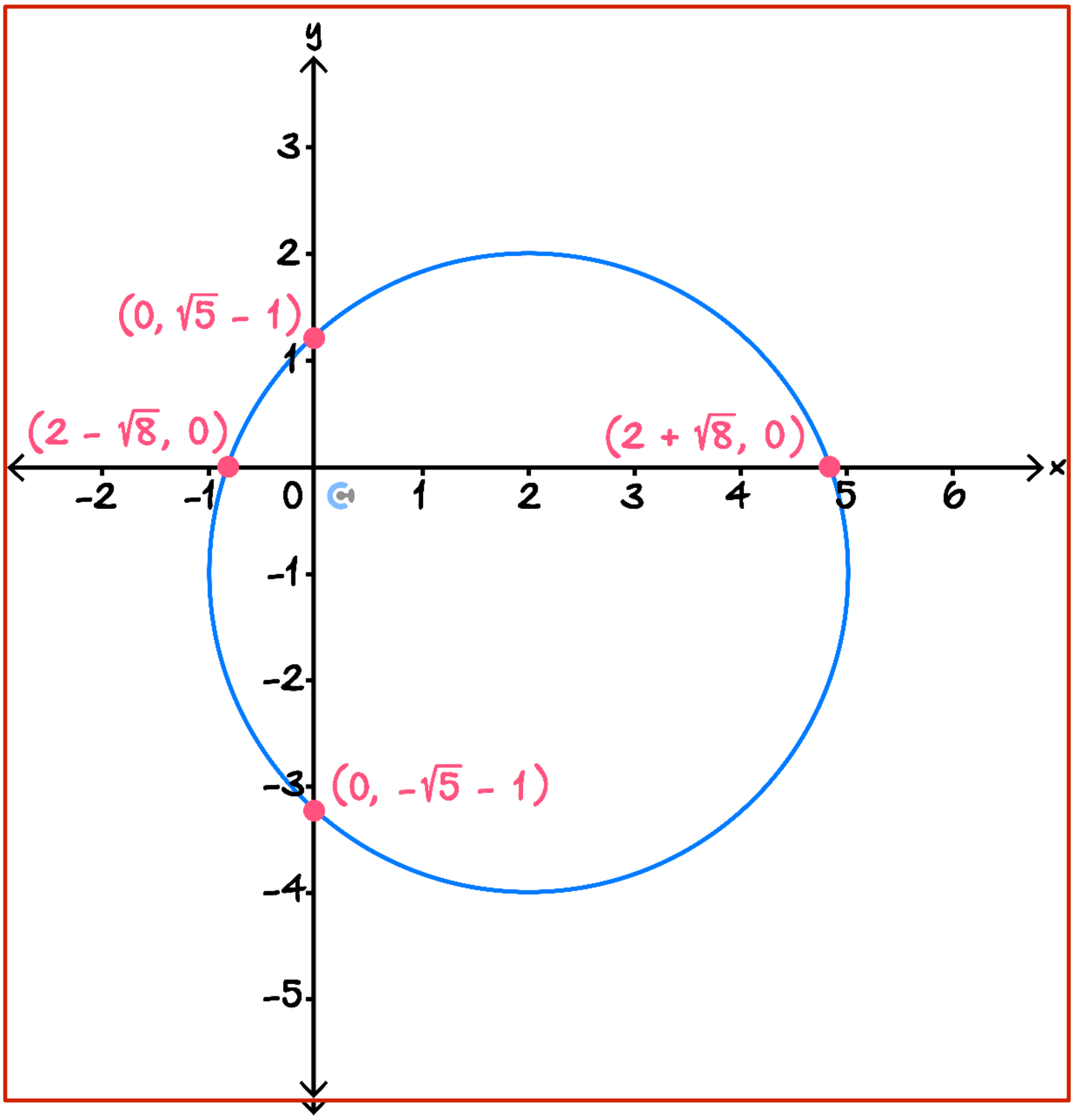
Question 15

Graph the following circle, label all intercepts.

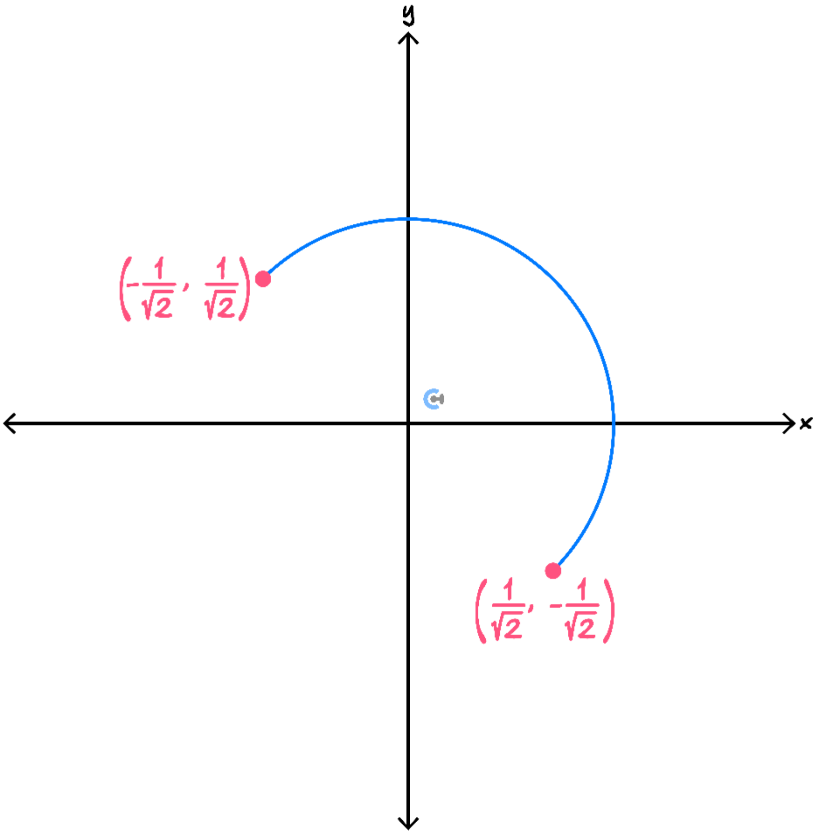
$$x^2 + y^2 + 2y - 4x = 4$$



Complete the square to get circle equation:
$(x-2)^2 + (y+1)^2 = 9.$

Question 16

Determine the equation of the semi-circle with radius 1, shown on the graph below.



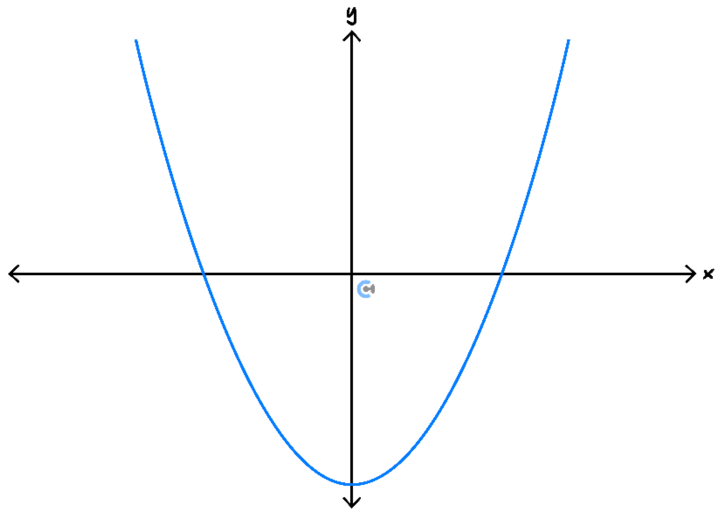
The line segment joining the two end points has a length of 2 units, which is the diameter of the circle.
Thus the center of our circle is the midpoint of our two end-points, the origin.
Thus our semi circle can be described with the following equation and restriction, $x^2+y^2=1$ and $x+y \geq 0$.
Now we can rearrange $x^2+y^2=1$ to be of the form, $(x+y)^2 = 1+2xy$.
Then to implement our desired restriction we simply square root both sides taking the positive square root, resulting in $x+y=\sqrt{1+2xy}$.
This forces $x+y>0$ giving us our desired graph.

Sub-Section [2.1.5]: Identify the Type of Relations and Identify whether the Relation is a Function

Question 17

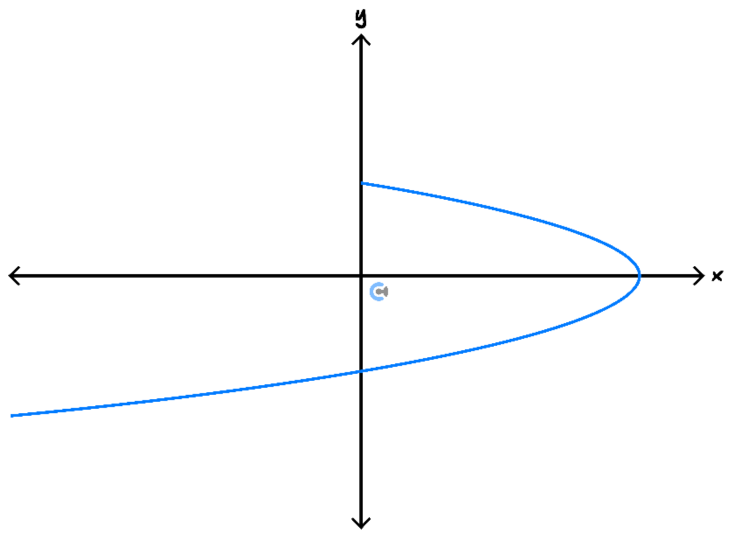
For each of the following graphs, identify the type of relation depicted and whether the relation is a function.

a.



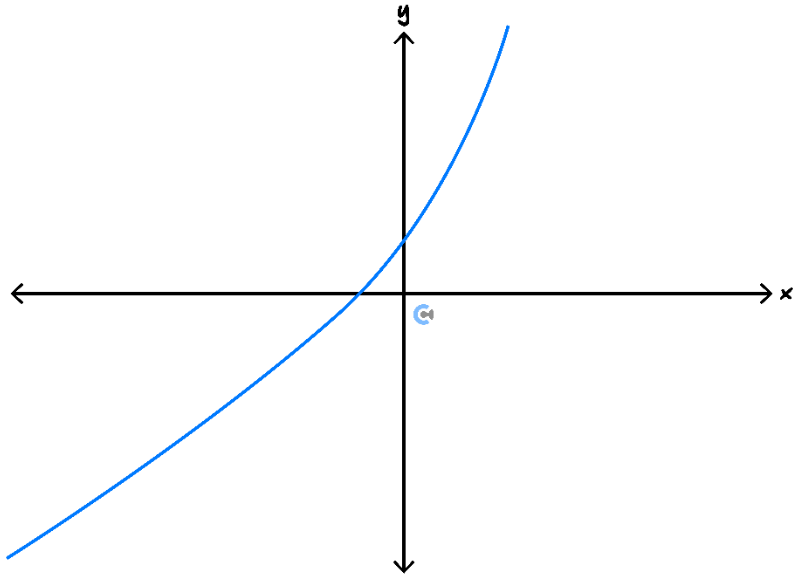
Many-to-one, is a function.

b.



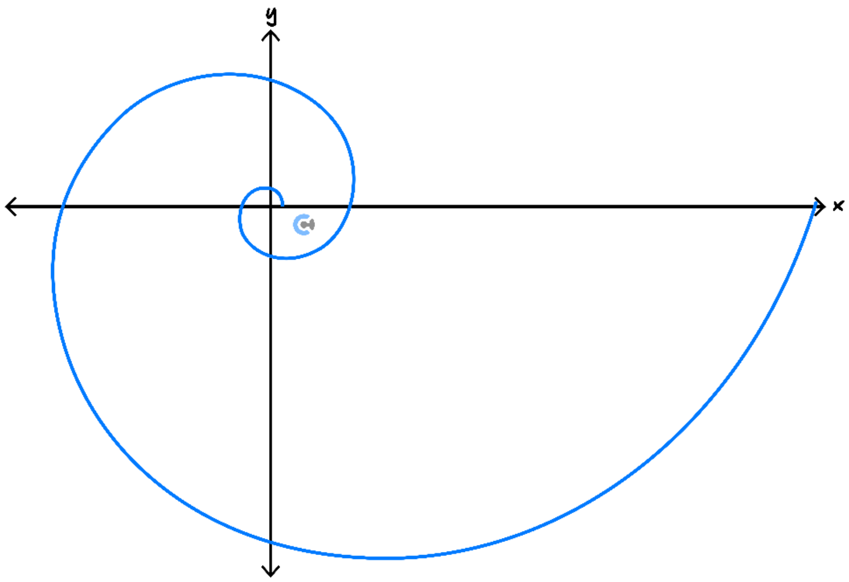
One-to-many, not a function.

c.



One-to-one, is a function.

d.



Many-to-many, not a function.

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Section B: [2.2] - Functions & Relations II (Checkpoints)

Sub-Section [2.2.1]: Find Domain and Range of Functions

Question 18

Find the domain of the following functions:

a. $y = \sqrt{5 - 2x}$

$$5 - 2x \geq 0$$
$$5 \geq 2x$$
$$x \leq \frac{5}{2}$$

b. $y = -\frac{3}{x^2 + 4x - 12}$

$$x^2 + 4x - 12 \neq 0$$
$$(x + 6)(x - 2) \neq 0$$
$$x \neq -6, 2$$

c. $y = 2\log_e(x + 1)$

$$x + 1 > 0$$
$$x > -1$$

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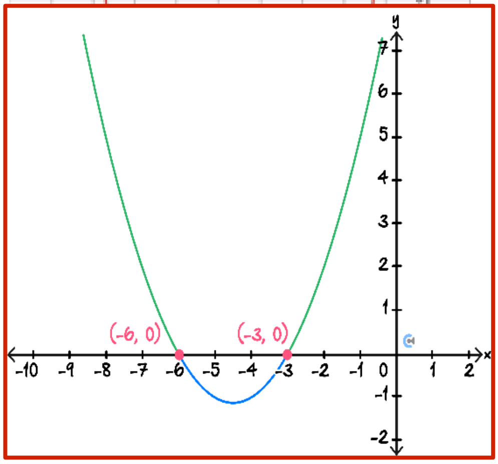
Question 19

Find the maximal domain of the following functions:

a. $y = \frac{(\sqrt{x^2+9x+18})}{2}$

$x^2 + 9x + 18 \geq 0$

$(x+6)(x+3) \geq 0$

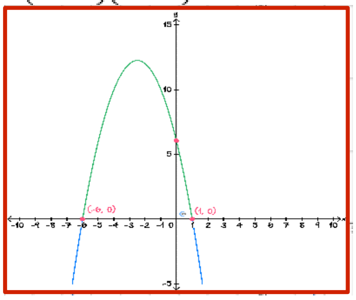


$x \in (-\infty, -6] \cup [-3, \infty)$

b. $y = \frac{3}{\sqrt{6-5x-x^2}} - 4$

$6 - 5x - x^2 > 0$

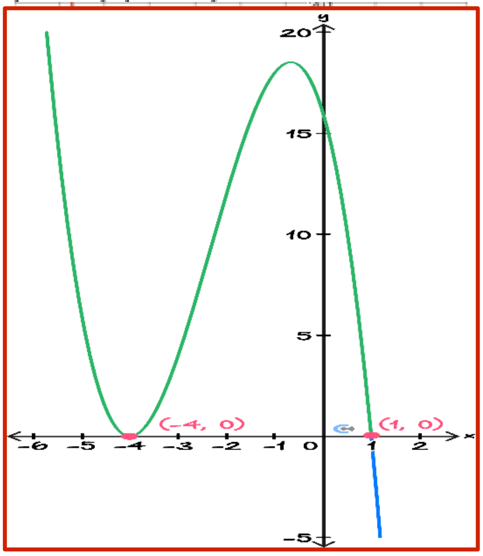
$-(x+6)(x-1) > 0$



$x \in (-6,1)$

c. $y = \log_5\left((1-x)(x+4)^2\right) + 1$

$(1-x)(x+4)^2 > 0$



$x \in (-\infty, 4) \cup (4,1)$

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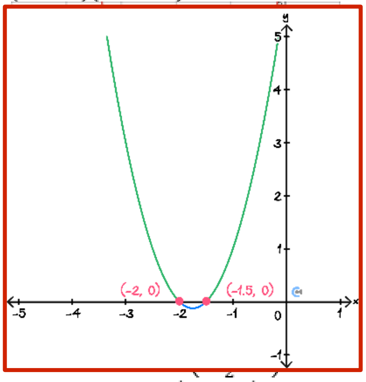
Question 20

Express $f(x)$ in full function mapping notation.

a. $f(x) = \log_2(2x^2 + 7x + 6)$

$2x^2 + 7x + 6 > 0$

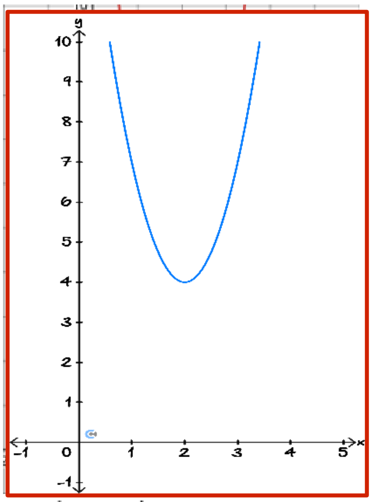
$(x + 2)(2x + 3) > 0$



$f: (-\infty, -2) \cup \left(-\frac{3}{2}, \infty\right) \to \mathbb{R}, f(x) = \log_2(2x^2 + 7x + 6)$

b. $f(x) = (3x^2 - 12x + 16)^{\frac{3}{2}}$

$3x^2 - 12x + 16 \geq 0$

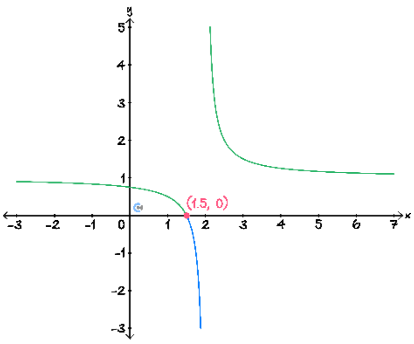


$x \in (-\infty, \infty)$

$f: \mathbb{R} \to \mathbb{R}, f(x) = (3x^2 - 12x + 16)^{\frac{3}{2}}$

c. $f(x) = 3\sqrt{-\frac{1}{4-2x} + 1}$

$$-\left(\frac{1}{4 - 2x}\right) + 1 \geq 0$$



$$x \in \left(-\infty, \frac{3}{2}\right] \cup (2, \infty)$$

$$f: \left(-\infty, \frac{3}{2}\right] \cup (2, \infty) \to \mathbb{R}, f(x) = 3\sqrt{-\frac{1}{4 - 2x} + 1}$$

Question 21

Find the maximal domain of the function $f(x) = x^2 + 4x + 12$ such that, the range of $f(x)$ is [8,17).

$f(x) = 8 \Rightarrow x = -2$
$f(x) = 17 \Rightarrow x = -5,1$
$x \in (-5,1)$

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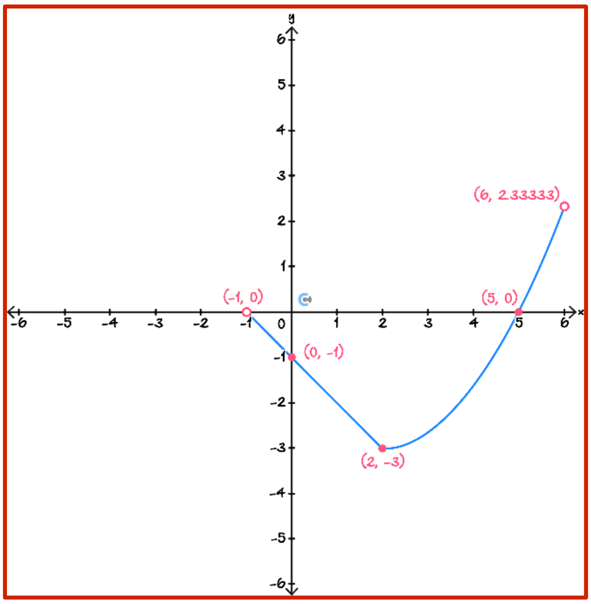
Sub-Section [2.2.2]: Sketch and Find the Domain and Range of Hybrid Functions

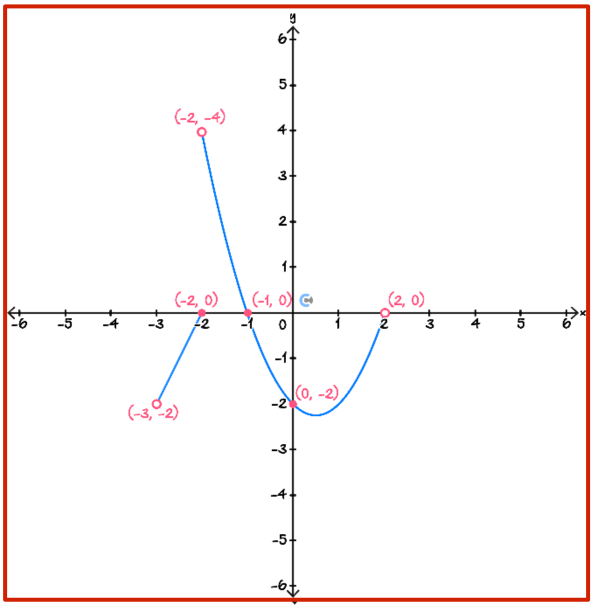
Question 22

Sketch the following graphs. Label all intercepts and endpoints.

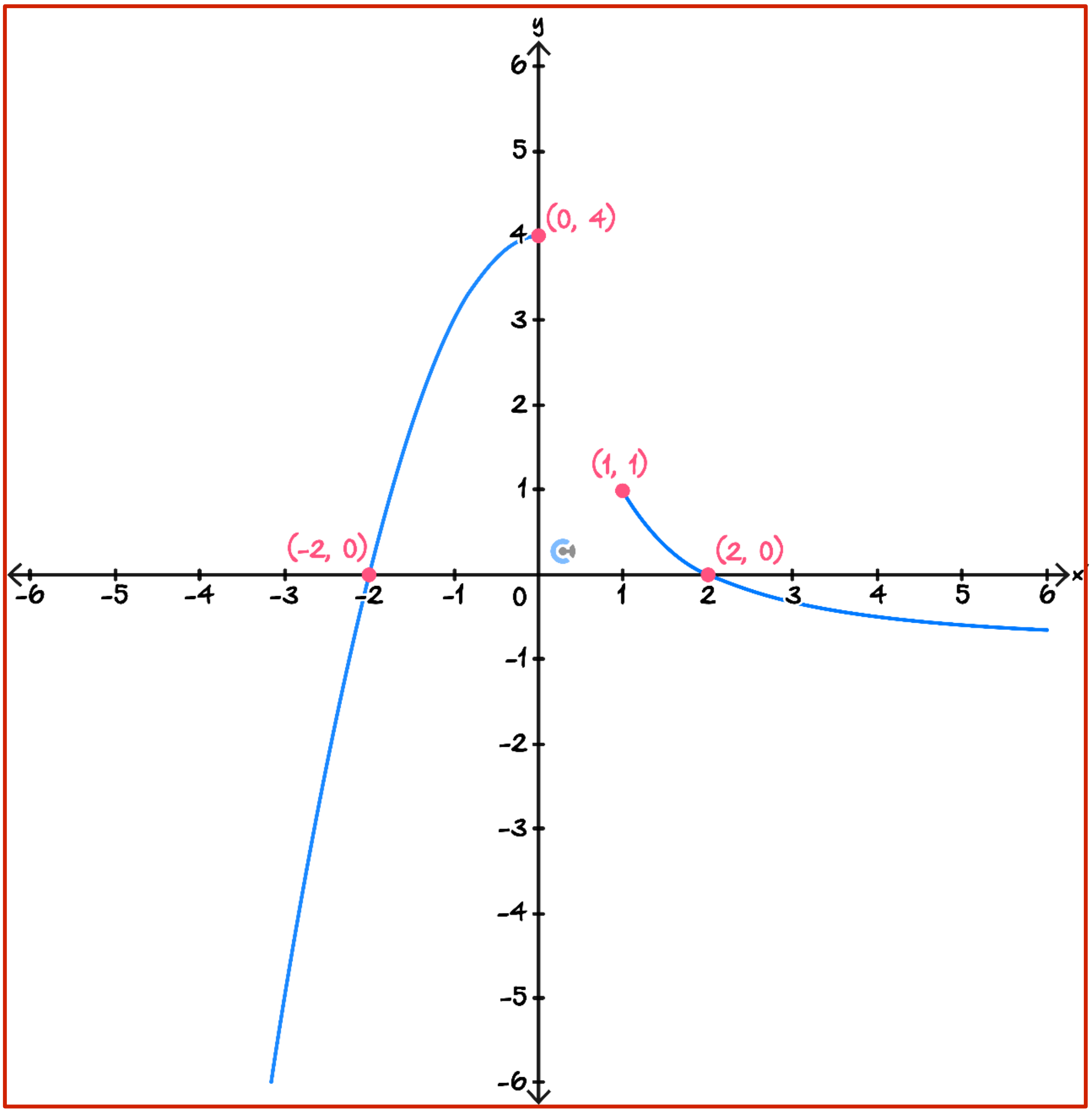
a. $f(x) = \begin{cases} \frac{1}{3}(x-2)^2 - 3, & 2 \leq x < 6 \\ -x - 1, & -1 < x < 2 \end{cases}$



b. $f(x) = \begin{cases} 2x + 4, & -3 < x \leq -2 \\ x^2 - x - 2, & -2 < x < 2 \end{cases}$



c. $f(x) = \begin{cases} 4 - x^2, & x \leq 0 \\ \frac{2}{x} - 1, & x \geq 1 \end{cases}$



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Question 23

Find the range of the following piecewise functions.

a. $f(x) = \begin{cases} x - 2, & -4 \le x < 1 \\ 2x - 2, & 1 \le x \le 3 \end{cases}$

Range of $x - 2,\ -4 \le x < 1$ is $y \in [-6, -1)$
Range of $2x - 2,\ 1 \le x \le 3$ is $y \in [0,4]$
Range of $f(x)$ is $y \in [-6, -1) \cup [0,4]$

b. $f(x) = \begin{cases} x^2 - 4x + 6, & 0 < x < 5 \\ \frac{1}{2}x + 6, & -6 < x < 0 \end{cases}$

Range of $x^2 - 4x + 6,\ 0 < x < 5$ is $y \in [2,11)$
Range of $\frac{1}{2}x + 6,\ -6 < x < 0$ is $y \in [3,6]$
Range of $f(x)$ is $y = [2,11)$

c. $f(x) = \begin{cases} x+4, & -3 < x < -1 \\ x^2, & -1 \leq x \leq 2 \\ x-4, & 2 < x < 8 \end{cases}$

Range of $x+4,\ -3 < x < -1$ is $y \in (1,3)$
Range of $x^2,\ -1 \leq x \leq 2$ is $y \in [0,4]$
Range of $x-4,\ 2 < x < 8$ is $y \in (-2,4)$
Range of $f(x)$ is $y \in (-2,4]$

Question 24

Find the maximal domain of the function: $f(x) = \begin{cases} \sqrt{8-2x} \\ \log_e(-x^2+5x+6) \end{cases}$

For $\sqrt{8-2x}$ to be defined, $8-2x \geq 0$. Therefore $x \in (-\infty, 4]$
For $\log_e(-x^2+5x+6)$ to be defined, $-x^2+5x+6 > 0$. Therefore $x \in (-1,6)$
Maximal domain is $(-\infty, 6)$

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Sub-Section [2.2.3]: Find the Rule, Domain, Range, and Intersections between Inverse Functions

Question 25

The function $f(x)$ is defined as $f: [-5,1) \to \mathbb{R}, f(x) = \frac{2}{3-x} + 6$.

a. Find the equation of $f^{-1}(x)$.

$$f^{-1}(x) = -\frac{2}{x-6} + 3$$

b. Determine the domain of $f^{-1}(x)$.

Domain of $f^{-1}(x)$ is Range of $f(x)$

$x \in \left[\frac{25}{4}, 7\right)$

c. State the range of $f^{-1}(x)$.

Range of $f^{-1}(x)$ is Domain of $f(x)$

$y \in [-5,1)$

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Question 26

Consider the function, $g: (-\infty, 0] \to \mathbb{R},\ g(x) = 2x^2 - 12x + 16$.

a. Find the equation of the inverse function.

$y = \pm\sqrt{\frac{1}{2}(x+2)} + 3$

Range of $g^{-1}(x)$ must be Domain of $g(x)$. Therefore we take the negative root

$g^{-1}(x) = -\sqrt{\frac{1}{2}(x+2)} + 3$

b. Find the domain of the inverse function.

Domain of $g^{-1}(x)$ is Range of $g(x)$

$x \in [16, \infty)$

c. State the range of the inverse function.

Range of $g^{-1}(x)$ is Domain of $g(x)$

$y \in (-\infty, 0]$

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Question 27

Consider the function, $f(x) = 1 - \sqrt{7 - x}$.

a. Define the inverse function of $f(x)$, using full function mapping notation.

$y = -x^2 + 2x + 6$

Domain of $f^{-1}(x)$ is Range of $f(x)$. Therefore $x \in (-\infty, 1]$

$f: (-\infty, 1] \to \mathbb{R}, f(x) = -x^2 + 2x + 6$

b. Find the point of intersection between $f(x)$ and $f^{-1}(x)$.

Intersection between $f(x)$ and $f^{-1}(x)$ is the intersection between $f^{-1}(x)$ and $y = x$

$$-x^2 + 2x + 6 = x$$

$$x = -2, 3$$

$x \neq 3$ since $x \in (-\infty, 1]$

Intersection at $(-2, -2)$

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Question 28

Find the values of k such that, the graph $f: [0, \infty) \to \mathbb{R}, f(x) = x^2 + k$ and $f^{-1}(x)$ have 2 solutions.

Maximum k value occurs when $y = x^2 + k$ has 1 solution with $y = x$

$x^2 + k = x$

$x^2 - x + k = 0$

$\Delta = (-1)^2 - 4(1)(k)$

$\Delta = 1 - 4k$

$\Delta = 0$

$k = \frac{1}{4}$

Minimum k value occurs when endpoint of $f(x)$ occurs on $y = x$

Endpoint of $f(x)$ is $(0, k)$

$k = 0$

$k \in \left[0, \frac{1}{4}\right)$

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Section C: [2.3] - Functions & Relations Exam Skills (Checkpoints)

Sub-Section [2.3.1]: Restrict Domain such that the Inverse Function Exists

Question 29

For each of the following functions, a domain restriction is given with an endpoint a or b. Determine the minimum value of a or maximum value of b such that, the inverse function, f^{-1}, exists.

a. $f:(-\infty,b]\to\mathbb{R}, f(x)=(x+1)^2-3$

$$a=-1$$

b. $f:[a,\infty)\to\mathbb{R}, f(x)=x^2-4x+7$

$f(x)=(x-2)^2+3$ and so $a=2$

c. $f:[a,\infty)\to\mathbb{R}, f(x)=-x^2+8x-11$

$f(x)=5-(x-4)^2$ and so $a=4$

Space for Personal Notes

Question 30

All the functions in this question are written in a non-standard form.

a. Consider the function:

$$f : (-\infty, a) \to \mathbb{R}, \quad f(x) = \frac{11 + 12x + 3x^2}{x^2 + 4x + 4}$$

Find the maximum value of a such that, $f(x)$ has an inverse.

$\boxed{f(x) = 3 - \frac{1}{(x+2)^2}. \text{ So } a = -2}$

b. Consider the function:

$$g : (a, \infty) \to \mathbb{R}, \quad g(x) = \frac{x^2 + 8x + 18}{x^2 + 8x + 16}$$

Find the minimum value of a such that, $g(x)$ has an inverse.

$\boxed{f(x) = 1 + \frac{2}{(x+4)^2}. \text{ So } a = -4}$

c. Consider the function:

$$h : (a, \infty) \to \mathbb{R}, \quad h(x) = \frac{3x^2 + 6x - 2}{x^2 + 2x + 1}$$

Find the minimum value of a such that, $h(x)$ has an inverse.

$\boxed{f(x) = -3 + \frac{5}{(x+1)^2}. \text{ So } a = -1}$

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Question 31

For each of the following semicircle functions, a domain restriction is given with an endpoint a.

Determine the minimum or maximum value of a such that, the inverse function exists.

a. Consider the semicircle function:

$$f : [-4, a] \to \mathbb{R}, f(x) = \sqrt{4 - (x+2)^2}$$

Find the minimum value of a such that, $f(x)$ has an inverse.

$\boxed{r = 4 \text{ so } a = -2.}$

b. Consider the semicircle function:

$$g : [a, 4] \to \mathbb{R}, \qquad g(x) = 2 - \sqrt{8 + 2x - x^2}$$

Find the maximum value of a such that, $g(x)$ has an inverse.

$g(x) = 2 - \sqrt{9 - (x-1)^2}.$
$r = 3 \text{ so } a = 4 - 3 = 1$

c. Consider the semicircle function:

$$h : [-5, a] \to \mathbb{R}, \qquad h(x) = \sqrt{20 - 16x - 4x^2} + 1$$

Find the maximum value of a such that, $h(x)$ has an inverse.

$h(x) = 2\sqrt{9 - (x+2)^2} + 1.$
So $a = -2$

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Question 32

Consider the function:

$$f : [a, \infty) \to \mathbb{R}, \qquad f(x) = \frac{2x^2 + 8x + 11}{5 + 4x + x^2}$$

Find the maximum value of a such that, $f(x)$ has an inverse.

$$f(x) = 2 + \frac{1}{(x+2)^2 + 1}, \text{ so } a = -2.$$

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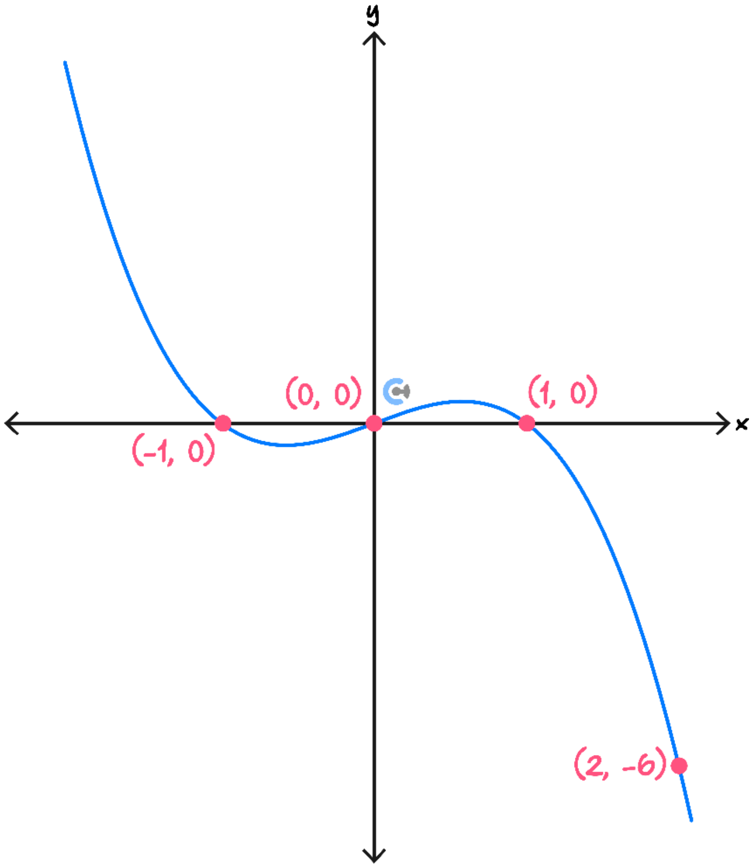
Sub-Section [2.3.2]: Figure Out Possible Rule of a Graph

Question 33

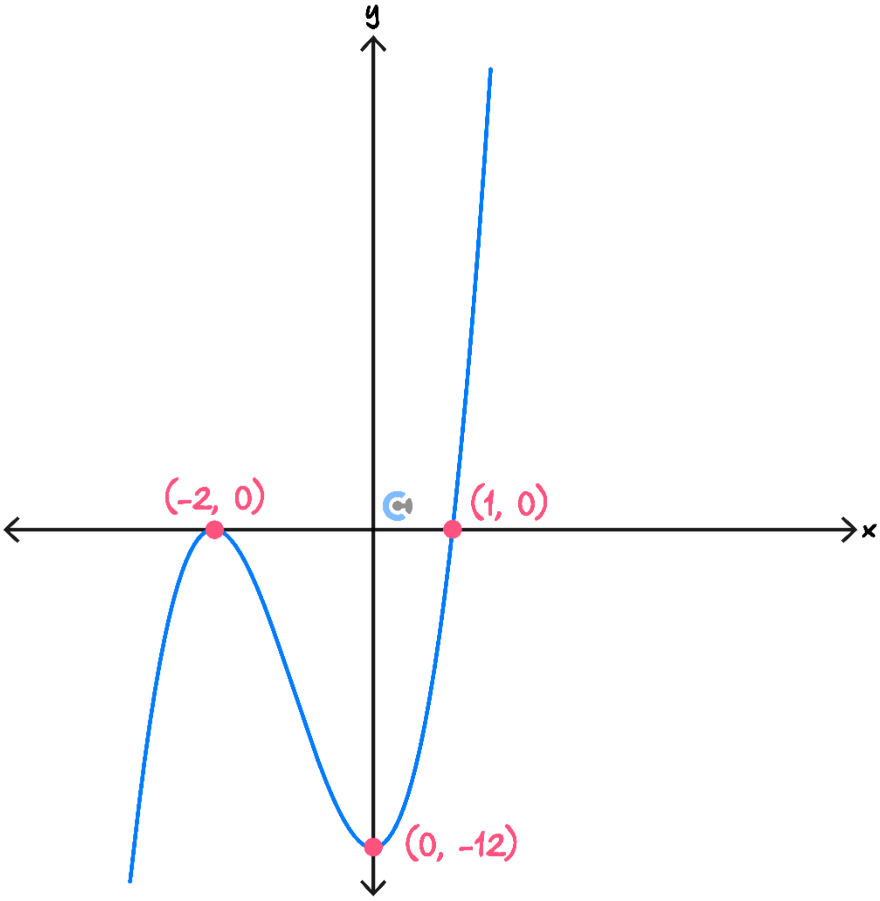
Determine a possible rule for the following graphs:

a.



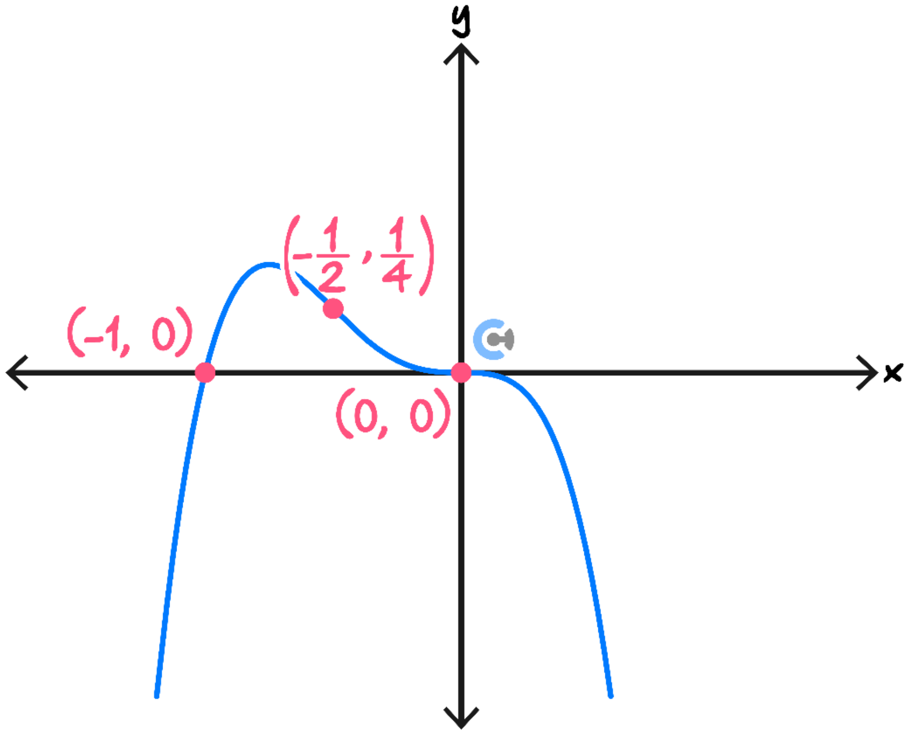
$$\boxed{f(x) = -x(x-1)(x+1)}$$

b.



$$\boxed{f(x) = 3(x+2)^2(x-1)}$$

c.



$$\boxed{f(x) = -4x^3(x+1)}$$

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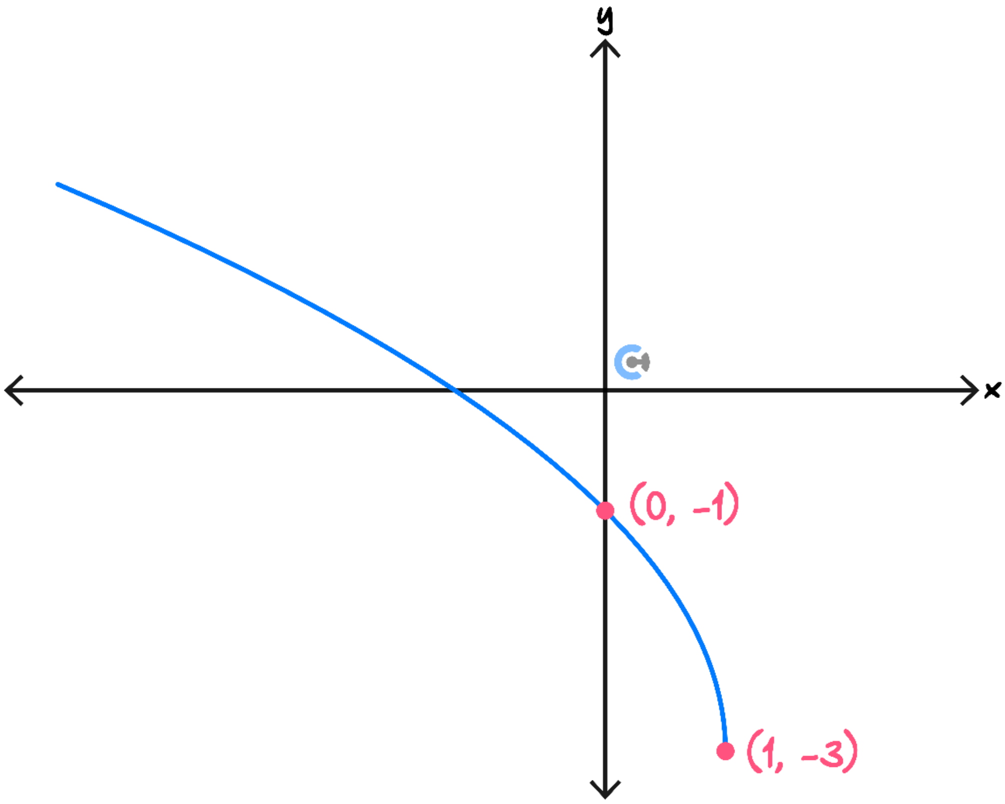
Question 34

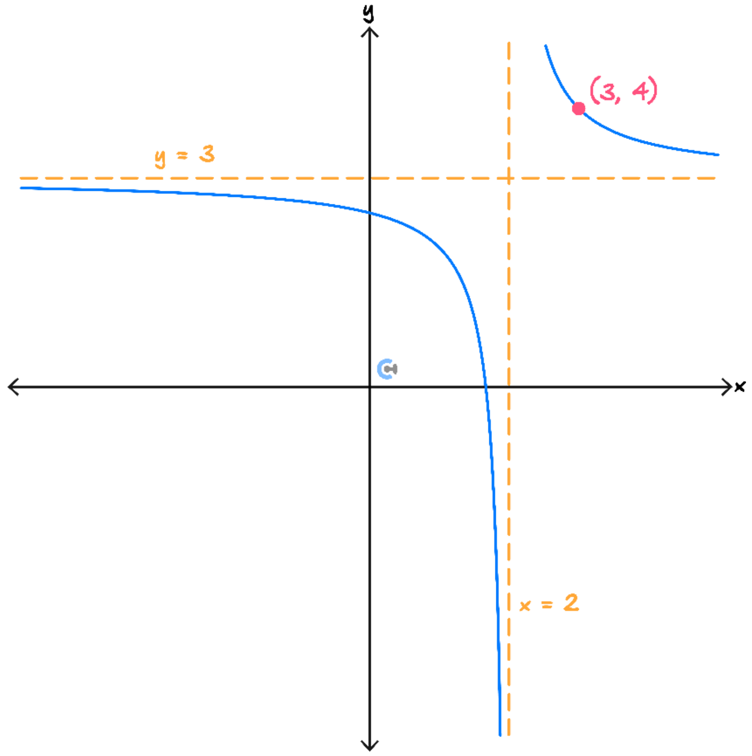
Determine a possible rule for the following graphs:

a.



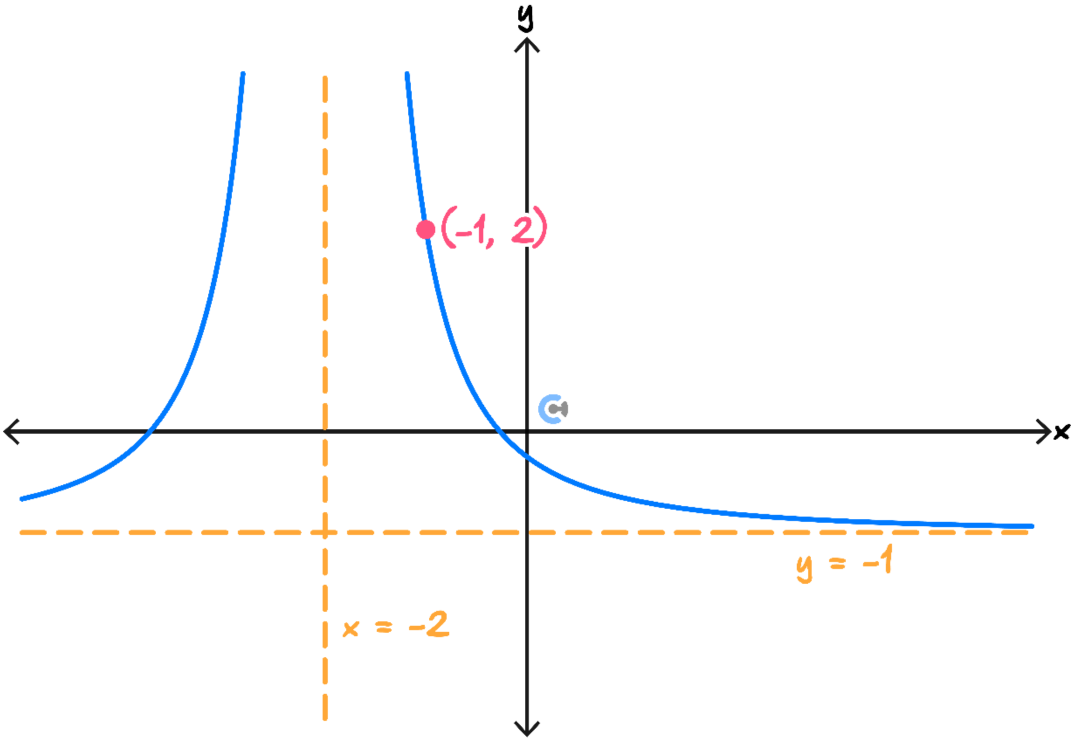
$$f(x) = 2\sqrt{1-x} - 3$$

b.



$$\boxed{f(x) = 3 - \frac{1}{2 - x}}$$

c.



$$\boxed{f(x) = \frac{3}{(x+2)^2} - 1}$$

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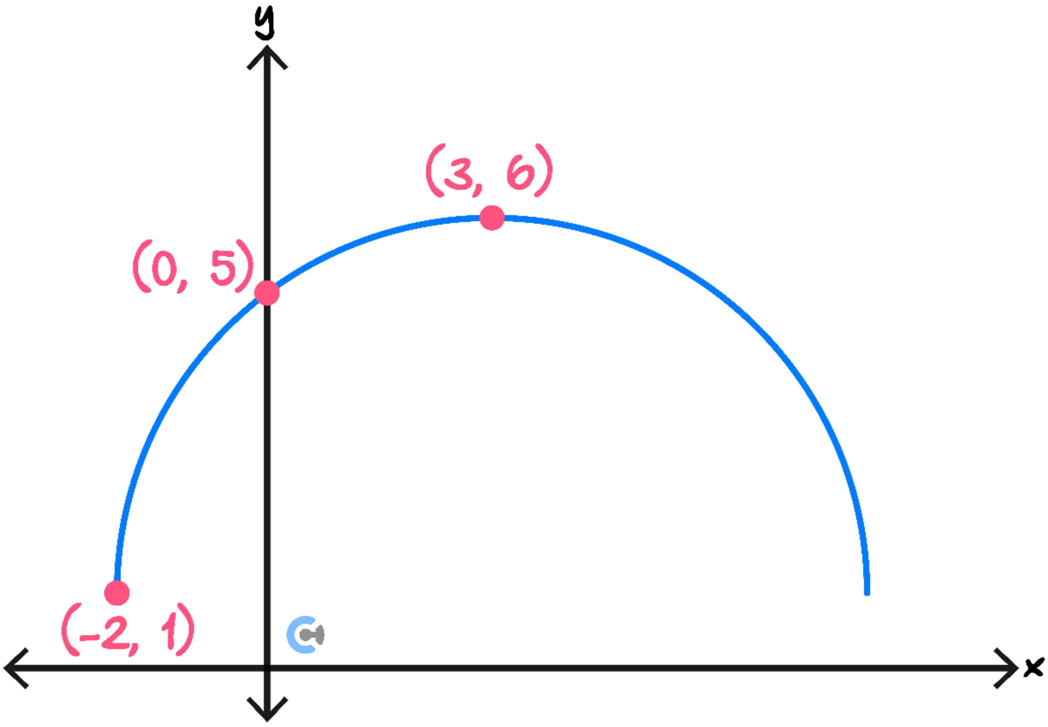
Question 35

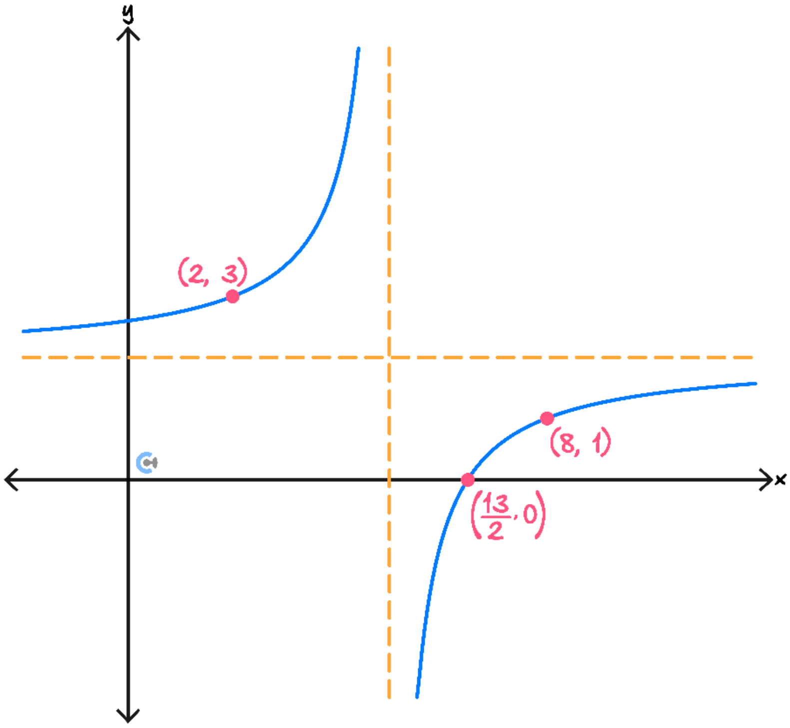
Determine a possible rule for the following graphs:

a.



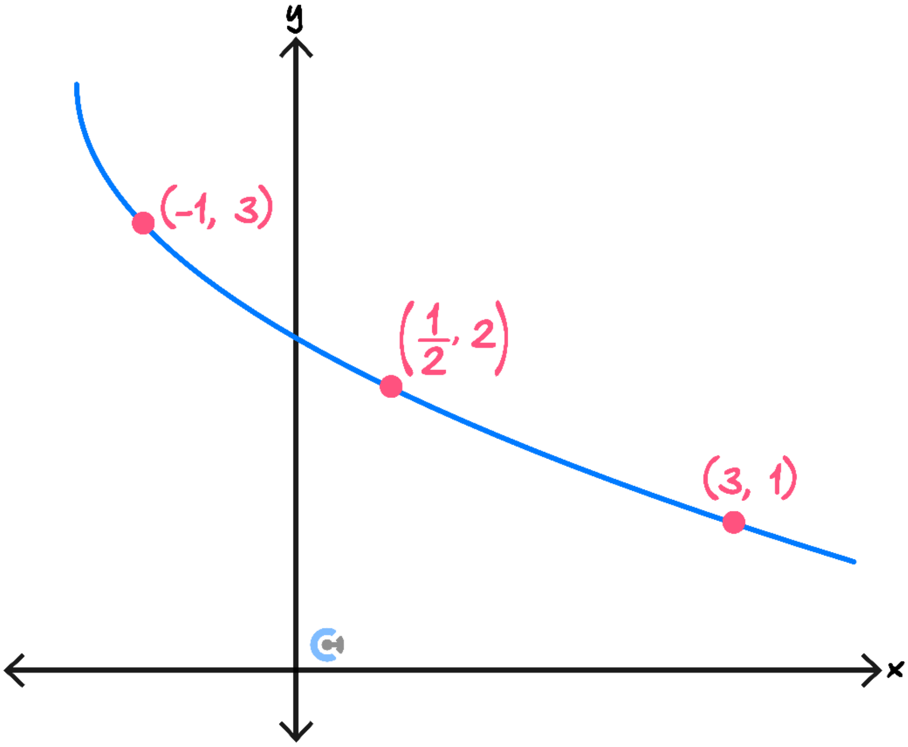
$$\boxed{f(x) = \sqrt{25 - (x-3)^2} + 1}$$

b.



$$f(x) = 2 - \frac{3}{x-5}$$

c.



$$\boxed{f(x) = 4 - \sqrt{3 + 2x}}$$

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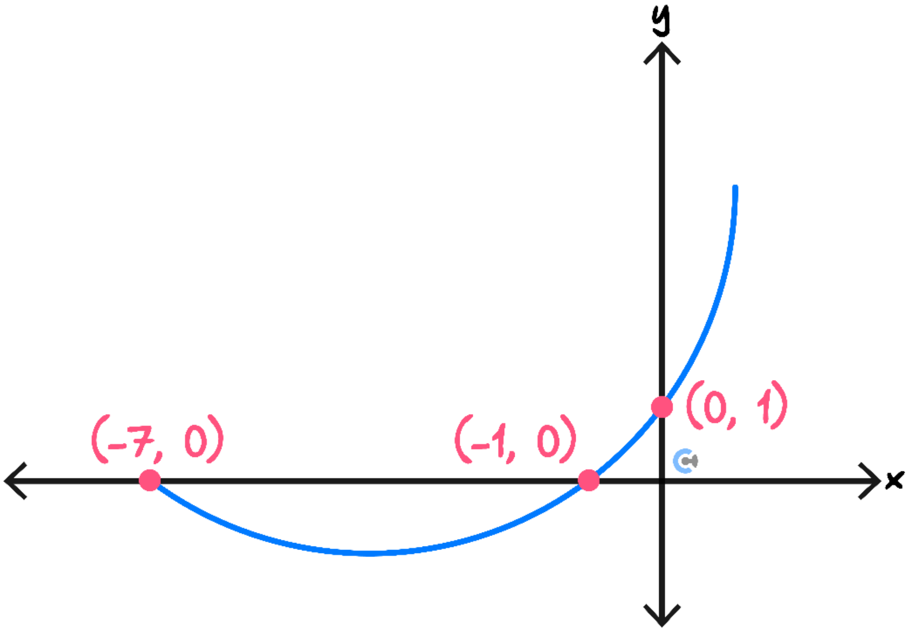
Question 36

Determine a possible function for the following graph:



$$\boxed{f: [-7, 1] \to \mathbb{R}, f(x) = 4 - \sqrt{25 - (x+4)^2}}$$

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Sub-Section [2.3.3]: Solve Number of Solution Problems Graphically

Question 37 Tech-Active.

Consider the function $f(x) = 4x^2 - 4x + 5$.

Determine the real values of k for which $f(x) = k$ has two solutions.

Complete the square / graph to see that $k > 4$

Question 38 Tech-Active.

Consider the function $f(x) = x^3 + 3x^2 - 9x + 2$.

Determine the real values of k for which $f(x) = k$ has:

a. Two solutions.

Graphing the function we see turning points at $(-3, 29)$ and $(1, -3)$, and so from the shape we conclude that $k = -3, 29$.

b. Three solutions.

$$\boxed{-3 < k < 29}$$

Question 39 Tech-Active.

Consider the function $f(x) = x^4 - 8x^3 + 6x^2 + 40x - 14$.

Determine the real values of k for which $f(x) = k$ has:

a. Three solutions.

Graphing the function we see turning points at $(-1, -39)$, $(5, -39)$ and $(2, 42)$, and so from the shape we conclude that $k = 42$.

b. Two solutions.

$$\boxed{k = -39 \text{ or } k > 42.}$$

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Question 40

Consider the function $f(x) = 3x^3 + k$.

Determine the real value of k for which $f(x) = f^{-1}(x)$ has three solutions.

A solution to $f(x) = f^{-1}(x)$ will lie on the line $y = x$.
Hence we just need to solve $f(x) = x \implies 3x^3 - x = -k$.
From the graph of $3x^3 - x$ we see that the equation $3x^3 - x = -k$ has 3 solutions if $-\frac{2}{9} < k < \frac{2}{9}$.
Hence $f(x) = f^{-1}(x)$ has three solutions if $-\frac{2}{9} < k < \frac{2}{9}$.

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Section D: [2.4] - Transformations (Checkpoints)

Sub-Section [2.4.1]: Applying x' and y' Notation to Find Transformed Points, Find Interpretation of Transformations and Altered Order of Transformation

Question 41

Find the coordinates of the image point for the following:

a. The point $(2, 3)$ undergoes a dilation by a factor of 6 from the y-axis, a reflection in the x-axis, followed by a translation 1 unit up.

$x' = 6x,\ y' = -y + 1$

$(x', y') = (6(2), -3 + 1)$

$(x', y') = (12, -2)$

b. The point $(1,5)$ undergoes a translation 2 units left, a dilation by a factor of $\frac{1}{4}$ from the y-axis, a translation 3 units up, followed by a reflection in the x-axis.

$x' = \frac{1}{4}(x - 2),\ y' = -(y + 3)$

$(x', y') = \left(\frac{1}{4}(1 - 2), -(5 + 3)\right)$

$(x', y') = \left(-\frac{1}{4}, -8\right)$

c. The point $(-4,2)$ is dilated by a factor of 3 from the x-axis, translated 1 unit right, reflected in the x-axis, reflected in the y-axis, dilated by a factor of 2 from the y-axis and then translated 5 units down.

$x' = -2(x + 1),\ y' = -3y - 5$

$(x', y') = (-2(-4 + 1), -3(2) - 5)$

$(x', y') = (6, -11)$

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Question 42

Consider the sequence of transformations:

- A dilation by a factor of $\frac{1}{2}$ from the y-axis.
- A reflection in the x-axis.
- A dilation by a factor of 6 from the x-axis.
- A translation 4 units down.
- A translation 1 unit right.
- A translation 9 units up.

a. Rewrite the transformations in the order of a dilation, a translation, a dilation, a reflection and then a translation.

A dilation by a factor of $\frac{1}{2}$ from the y-axis
A translation 1 unit right
A dilation by a factor of 6 from the x-axis
A reflection in the x-axis
A translation 5 units up

b. Express the transformations as a sequence of two translations, followed by two dilations and a reflection.

$x' = \frac{1}{2}x + 1, y' = -6y + 5$

$x' = \frac{1}{2}(x + 2), y' = -6\left(y - \frac{5}{6}\right)$

A translation 2 units right

A translation $\frac{5}{6}$ units left

A dilation by a factor of $\frac{1}{2}$ from the y-axis

A dilation by a factor of 6 from the x-axis

A reflection in the x-axis

c. Express the transformations in the order of a dilation, a translation, a dilation, a translation and then a reflection.

$x' = \frac{1}{2}(x + 2), y' = -(6y - 5)$

A dilation by a factor of $\frac{1}{2}$ from the y-axis

A translation 2 units right

A dilation by a factor of 6 from the x-axis

A translation 5 units down

A reflection in the x-axis

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Question 43

The transformation T is defined as $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (5 - 2x, 6y + 1)$.

a. Evaluate $T(-3,8)$.

$T(-3,8) = (\boxed{5 - 2(-3)}, 6(8) + 1)$
$T(-3,8) = (11,49)$

b. Find the pre-image of $(7, -35)$ under the transformation T.

$T(x, y) = (7, -35)$
$(5 - 2x, 6y + 1) = (7, -35)$
$x = -1, y = -6$
$(-1, -6)$

c. Express T as a sequence of two translations, two dilations and a reflection.

$x' = -2\left(x - \frac{5}{2}\right), y' = 6\left(y + \frac{1}{6}\right)$
Translation $\frac{5}{2}$ units left
Translation $\frac{1}{6}$ units up
Dilation by a factor of 2 from the y-axis
Dilation by a factor of 6 from the x-axis
Reflection in the y-axis

d. Identify a sequence of transformations that maps the point $(-3,8)$ to the image of $(-3,8)$ under T and also maps the point $(1,-2)$ to the point $(23,-1)$.

$x' = ax + b, y' = cy + d$

$(-3,8) \to (11,49)$: $11 = a(-3) + b$, $\boxed{49 = c(8) + d}$

$(1,-2) \to (23,-1)$: $23 = a(1) + b$, $-1 = c(-2) + d$

Solve equations simultaneously: $a = 3,\ b = 20,\ c = 5,\ d = 9$

$x' = 3x + 20, y' = 5y + 9$

A dilation by a factor of 3 from the y-axis

A translation 20 units right

A dilation by a factor of 5 from the x-axis

A translation 9 units up

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Question 44

a. Consider the transformation T described by:

- A translation 2 units left.
- A dilation by a factor of 3 from the x-axis.
- A dilation by a factor of $\frac{1}{4}$ from the y-axis.
- A reflection in the x-axis.
- A translation 1 unit up.
- A reflection in the line $y = x$.
- A translation 4 units right.

i. Apply T to the point $(5,2)$.

$$x' = -3y + 1 + 4, y' = \frac{1}{4}(x-2)$$
$$(x', y') = \left(-3(2) + 1 + 4, \frac{1}{4}(5-2)\right)$$
$$(x', y') = \left(-1, \frac{3}{4}\right)$$

ii. Express T as a sequence of 2 dilations followed by 2 reflections and then 2 translations.

A reflection in $y = x$ will swap x and y values
$x' = -3y + 5, y' = \frac{1}{4}x - \frac{1}{2}$
A dilation by a factor of 3 from the x-axis
A dilation by a factor of $\frac{1}{4}$ from the y-axis
A reflection in $y = x$
A reflection in the y-axis
A translation 5 units right
A translation $\frac{1}{2}$ units left

b. Consider the transformation S described by:

- A dilation by a factor of 2 from the x-axis.
- A reflection in the y-axis.
- A dilation by a factor of $\frac{1}{3}$ from the y-axis.
- A reflection in the line $y = 4$.
- A translation 5 units down.
- A translation 1 unit right.

i. S can also be defined $S : \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (ax + b, cy + d)$. Find the values of a, b, c and d.

A reflection in the line $y = 4$ can be treated as a translation 4 units down, followed by a reflection in the x axis and then a translation 4 units up

$x' = -\frac{1}{3}x + 1, y' = -(2y - 4) + 4 - 5$

$x' = -\frac{1}{3}x + 1, y' = -2y + 3$

$a = -\frac{1}{3}, b = 1, c = -2, d = 3$

ii. Hence, evaluate $S(-2,4)$.

$$S(-2,4) = \left(-\frac{1}{3}(-2) + 1, -2(4) + 3\right)$$

$$S(-2,4) = \left(\frac{5}{3}, -5\right)$$

c. A point (x, y) undergoes the transformations T followed by S. Find the image point.

$$T(x,y) = \left(-3y + 5, \frac{1}{4}x - \frac{1}{2}\right)$$
$$S\left(-3y + 5, \frac{1}{4}x - \frac{1}{2}\right) = \left(-\frac{1}{3}(-3y + 5) + 1, -2\left(\frac{1}{4}x - \frac{1}{2}\right) + 3\right)$$
$$S\left(-3y + 5, \frac{1}{4}x - \frac{1}{2}\right) = \left(y - \frac{2}{3}, -\frac{1}{2}x + 4\right)$$
$$(x', y') = \left(y - \frac{2}{3}, -\frac{1}{2}x + 4\right)$$

d. Given that the image point from **part c.** is $(-4, 6)$, find the pre-image.

$$(-4,6) = \left(y - \frac{2}{3}, -\frac{1}{2}x + 4\right)$$
$$-4 = y - \frac{2}{3}, 6 = -\frac{1}{2}x + 4$$
$$y = -\frac{10}{3}, x = -4$$
$$\left(-4, -\frac{10}{3}\right)$$

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Sub-Section [2.4.2]: Find Transformed Functions

Question 45

Find the resultant function when:

a. $y = x^2$ is dilated by a factor of 2 from the y-axis, reflected in the x-axis, translated 3 units up and translated 1 unit left.

$$x' = 2x - 1, y' = -y + 3$$
$$x = \frac{1}{2}(x' + 1), y = -(y' - 3)$$
$$-(y' - 3) = \left(\frac{1}{2}(x' + 1)\right)^2$$
$$y' = -\frac{1}{4}(x' + 1)^2 + 3$$

b. $y = \frac{1}{x}$ is reflected in the y-axis, translated 3 units up, dilated by a factor of 2 from the x-axis, dilated by a factor of $\frac{1}{4}$ from the y-axis and translated 2 units right.

$$x' = -\frac{1}{4}x + 2, y' = 2(y + 3)$$
$$x = -4(x' - 2), y = \frac{1}{2}(y' - 6)$$
$$\frac{1}{2}(y' - 6) = \frac{1}{-4(x' - 2)}$$
$$y' = -\frac{1}{2(x - 2)} + 6$$

c. $y = \sqrt{x}$ is translated 3 units down, translated 5 units right, reflected in the y-axis, dilated by a factor of 3 from the x-axis, dilated by a factor of 2 from the y-axis, and reflected in the x-axis.

$$x' = -2(x+5),\ y' = -3(y-3)$$
$$x = -\frac{1}{2}x' - 5,\ y = -\frac{1}{3}(y' - 9)$$
$$-\frac{1}{3}(y'-9) = \sqrt{-\frac{1}{2}x' - 5}$$
$$y' = -3\sqrt{-\frac{1}{2}x' - 5} + 9$$

Question 46

Find the resultant function when:

a. $y = -2(x+5)^2 + 1$ is dilated by a factor of $\frac{1}{3}$ from the x-axis, translated 4 units right, translated 1 unit down, reflected in the y-axis and dilated by a factor of 2 from the y-axis.

$$x' = -2(x+4),\ y' = \frac{1}{3}y - 1$$
$$x = -\frac{1}{2}x' - 4,\ y = 3(y'+1)$$
$$3(y'+1) = -2\left(-\frac{1}{2}x' - 4 + 5\right)^2 + 1$$
$$y' = -\frac{1}{6}(x'-2)^2 - \frac{2}{3}$$

b. $y = \frac{2}{(5-x)^2} + 7$ is reflected in the x-axis, translated 2 units up, dilated by a factor of 3 from the y-axis, reflected in the y-axis, translated 4 units right, and dilated by a factor of $\frac{1}{4}$ from the x-axis.

$$x' = -3x + 4, \; y' = \frac{1}{4}(-y + 2)$$
$$x = -\frac{1}{3}(x' - 4), \; y = -4\left(y' - \frac{1}{2}\right)$$

$$-4\left(y' - \frac{1}{2}\right) = \frac{2}{\left(5 - \left(-\frac{1}{3}x' - 4\right)\right)^2} + 7$$
$$y' = -\frac{9}{(x' + 11)^2} - \frac{5}{4}$$

c. $y = 4 - 2(x+1)^3$ is translated 4 units right, dilated by a factor of 3 from the x-axis, reflected in the y-axis, translated 5 units up, reflected in the x-axis and dilated by a factor of 2 from the y-axis.

$$x' = -2(x + 4), \; y' = -(3y + 5)$$
$$x = -\frac{1}{2}x' - 4, \; y = -\frac{y'+5}{3}$$
$$-\frac{y'+5}{3} = 4 - 2\left(-\frac{1}{2}x' - 4 + 1\right)^3$$
$$y' = \boxed{-\frac{3}{4}}(x' + 6)^3 - 17$$

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Question 47

Find the resultant function when:

a. $(x-2)^2+(y+5)^2=9$ is dilated by a factor of 3 from the y-axis, reflected in the x-axis, translated 4 units up, translated 1 unit left and dilated by a factor of 3 from the x-axis.

$x' = 3x - 1,\ y' = 3(-y+4)$
$x = \frac{1}{3}x' + \frac{1}{3},\ y = -\frac{1}{3}y' + 4$
$\left(\frac{1}{3}x' + \frac{1}{3} - 2\right)^2 + \left(-\frac{1}{3}y' + 4 + 5\right)^2 = 9$
$(x-5)^2 + (y-27)^2 = 81$

b. $y = 2x^2 + 3x - 6$ is reflected in the y-axis, dilated by a factor of 4 from the x-axis, translated 5 units down, translated 1 unit right, dilated by a factor of $\frac{1}{2}$ from the y-axis.

$x' = \frac{1}{2}(-x+1),\ y' = 4y - 5$
$x = -2x' + 1,\ y = \frac{y'+5}{4}$
$\frac{y'+5}{4} = 2(-2x'+1)^2 + 3(2x'+1) - 6$
$y' = 32x' - 8x' - 9$

c. $x = -\sqrt{-y^2 + 6y + 15} + 4$ is translated 2 units down, dilated by a factor of 3 from the x-axis, reflected in the y-axis, translated 5 units left, dilated by a factor of $\frac{1}{4}$ from the y-axis, translated 5 units up, reflected in the y-axis and dilated by a factor of 2 from the x-axis.

$$x' = -\frac{1}{4}(-x - 5),\ y' = 2(3(y-2)+5)$$

$$x = 4\left(x' - \frac{5}{4}\right),\ y = \frac{y'}{6} + \frac{1}{3}$$

$$4\left(x' - \frac{5}{4}\right) = -\sqrt{-\left(\frac{y'}{6} + \frac{1}{3}\right)^2 + 6\left(\frac{y'}{6} + \frac{1}{3}\right) + 15} + 4$$

$$x' = \frac{-\sqrt{-y^2 + 32y + 608}}{24} + \frac{9}{4}$$

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Question 48

a. When the graph $y = 6 - 2(x+1)^2$ undergoes the transformation T, described as:

- A translation 4 units right.
- A dilation by a factor of 2 from the y-axis.
- A translation 4 units down.
- A reflection in the y-axis.
- A dilation by a factor of 3 from the x-axis.

It is mapped onto an equation $y = a(x-h)^2 + k$, where $a, h, k \in \mathbb{R}$.

Find the values of a, h and k.

$$x' = -2(x+4),\ y' = 3(y-4)$$
$$x = -\frac{1}{2}x' - 4,\ y = \frac{y'+12}{3}$$
$$\frac{y'+12}{3} = 6 - 2\left(-\frac{1}{2}x' - 4 + 1\right)^2$$
$$y' = -\frac{3}{2}(x+6)^2 + 6$$
$$a = -\frac{3}{2}, b = -6, k = 6$$

b. The graph $y = 6 - 2(x+1)^2$ can also be mapped to the same equation from **part a.**, by a sequence of 2 dilations, a reflection and 2 translations. Describe this sequence of transformations.

$x' = -2x - 8,\ y' = 3y - 12$
A dilation by a factor of 2 from the y-axis.
A dilation by a factor of 3 from the x-axis.
A reflection in the y-axis.
A translation 8 units left.
A translation 12 units down.

c. Find the pre-image, that when undergoes the transformation T, results in the equation $y = 6 - 2(x+1)^2$.

$$x' = -2x - 8,\ y' = 3(y - 4)$$
$$3(y-4) = 6 - 2(-2x - 8 + 1)^2$$
$$y = \frac{-4}{3}(2x^2 + 14x + 23)$$

d. The graph of $y = 6 - 2(x+1)^2$ undergoes the transformation T, followed by a dilation by a factor of 2 from the x-axis, a reflection in the line $x = 6$, a reflection in the line $y = x$, and a translation 1 unit up. Find the image equation.

$x' = 2y, y' = -(x-6) + 6 + 1$

$y = \frac{1}{2}x', x = -y' + 13$

$\frac{x'}{2} = -2(-y' + 13 + 1)^2 + 6$

$x' = -4(-y' + 14)^2 + 12$

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Sub-Section [2.4.3]: Find Transformations from Transformed Function (Reverse Engineering)

Question 49

Find the sequence of transformations that map:

a. $y = x^2$ to $y = -3(x+1)^2 + 7$.

$\frac{y'-7}{-3} = (x'+1)^2$

$y = \frac{y'-7}{-3}, x = x' + 1$

$y' = -3y + 7, x' = x - 1$

Dilation by a factor of 3 from the x-axis

Reflection in the x-axis

Translation 7 units up

Translation 1 unit down

b. $y = \frac{1}{x}$ to $y = \frac{3}{5-2x} + 6$.

$\frac{y'-6}{3} = \frac{1}{5-2x'}$

$y = \frac{y'-6}{3}, x = 5 - 2x'$

$y' = 3y + 6, x' = -\frac{1}{2}x + \frac{5}{2}$

Dilation by a factor of 3 from the x-axis

Translation 6 units up

Dilation by a factor of $\frac{1}{2}$ from the y-axis

Reflection in the y-axis

Translation $\frac{5}{2}$ units right

c. $y = \sqrt{x}$ to $y = 1 - \frac{\sqrt{4-3x}}{2}$.

$-2(y'-1) = \sqrt{4-3x'}$
$y = -2(y'-1), x = 4-3x'$
$y' = -\frac{1}{2}y + 1, x' = -\frac{1}{3}x + \frac{4}{3}$
Dilation by a factor of $\frac{1}{2}$ from the x-axis
Reflection in the x-axis
Translation 1 unit up
Dilation by a factor of $\frac{1}{3}$ from the y-axis
Reflection in the y-axis
Translation $\frac{4}{3}$ units right

Question 50

Find the sequence of transformations that map:

a. $y = 4(x+8)^3 - 5$ to $y = 5 - 2(6x-1)^3$.

$\frac{y+5}{4} = (x+8)^3$
$\frac{y'-5}{-2} = (6x'-1)^3$
$\frac{y+5}{4} = \frac{y'-5}{-2}, x+8 = 6x'-1$
$y' = -\frac{1}{2}y + \frac{5}{2}, x' = \frac{1}{6}x + \frac{3}{2}$
Dilation by a factor of $\frac{1}{2}$ from the x-axis
Reflection in the x-axis
Translation $\frac{5}{2}$ units up
Dilation by a factor of $\frac{1}{6}$ from the y-axis
Translation $\frac{3}{2}$ units right

b. $y = 3\sqrt{16-(x+1)^2} + 5$ to $y = 1 - 2\sqrt{16-(3x+5)^2}$.

$\frac{y-5}{3} = \sqrt{16-(x+1)^2}$

$\frac{y'-1}{-2} = \sqrt{16-(3x'+5)^2}$

$\frac{y-5}{3} = \frac{y'-1}{-2}, x+1 = 3x'+5$

$y' = \boxed{-\frac{2}{3}y + \frac{13}{3}}$ $x' = \frac{1}{3}x - \frac{4}{3}$

Dilation by a factor of $\frac{2}{3}$ from the x-axis

Reflection in the x-axis

Translation $\frac{13}{3}$ units up

Dilation by a factor of $\frac{1}{3}$ from the y-axis

Translation $\frac{4}{3}$ units left

c. $y = \frac{3}{(4-2x)^2} + 7$ to $y = -\frac{6}{(x+1)^2} + 5$.

$\frac{y-7}{3} = \frac{1}{(4-2x)^2}$

$\frac{y'-5}{-6} = \frac{1}{(x'+1)^2}$

$\frac{y-7}{3} = \frac{y'-5}{6}, 4-2x = x'+1$

$y' = 2y - 9, x' = -2x + 3$

Dilation by a factor of 2 from the x-axis

Translation 9 units left

Dilation by a factor of 2 from the y-axis

Reflection in the y-axis

Translation 3 units right

Space for Personal Notes

Question 51

a. The function $y = -2(3(x-1))^4 + 5$ undergoes a sequence of 2 transformations, a reflection, and 2 dilations to become the graph $y = 6(2-x)^4 - 1$.

$\frac{y-5}{-2} = (3(x-1))^4$

$\frac{y'+1}{6} = (x'-2)^4$

$\frac{y-5}{-2} = \frac{y'+1}{6}$, $3(x-1) = x' - 2$

$y' = -3\left(y - \frac{14}{3}\right)$, $x' = 3(x - \frac{1}{3})$

Translation $\frac{14}{3}$ units down

Translation $\frac{1}{3}$ units left

Reflection in the x-axis

Dilation by a factor of 3 from the x-axis

Dilation by a factor of 3 from the y-axis

b. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + b, y + c)$ maps the equation $y = 11 + 5(x+3)^2$ onto the equation $y = 20(x-6)^2 + 9$. Find the values of a, b and c.

$\frac{y-11}{5} = (x+3)^2$

$\frac{y'-9}{5} = (2(x'-6))^2$

$\frac{y-11}{5} = \frac{y'-9}{5}$, $x + 3 = 2(x'-6)$

$y' = y - 2$, $x' = \frac{1}{2}x + \frac{15}{2}$

$a = \frac{1}{2}$, $b = \frac{15}{2}$, $c = -2$

c. The graph $y = \frac{\sqrt{6x-4}}{3} + 2$ is mapped onto $y = 5 - 2\sqrt{-1-x}$ by a sequence of 2 dilations and 2 reflections, followed by a translation.

$\frac{3(y-2)}{2} = \sqrt{\frac{3}{2}x - 1}$

$\frac{y'-5}{-2} = \sqrt{-x'-1}$

$\frac{3(y-2)}{2} = \frac{y'-5}{-2}, \boxed{\frac{3}{2}x - 1} = -x' - 1$

$y' = -3y + 11, x' = \boxed{-\frac{3}{2}x}$

Dilation by a factor of 3 from the x-axis

Dilation by a $\boxed{\text{factor of } \frac{3}{2}}$ from the y-axis

Reflection in the x-axis

Reflection in the y-axis

Translation 11 units up

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Question 52

Find the sequence of transformations that map:

a. $y = x^2 - 4x + 6$ onto $y = -2x^2 + 10x - 7$.

$y = (x-2)^2 + 2$
$y - 2 = (x-2)^2$
$y' = -2\left(x' - \frac{5}{2}\right)^2 + \frac{11}{2}$

$\frac{y' - \frac{11}{2}}{-2} = \left(x' - \frac{5}{2}\right)^2$

$y - 2 = \frac{y' - \frac{11}{2}}{-2}, x - 2 = \left(x' - \frac{5}{2}\right)$

$y' = -2y + \frac{9}{2}, x' = x + \frac{1}{2}$

Dilation by a factor of 2 from x-axis.
Reflection in the x-axis.
Translation $\frac{9}{2}$ units up.
Translation $\frac{1}{2}$ units right.

b. $y = 2\sqrt{(x+4)^2 + 1} - 5$ onto $y = 3 - \sqrt{(2x-6)^2 + 9}$.

$\frac{y+5}{2} = \sqrt{(x+4)^2 + 1}$

$y' = 3 - 3\sqrt{\left(\frac{2}{3}x' - 2\right)^2 + 1}$

$\frac{y'-3}{-3} = \sqrt{\left(\frac{2}{3}x' - 2\right)^2 + 1}$

$\frac{y+5}{2} = \frac{y'-3}{-3}, x + 4 = \frac{2}{3}x' - 2$

$y' = -\frac{3}{2}y - \frac{9}{2}, x' = \frac{3}{2}x + 9$

Dilation by a factor of $\frac{3}{2}$ from the x-axis
Reflection in the x-axis
Translation $\frac{9}{2}$ units down
Dilation by a factor of $\frac{3}{2}$ from the y-axis
Translation 9 units right

c. $f : [-4, \infty) \to \mathbb{R}, f(x) = -x^2 - 8x + 9$ onto $g : (-\infty, 5] \to \mathbb{R}, g(x) = 2x^2 - 20x + 13$.

$y = -(x+4)^2 + 25$
$-(y-25) = (x+4)^2$
$y' = 2(-(x'-5))^2 - 37$
$\frac{y'+37}{2} = (-(x'-5))^2$
$-(y-25) = \frac{y'+37}{2}, x+4 = -(x'-5)$
$y' = -2y + 13, x' = -x + 1$
Dilation by a factor of 2 from the x-axis
Reflection in the x-axis
Translation 13 units up
Reflection in the y-axis
Translation 1 units right

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Section E: [2.5] - Transformations Exam Skills (Checkpoints)

Sub-Section [2.5.1]: Apply Quick Method to Find Transformations

Question 53

Find the image of the graph of $y = x^2$ under the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (1 - 2x, y + 5)$.

Apply the transformation $x \to 1 - 2x$ in an opposite way, replacing x with $\frac{1-x}{2}$. After applying y-axis transformations as well, we get: $y = \left(\frac{1-x}{2}\right)^2 + 5$.

Question 54

Describe a sequence of transformations that maps the graph of $y = x^3$ onto the graph of $y = 2(3x + 2)^3 - 3$.

In the equation x is replaced with $3x + 2$, so we apply those transformations in reverse;

- Translate 2 units left.
- Dilate by a factor of $\frac{1}{3}$ from the y-axis.

Then the y-transformations as normal;

- Dilate by a factor of 2 from the x-axis.
- Translate 3 units down.

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Question 55

Find the image of the graph of $y = \log_2(x)$ under the following sequence of transformations:

- A dilation by a factor of 3 from the x-axis, followed by,
- A translation of 2 units left and 3 units up, followed by,
- A reflection in the y-axis, followed by,
- A dilation by a factor of 5 from the y-axis.

The last 3 transformations apply to x, and applying them in reverse gives:

$$x \to \frac{1}{5}x \to -\frac{1}{5}x \to -\frac{1}{5}x + 2$$

Applying the y-axis transformation in order gives:

$$y \to 3y + 3$$

As such, the rule for the image of our graph after the transformations is:

$$y = 3\left(\log_2\left(-\frac{1}{5}x + 2\right)\right) + 3$$

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Sub-Section [2.5.2]: Find Opposite Transformations

Question 56

Describe a sequence of transformations that maps the graph of $y = 4(x-2)^2 - 3$ onto the graph of $y = x^2$.

1. Translate 2 units left.
2. Translate 3 units up
3. Dilate by a factor of $\frac{1}{4}$ from the x-axis.

Question 57

The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = \left(2x + 3, \frac{1}{3}y - 4\right)$ maps the graph of $y = f(x)$ onto the graph of $y = x^3$.

Find the rule of f.

Let (x', y') be the image of any point (x, y) under T on the graph of $y = f(x)$.

We can substitute $x' = 2x + 3$ and $y' = \frac{1}{3}y - 4$ into the equation $y' = (x')^3$ to get:

$$\frac{1}{3}y - 4 = (2x+3)^3 \to y = f(x) = 3(2x+3)^3 + 12$$

Question 58

The following sequence of transformations maps the graph of f onto the graph of $y = \sqrt{x}$, for $x \in (2, \infty)$:

A dilation by a factor of 3 from the x-axis, followed by,

A translation of 2 units left and 4 units up, followed by,

A reflection in both the x-axis and the y-axis.

State the rule and domain of f.

From the transformation, we can see that:
$(x, y) \to (x, 3y) \to (x - 2, 3y + 4) \to (2 - x, -3y - 4)$
Let (x', y') be the image of any point (x, y) under T on the graph of $y = f(x)$.
Therefore $y' = \sqrt{x'}$, and as such substituting the transformed values of y and x (y' and x'), we get:

$$-3y - 4 = \sqrt{2 - x} \to y = f(x) = -\frac{\sqrt{2 - x}}{3} - \frac{4}{3}$$

Now to get the domain of f, we simply use the equation
$x' > 2 \to 2 - x > 2 \to x < 0$ therefore, the domain of f is $(-\infty, 0)$.

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Sub-Section [2.5.3]: Apply Transformations of Functions to Find its Domain, Range, Transformed Points

Question 59

The function $f : \mathbb{R} \to \mathbb{R}$ has a range of $[2, \infty]$.

The transformation, $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (5 - 2x, 3 + y)$ maps the graph of f onto the graph of g. State the domain and range of g.

Apply T to both the domain and range.
Since, x is any real number $5 - 2x$ can be any real number
Since, $y \geq 2$, as a result $y + 3 \geq 5$.
Therefore, $\text{dom}(g) = \mathbb{R}$ and $\text{ran}(g) = [5, \infty]$.

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Question 60

The function $f : (-\infty, -1) \to \mathbb{R}$ has a range of $(-2, \infty)$.

Describe a sequence of transformations that maps the graph of f onto a graph of a function with a domain of $[0, \infty]$ and a range of $(-\infty, 2)$.

Both functions swap ∞ signs therefore a reflection in both axes are required;

- Reflect in the x-axis.
- Reflect in the y-axis.

After applying the above the domain is now $[1, \infty]$ and the range is now $(-\infty, 2)$. Therefore;

- Translate 1 unit left (to fix domain).

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Question 61

Consider the function, $f : [-2, \infty] \to R, f(x) = 3\sqrt{x+2} - 5$.

The following sequence of transformations maps the graph of f onto the graph of g:

- A reflection in the x-axis, followed by,
- A dilation by a factor of 3 from the x-axis, followed by,
- A dilation by a factor of $\frac{1}{2}$ from the y-axis, followed by,
- A translation of 3 units up and 2 units left.

State the domain and range of g.

The domain of f is $[-2, \infty]$ and the range is $[-5, \infty]$.
Under the above transformations,

$$(x, y) \to (x, -y) \to \left(\frac{1}{2}x, -3y\right) \to \left(\frac{1}{2}x - 2, 3 - 3y\right)$$

Now apply those transformations to the domain and range:

$$\text{dom}(f) = [-2, \infty] \therefore \text{dom}(g) = \left[\frac{1}{2}(-2) - 2, \infty\right) = [-3, \infty)$$

$$\text{ran}(f) = [-5, \infty] \therefore \text{ran}(g) = (-\infty, 3 - 3(-5)] = (-\infty, 18]$$

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Sub-Section [2.5.4]: Find Transformations of Inverse Functions

Question 62

Consider the function, $f: \mathbb{R}\{1\} \to \mathbb{R}, f(x) = \frac{2}{x-1} + 4$. The transformation, $T: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (x + a, y + b)$ maps the graph of f onto the graph of its inverse function. Find the values of a and b.

The horizontal asymptote of f is $y = 4$, whilst the horizontal asymptote of f^{-1} is $y = 1$.
Therefore, a translation of 3 units down is needed for the graph of $f, \therefore b = -3$.
The vertical asymptote of f is $x = 1$, whilst the vertical asymptote of f^{-1} is $x = 4$.
Therefore, a translation of 3 units right is needed for the graph of $f, \therefore a = 3$.

Question 63

Consider the one-to-one functions, $f(x)$ and $g(x)$. The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (3 - x, 2y + 7)$ maps the graph of f onto the graph of g.

Describe a sequence of transformations that maps the graph of f^{-1} onto the graph of g^{-1}.

Swap x and y in the equation of T to get a transformation $S: \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (2x + 7, 3 - y)$ that maps the graph of f^{-1} onto the graph of g^{-1}. It is possible to read off a sequence of transformations from here using DRT;

- A dilation by a factor of 2 from the y-axis then,
- A reflection in the x-axis then,
- A translation of 7 units right and 3 units up.

Question 64

Let $f: [1, \infty] \to \mathbb{R}, f(x) = 3x^2 - 6x + 8$ and $g: [-3, \infty] \to \mathbb{R},\ g(x) = \sqrt{x\ +\ 3} + 4$.

Describe a sequence of transformations that maps the graph of f onto the graph of g^{-1}.

First find the rule for g^{-1} by solving $g(y) = x$ for y;
$\sqrt{y+3} + 4 = x \to x - 4 = \sqrt{y+3} \to y = (x-4)^2 - 3$

The domain of g^{-1} is the range of g which is $[4, \infty)$.
Similarly, the range of g^{-1} is $[-3, \infty]$.

By completing the square of $f(x)$, we get $f(x) = 3(x-1)^2 + 5$
Now we can transform:

1. Translate 5 units down $f_1(x) = 3(x-1)^2$.
2. Dilate by a factor of $\frac{1}{3}$ from the x-axis $f_2(x) = (x-1)^2$.
3. Translate 3 units to the right $f_3(x) = (x-4)^2$.
4. Translate 3 units down $g^{-1}(x) = (x-4)^2 - 3$.

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Sub-Section [2.5.5]: Find Multiple Transformations for the Same Functions

Question 65

Describe a sequence of transformations that map the graph of $f(x) = 4(x-3)^2 + 5$ to $g(x) = x^2$ without using a dilation from the x-axis.

1. Translate 5 units down and 3 units left.
2. Dilate by a factor of 2 from the y-axis.

Question 66

Consider the functions $f(x) = x^2 - 8x + 10$ and $g(x) = 4(x+2)^2 - 5$. Find 2 different sets of transformations, one using a dilation from the x-axis and one using a dilation from the y-axis to map the graph of $f(x)$ to the graph of $g(x)$.

Write $f(x)$ in T. P form: $f(x) = (x-4)^2 - 6$.

Set 1:

1. Dilate by a factor of 4 from x-axis.
2. Translate 19 units up.
3. Translate 6 units left.

Set 2:

1. Translate 4 units right $f_1(x) = x^2 - 6$.
2. Dilate by a factor of $\frac{1}{2}$ from y-axis $f_2(x) = (2x)^2 - 6$.
3. Translate 2 units left $f_3(x) = 4(x+2)^2 - 6$.
4. Translate 1 unit up $g(x) = 4(x+2)^2 - 5$.

Question 67

Consider the functions $f(x) = x^2 + 6x + 7$ and $g(x) = 16x^2 - 32x + 6$. Find 2 different sequences of 3 transformations, one using a dilation from the x-axis and one using a dilation from the y-axis to map the graph of $f(x)$ to the graph of $g(x)$.

Convert both functions to T. P form:
$f(x) = (x+3)^2 - 2,\ g(x) = 16(x-1)^2 - 26$

Set 1:
1. Dilate by a factor of 16 from the x-axis.
2. Translate 6 units up.
3. Translate 4 units right.

Set 2:
1. Translate 7 units right $f_1(x) = (x-4)^2 - 2$.
2. Dilate by a factor of $\frac{1}{4}$ from the y-axis.
$f_2(x) = (4x-4)^2 - 2 = (4)^2(x-1)^2 - 2 = 16(x-1)^2 - 2$
3. Translate 24 units down.

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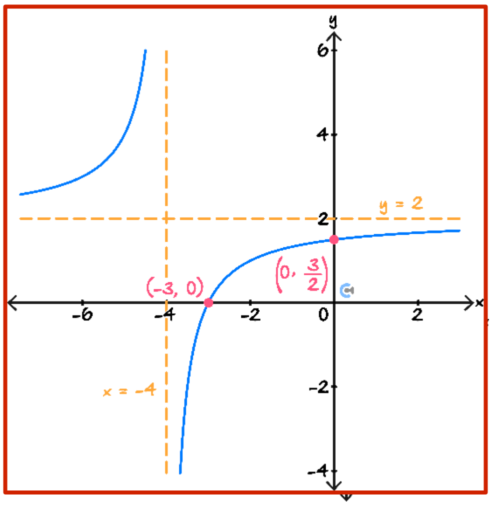
Section F: [2.1-2.5] - Exam 1 Questions (Checkpoints) (14 Marks)

Question 68

Let $f(x) = \frac{2x+6}{x+4}$ be defined on its maximal domain.

a. Sketch the graph of $f(x)$ on the axes below. Label all asymptotes with their equations and axial intercepts with their coordinates.



b. State the domain and range of f^{-1}.

$\text{Dom} = \mathbb{R}\backslash\{2\}$
$\text{Range} = \mathbb{R}\backslash\{-4\}$

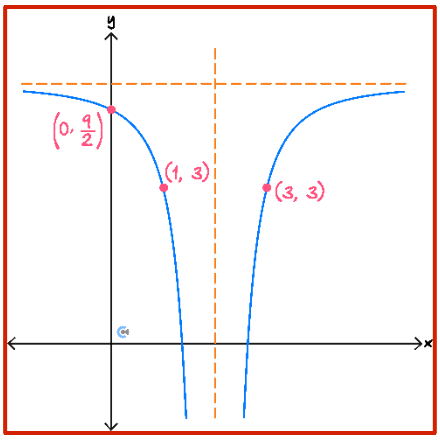
c. Find the values of x for which $f(x) > 1$.

$f(x) = 2 \implies x = -2.$
Hence $x > -2$ or $x < -4.$

Question 69

Consider the function $f : \mathbb{R}\backslash\{h\} \to \mathbb{R},\ \ f(x) = \frac{a}{(x-h)^2} + k$.

The graph of f is drawn below.



a. Show that $a = -2, h = 2$, and $k = 5$.

The graph of f must be symmetrical on the horizontal asymptote $x = h$.
Thus h must be the average of 1 and 3 $\implies h = 2$.
Substituting the point $(1, 3)$ into the equation $y = f(x)$ yields, $3 = a + k$.
Substituting the point $\left(0, \frac{9}{2}\right)$ into the equation $y = f(x)$ yields, $9 = \frac{1}{2}a + 2k$.
Solving simultaneously yields $a = -2$ and $k = 5$.

b. Find the maximal domain of $g(x) = \sqrt{4 - (f(x) - 1)^2}$.

As the maximal domain of $\sqrt{4-(x-1)^2}$ is $[-1, 3]$, we require $f(x) \leq 3$ and $f(x) \geq -1$.
From the graph we see that $f(x) = 3 \implies x = 1, 3$.
If $f(x) = -1$ we see that $\frac{2}{(x-2)^2} = 6 \implies x - 2 = \pm\frac{1}{\sqrt{3}}$.
From the graph of f, we see that $f(x) \in [-1, 3]$ if $x \in \left[1, 2 - \frac{1}{\sqrt{3}}\right] \cup \left[2 + \frac{1}{\sqrt{3}}, 3\right]$.
Hence the maximal domain of $g(x)$ is $\left[1, 2 - \frac{1}{\sqrt{3}}\right] \cup \left[2 + \frac{1}{\sqrt{3}}, 3\right]$.

Question 70

Consider the function $f : [a, \infty) \to \mathbb{R}, f(x) = x^2 - 2x + 2$.

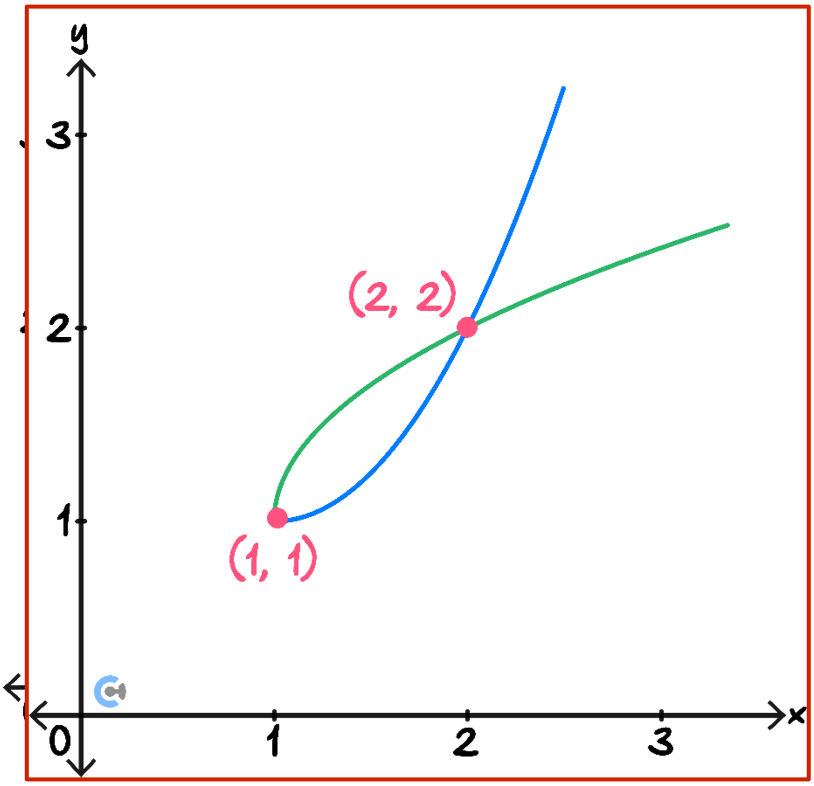
a. Find the smallest value of a for which the inverse function of f, f^{-1} exists.

$f(x) = (x-1)^2 + 1$, hence $a = 1$.

b. State the domain and range of f^{-1}.

Dom $= [1, \infty)$
Range $= [1, \infty)$

c. The graph of $y = f(x)$ is drawn on the axis below, sketch the graph of $y = f^{-1}(x)$ on the same axis, labelling points of intersection with their coordinates.



d. Let $g: [1, \infty) \to \mathbb{R}, g(x) = (x-1)^2 + k$.

i. Find the values of k for which $g(x) = g^{-1}(x)$ has no solutions.

Observe that solving $g(x) = g^{-1}(x)$ is equivalent to solving $g(x) = x \implies (x-1)^2 - x = x^2 - 3x + 1 = -k$.

By completing the square of $x^2 - 3x + 1 = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$, we see that $x^2 - 3x + 1 = -k$ has no solutions if $-k < \frac{5}{4} \implies k > \frac{5}{4}$.

ii. Find the values of k for which $g(x) = g^{-1}(x)$ has two solutions.

$1 < k < \frac{5}{4}$.

iii. Find the values of k for which $g(x) = g^{-1}(x)$ has one solution.

$k = \frac{5}{4}$ or $k < 1$.

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Question 71

Consider the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = \left(\frac{1}{2}x - 3, 4y + 2\right)$.

a. Find the image of the point $(4, 1)$ under T.

$$P' = \left(\frac{1}{2}(4) - 3, 4(1) + 2\right) = (-1, 6)$$

b. Write out what the transformation T does in the order DRT.

1. Dilation by a factor of $\frac{1}{2}$ from the y-axis.
2. Dilation by a factor of 4 from the x-axis.
3. Translation 3 units to the left.
4. Translation 2 units up.

c. Find the image of the curve $y = x^3$ under the transformation T. Give your answer in the form $y = a(x + b)^3 + c$.

$$x' = \frac{1}{2}x - 3 \to x = 2(x' + 3).$$

Therefore, image is $y = 4\left(2(x + 3)\right)^3 + 2 = 32(x + 3)^3 + 2.$

Space for Personal Notes

Question 72

Consider the function $f: \mathbb{R} \to \mathbb{R},\ f(x) = 4x^2 - 16$.

a. Find the coordinates of all axes intercepts of f.

x-intercepts: $(2, 0)$ and $(-2, 0.)$
y-intercept: $(0, -16)$

b. Let the graph of g be a transformation of the graph of f where the transformations have been applied in the following order:

1. Dilation by a factor of 2 from the y-axis.

2. Dilation by a factor of 3 from the x-axis.

3. Translation 6 units to the right.

Find the rule for $g(x)$.

$$g(x) = 3f\left(\frac{1}{2}(x-6)\right) = 3\left(4\left(\frac{1}{2}(x-6)\right)^2 - 16\right) = 3(x-6)^2 - 48$$

c. State the coordinates of the axes intercepts of g.

x-intercepts: $(2, 0)$ and $(10, 0)$.
y-intercept: $(0, 60)$

Question 73

Consider the function $f(x) = 4\sqrt{3x+7} + 2$.

Apply the following transformations to $f(x)$:

1. Dilation by a factor of $\frac{1}{2}$ from the x-axis.

2. Translated 3 units in the positive direction of the y-axis.

3. Reflection in the x-axis.

4. Translated 2 units in the negative direction of the x-axis.

5. Dilated by a factor of 2 from the y-axis.

$$f(x) \to 2\sqrt{3x+7} + 1 \to 2\sqrt{3x+7} + 4 \to -2\sqrt{3x+7} - 4$$

$$\to -2\sqrt{3(x+2)+7} - 4 \to -2\sqrt{3\left(\frac{1}{2}x+2\right)+7} - 4$$

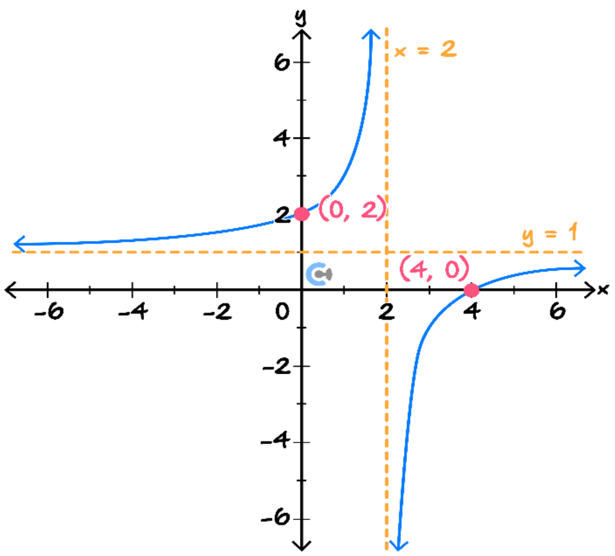
Therefore, the image is $y = -2\sqrt{3\left(\frac{1}{2}x+2\right)+7} - 4 = -2\sqrt{\frac{3}{2}x+13} - 4$

Space for Personal Notes

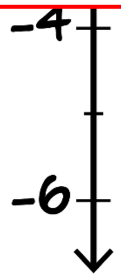
Question 74 (4 marks)

a. Sketch the graph of $y = 1 - \frac{2}{x-2}$ on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates. (3 marks)

Marks	0	1	2	3	Average
%	11	9	24	56	2.3



Most students recognised that the graph was a rectangular hyperbola and presented a neatly drawn curve with branches correctly positioned. Students generally paid attention to curvature and asymptotic behaviour. Asymptotes were sometimes correctly positioned but labelled inaccurately or not at all. The axial intercepts were generally given as coordinates with occasional errors seeing the x-intercept labelled (4,0) but positioned at (3,0) or the y-intercept given as (0,3).



b. Find the values of x for which $1 - \frac{2}{x-2} \geq 3$. (1 mark)

Marks	0	1	Average
%	69	32	0.3

$x \in [1, 2)$

This question, while well attempted, was not done well. Most students attempted to solve algebraically instead of using the graph, and only obtained the lower bound of inequality. Other errors saw students write the interval as $(2,1]$. Others had the values but incorrect brackets.

Question 75 (3 marks)

The transformation $T: R^2 \to R^2$ with rule $T(x, y) = (x + b, ay + c)$, where a, b, and c are integers, maps the graph of $y = 2 - 4x^3$ onto the graph of $y = 1 - 16(x - 2)^3$.

Find the values of a, b and c.

$g(f(x)) = 2 - 4x^3$ to $f(g(x)) = 1 - 16(x-2)^3$

$x' = x + b,\ y' = ay + c$

$\frac{y' - c}{a} = 2 - 4(x' - b)^3$

$y = 2a - 4a(x' - b)^3 + c$ equate to $y = 1 - 16(x-2)^3$

$a = 4,\ b = 2,\ c = -7$

Alternatively, dilation of 4 from the x-axis: $4(2 - 4x^3) = 8 - 16x^3$,

translation of 2 in the positive direction of the x-axis:

$8 - 16(x-2)^3$,

translation of 7 in the negative y direction: $1 - 16(x-2)^3$

$$\text{Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

So $a = 4,\ b = 2,\ c = -7$

Space for Personal Notes

Question 76 (3 marks)

Let $f: R \to R, f(x) = 2e^x + 1$ and let $g: (-2, \infty) \to R, g(x) = \log_e(x+2)$.

Let $T: R^2 \to R^2, T(x, y) = (x + c, y + d)$ and let the graph of the function h be the transformation of the graph of the function g under T.

If $h = f^{-1}$, then find the values of c and d.

$x = 2e^y + 1$

$f^{-1}(x) = \log_e(\frac{x-1}{2})$

$h(x) = \log_e\left(\frac{x-1}{2}\right) = g(x-c) + d = \log_e(x + 2 - c) + d$

$\log_e\left(\frac{x-1}{2}\right) = \log_e(x-1) - \log_e(2) = \log_e(x + 2 - 3) - \log_e(2)$

$c = 3, \quad d = -\log_e(2)$ or $d = \log_e(\frac{1}{2})$

Question 77 (4 marks)

a. Let $f: R\backslash\left\{\frac{1}{3}\right\} \to R, f(x) = \frac{1}{3x-1}$.

Find the rule of f^{-1}. (2 marks)

Marks	0	1	2	Average
%	6	37	57	1.5

$x = \frac{1}{3(f^{-1}(x)) - 1}$

Thus $f^{-1}(x) = \frac{1}{3x} + \frac{1}{3}$ or $f^{-1}(x) = \frac{1+x}{3x}$

This question was well attempted and generally well done; however, in some cases progression to the correct answer was hindered by errors with algebraic manipulation (transposition) or poor use of notation.

b. State the domain of f^{-1}. (1 mark)

Marks	0	1	Average
%	36	64	0.7

Domain = $R \setminus \{0\}$

In general students knew that the domain of f^{-1} = range of f.

c. Let g be the function obtained by applying the transformation T to the function f, where:

$$T(x, y) = (x + c, y + d)$$

and $c, d \in R$.

Find the values of c and d given that $g = f^{-1}$. (1 mark)

Marks	0	1	Average
%	76	24	0.3

$c = -\frac{1}{3}$ and $d = \frac{1}{3}$

This question, while well attempted, was not done well. Some students had the incorrect sign for *c* and *d*. Other students attempted dilations rather than translations as specified by the question.

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