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VCE Mathematical Methods ½
Polynomials Exam Skills [1.6]
Workbook

Outline:

Learning Objectives:

- ☐ **MM12 [1.6.1]** – Solve polynomial inequalities.
- ☐ **MM12 [1.6.2]** – Solve number of solution problems.
- ☐ **MM12 [1.6.3]** – Apply bisection method to approximate x-intercepts.

Section A: Recap

If you were here last week, skip to Section B Warmup Test.

Degree of Polynomial Functions

Degree = Highest Power of the Polynomial

Question 1 5s

State the degree of each polynomial.

a. $x^3 - 4x^2 + 5x + 6$

3

b. $3x + 5x^2 - x^7$

7

Roots of Polynomial Functions

Roots = x-intercept

Space for Personal Notes

Question 2 5s

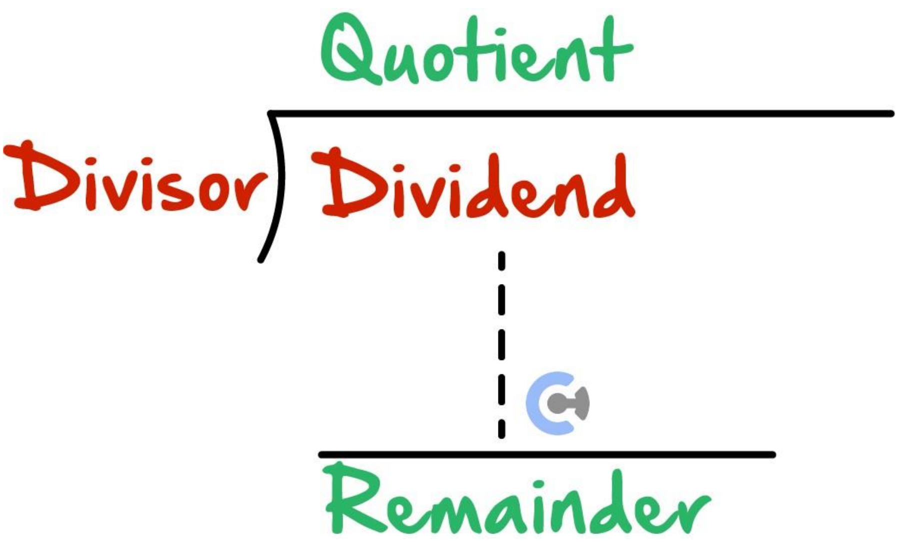
Find the roots of the following polynomial:

$$(x-1)^2(x+3)^4$$

$\therefore x = 1, -3$

Polynomial Long Division

- Division of polynomials:



$$\frac{\textit{Dividend}}{\textit{Divisor}} = \textit{Quotient} + \frac{\textit{Remainder}}{\textit{Divisor}}$$

Question 3

Simplify the following using polynomial long division:

$$\frac{x^3 + x^2 + 2}{x - 3}$$

$$\begin{array}{r l}
 & x^2 + 4x + 12 \\
x-3 & \overline{)\, x^3 + x^2 + 2} \\
 & -(x^3 - 3x^2) \\
 & \quad 4x^2 + 2 \\
 & \quad -(4x^2 - 12x) \\
 & \qquad 12x + 2 \\
 & \qquad -(12x - 36) \\
 & \qquad\quad 38
\end{array}$$

$$= x^2 + 4x + 12 + \frac{38}{x-3} \quad //$$

TIP: Always remember to fill in any missing powers of x in the numerator or denominator with "placeholders" that have a coefficient of 0.

<u>Remainder Theorem</u>

- **Definition**: Finds the remainder of long division without the need for long division.

When $P(x)$ is divided by $(x - \alpha)$, the remainder is $P(\alpha)$

- **Steps:**

  1. Find x values, which makes the divisor equal to 0.
  2. Substitute it into the dividend function.

Question 4 30s

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where, $f(x) = x^3 - 2x^2 + 3x + 1$ and $g(x) = 2x + 4$.

let $g(x) = 0$:

$\therefore x = -2$

$\therefore R = f(-2)$

$= -8 - 8 - 6 + 1$

$= -21$

Factor Theorem

- For every x-intercept, there is a factor:

if $P(\alpha) = 0$ then, $(x - \alpha)$ is a factor of $P(x)$

Question 5 30s

Determine if $x + 2$ is a factor of $P(x) = 2x^3 - 7x^2 + 7x - 2$.

let $x + 2 = 0$:

$\therefore x = -2$

$R = P(-2) = 2(-8) - 7(4) - 14 - 2$

$= -16 - 28 - 14 - 2$

$= -60$

$\therefore (x+2)$ ISN'T a factor of $P(x)$ as $R \neq 0$.

Factorising Cubic Polynomials

- The steps are:
  - Find a single root by trial and error.

    (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

Question 6

Find all the roots of $f(x) = x^3 + 3x^2 - 6x - 8$.

1. Finding a Single Root (T4E):

Sub $x=1$:

$f(1) = 1+3-6-8 = -10$

Sub $x=2$:

$f(2) = 8+12-12-8 = 0$

$\therefore (x-2)$ is a factor

2. Synthetic Division:

2	1	3	-6	-8
	↓	2	10	8
	1	5	4	0

3. Factorise:

$= (x-2)(x^2+5x+4)$

$= (x-2)(x+4)(x+1)$

$\therefore x = 2, -4, -1$

NOTE: When the question asks for all roots, you cannot just factorise and end it there!

Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$Potential\ root = \pm\frac{Factors\ of\ constant\ term\ a_0}{Factors\ of\ leading\ coefficient\ a_n}$$

- If the roots are rational numbers, the roots can only be $\pm\frac{factors\ of\ constant\ term\ a_0}{factors\ of\ leading\ coefficient\ a_n}$.

NOTE: All the roots are part of the suggestion given by the rational root theorem.

Question 7

Find all the roots of $f(x) = 2x^3 - x^2 - 22x - 24$.

1. Finding a Single Root (RRT):

$RRT = \pm \frac{\{1,2,3,4,6,8,12,24\}}{\{1,2\}} = \pm\{1,2,3,4,6,8,12,24,\frac{1}{2},\frac{3}{2}\}$

Sub $x=4$:

$f(4) = 2(4)^3 - (4)^2 - 22(4) - 24$

$= 128 - 16 - 88 - 24 = 0$

$\therefore (x-4)$ is a factor

2. Synthetic Division:

4	2	−1	−22	−24
		8	28	24
	2	7	6	0

3. Factorise:

$= (x-4)(2x^2+7x+6)$

$= (x-4)(2x+3)(x+2)$

4. Find Root:

$\therefore x = 4, -\frac{3}{2}, -2$

Sum and Difference of Cubes

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Question 8

Factorise the following polynomial as much as possible:

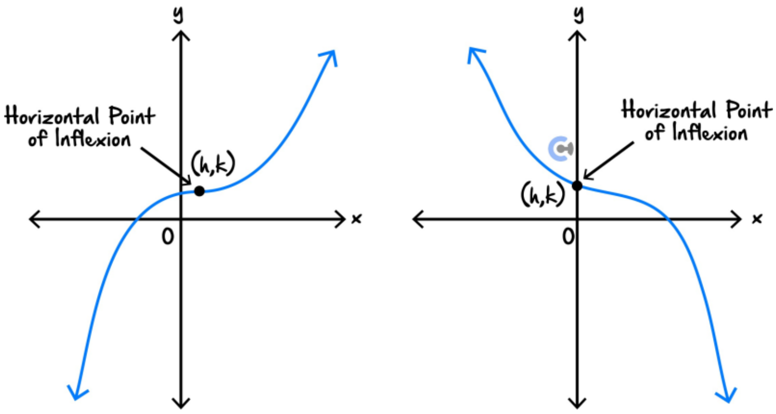
$$8x^3 - 216$$

$= 8(x^3 - 27)$

$= 8(x-3)(x^2+3x+9)$

Graphs of $a(x-h)^n + k$, Where n is an Odd Positive Integer

- All graphs look like a "cubic".



- The point (h, k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

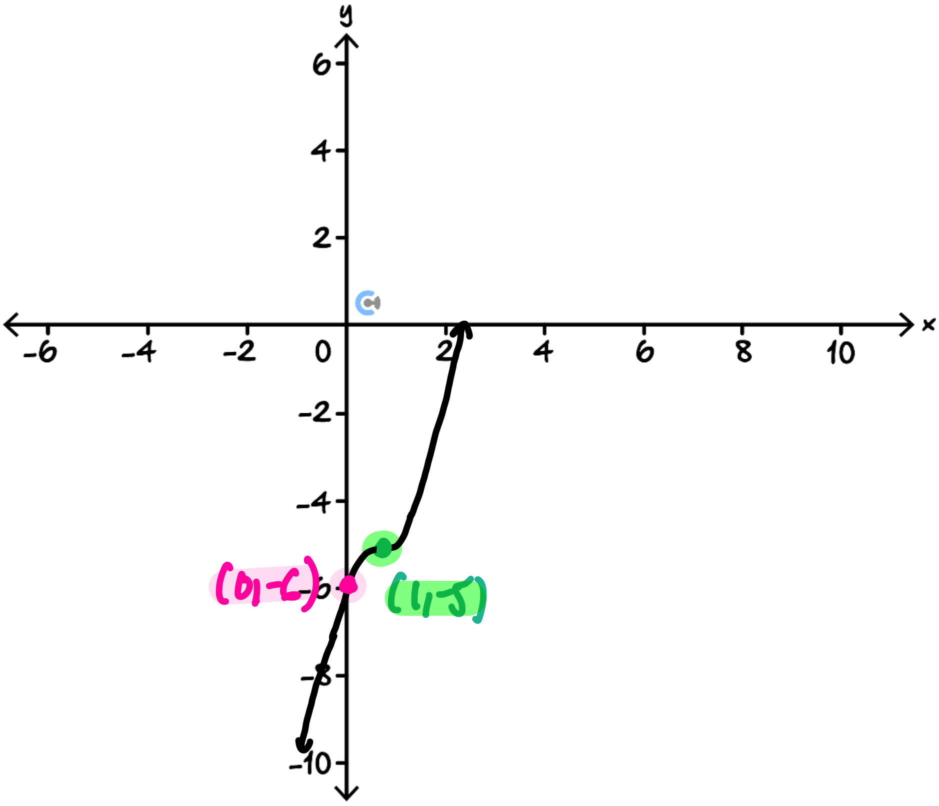
Question 9

POI: (1,-5)

Y-int: $y=(-1)^3-5$

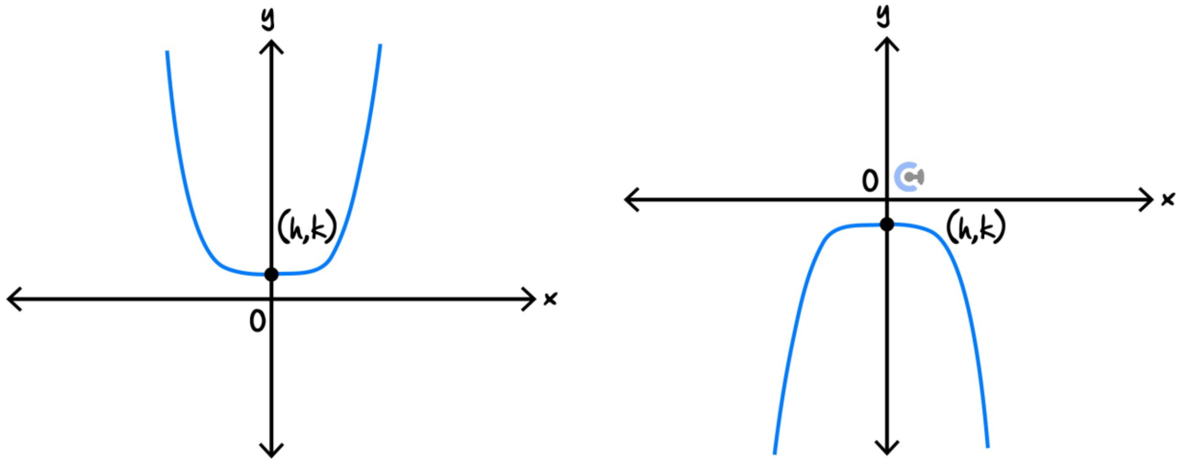
$=-6$

Sketch the graph of $y = (x-1)^3 - 5$ on the axes below.



Graphs of $a(x-h)^n + k$, Where n is an Even Positive Integer

- All graphs look like a "quadratic".



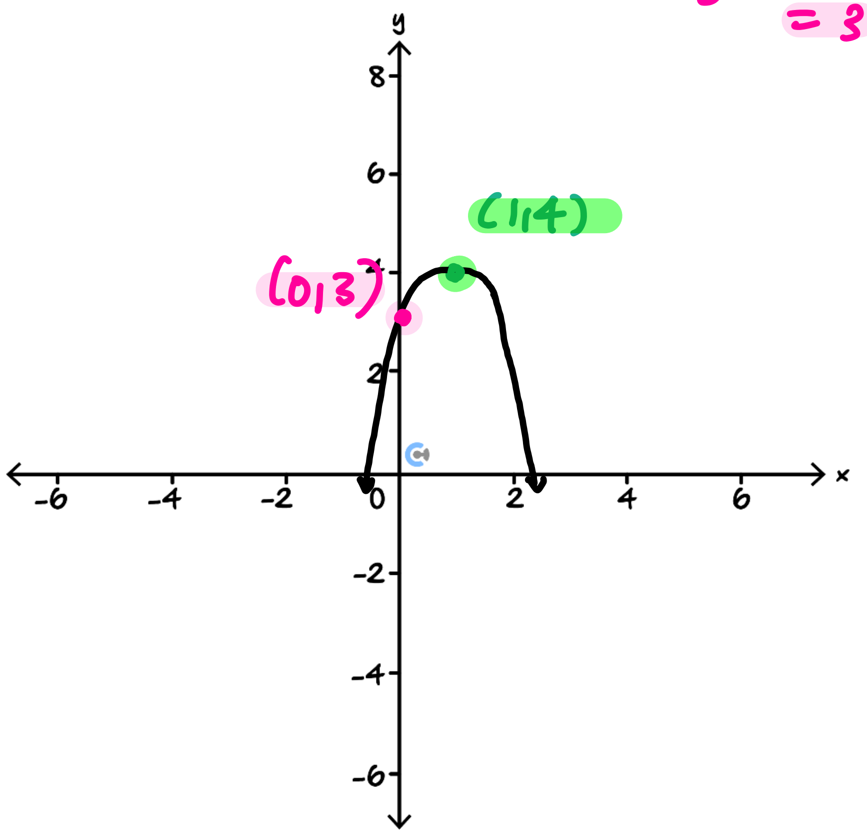
- The point (h, k) gives us the turning point.

Question 10

Sketch the graph of $y = -(x-1)^4 + 4$ on the axes below.

TP: (1,4)

y-int: $y = -(-1)^4 + 4$
$= 3$



Definition

Graphs of Factorised Polynomials

- **Steps:**

1. Plot x-intercepts.

2. Determine whether the polynomial is positive or negative.

3. Use the repeated factors to deduce the shape.

Non-Repeated: Only x-intercept.

Even Repeated: x-intercept and a turning point.

Odd Repeated: x-intercept and a stationary point of inflection.

Space for Personal Notes

Question 11

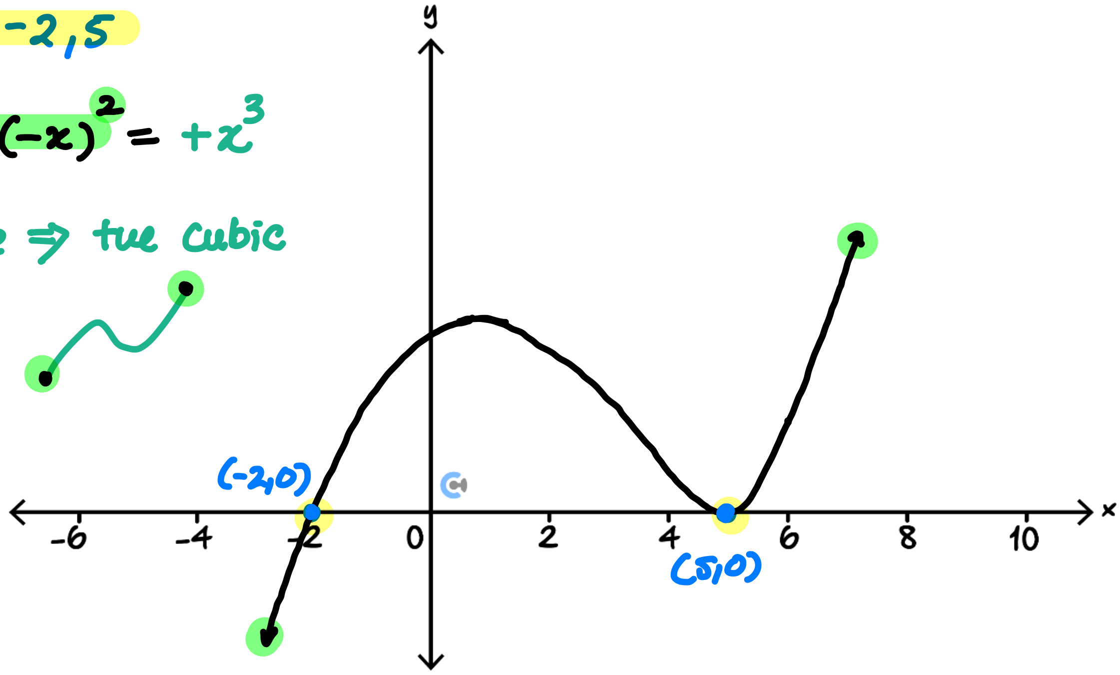
Sketch the graphs of the following functions on the axes provided. Ignore the y-axis scale.

a. $y = (2 + x)(5 - x)^2$

$x = -2, 5$

$x \cdot (-x)^2 = +x^3$

Shape $\Rightarrow$ +ve cubic

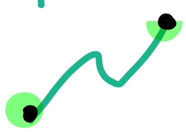


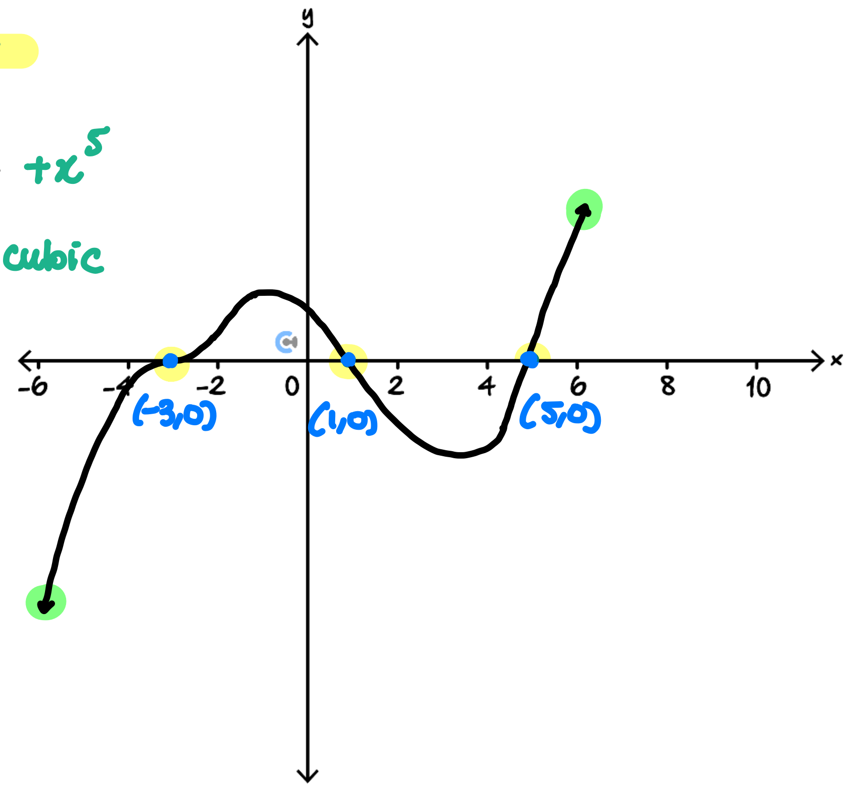
b. $y = (x + 3)^3(x - 1)(x - 5)$

$x = -3, 1, 5$

$x^3 \cdot x \cdot x = +x^5$

Shape $\Rightarrow$ +ve cubic





Section B: Warmup Test

INSTRUCTION: 15 Marks. 15 Minutes Writing.

Question 12 (3 marks)

Consider the function $f(x) = x^3 + ax^2 + bx - 2$. If $x - 1$ is a factor of $f(x)$ and the remainder of $f(x) \div (x - 2)$ is given by 12, find the value(s) of a and b.

$\because f(1) = 0 \Rightarrow 1 + a + b - 2 = 0$

$\because f(2) = 12$ $\quad \boxed{a + b = 1} \ldots (1)$

$8 + 4a + 2b - 2 = 12$

$4a + 2b = 6$

$\boxed{2a + b = 3} \ldots (2)$

$2a + b = 3$ $\quad (2) - (1)$

$-(a + b = 1)$

$\therefore a = 2 \rightarrow \therefore 2 + b = 1$

$\therefore b = -1$

Space for Personal Notes

Question 13 (3 marks)

Solve the following equation for x:

$$2x^3 - 5x^2 = 4x - 3$$

$2x^3 - 5x^2 - 4x + 3 = 0$

1. Finding a Single Root (T&E):

Sub $x=1$:

$f(1) = 2 - 5 - 4 + 3 = -4$

Sub $x=-1$:

$f(-1) = -2 - 5 + 4 + 3 = 0$

$\therefore (x+1)$ is a factor

2. Synthetic Division:

-1	2	-5	-4	3
	$\downarrow$	-2	7	-3
	2	-7	3	0

3. Factorise:

$= (x+1)(2x^2 - 7x + 3)$

$= (x+1)(2x-1)(x-3)$

$\therefore x = -1, \frac{1}{2}, 3$

Space for Personal Notes

Question 14 (3 marks)

Sketch the graph of the following function on the axes below. Label all axis intercepts with their coordinates.

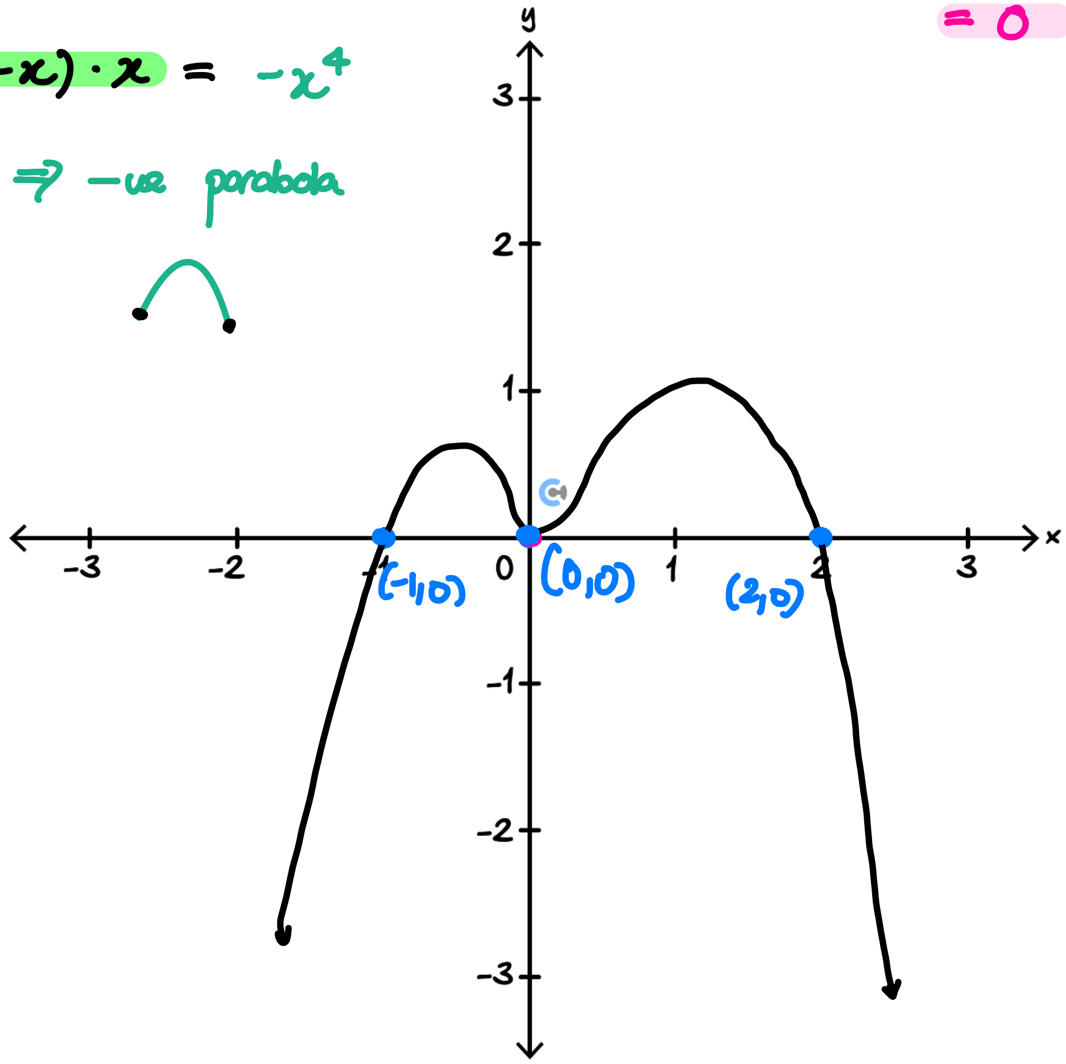
$$y = x^2(2 - x)(x + 1)$$

$x = -1, 0, 2$

$y = 0(2)(1)$
$= 0$

$x^2 \cdot (-x) \cdot x = -x^4$

Shape ⇒ −ve parabola



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Question 15 (6 marks)

Consider the function $f(x) = 2x^3 - 3x^2 - ax + 2$.

It is known that the remainder, when $f(x)$ is divided by $x - 3$, is 20.

a. Show that $a = 3$. (1 mark)

$\because R = f(3) = 20$

$f(3) = 2(27) - 3(9) - 3a + 2 = 20$

$54 - 27 - 3a = 18$

$27 - 3a = 18$

$3a = 9$

$\therefore a = 3$ Q.E.D

b. Hence, solve $f(x) = 0$. (3 marks)

$f(x) = 2x^3 - 3x^2 - 3x + 2$

1. Finding a Single Root (T&E):

Sub $x=1$:

$f(1) = 2 - 3 - 3 + 2 = -2$

Sub $x=-1$:

$f(-1) = -2 - 3 + 3 + 2 = 0$

$\therefore (x+1)$ is a factor

2. Synthetic Division:

$$\begin{array}{c|cccc} -1 & 2 & -3 & -3 & 2 \\ & \downarrow & -2 & 5 & -2 \\ \hline & 2 & -5 & 2 & 0 \end{array}$$

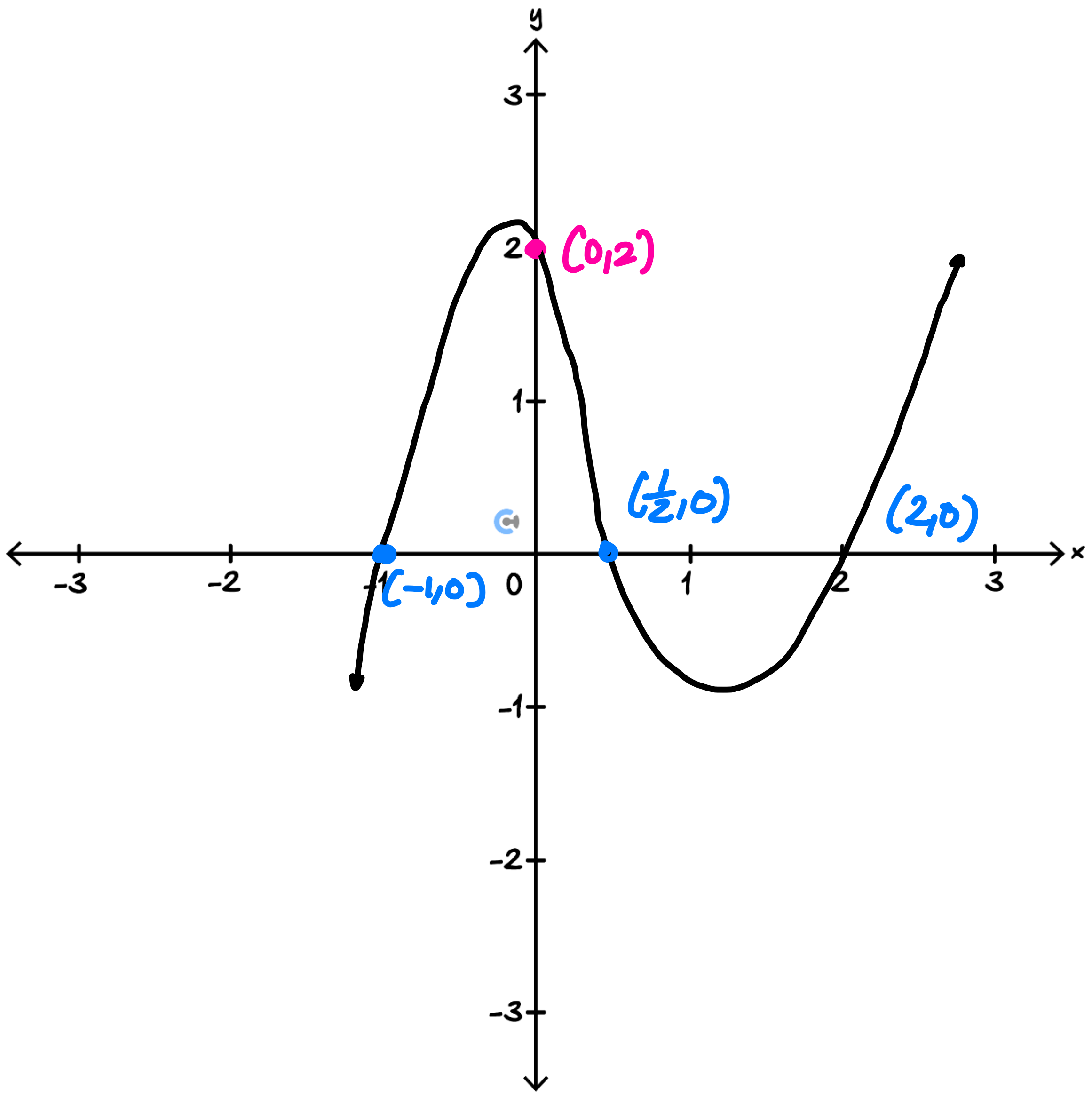
3. Factorise:

$= (x+1)(2x^2 - 5x + 2)$

$= (x+1)(2x-1)(x-2)$

$\therefore x = -1, \frac{1}{2}, 2$

c. Sketch the graph of $y = f(x)$ on the axes below. Label all axis intercepts with coordinates. (2 marks)



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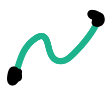
$x = -1, \frac{1}{2}, 2$

$+2x^3$

$y = 2$

Shape ⇒ true cubic



Section C: Polynomials Exam Skills

Sub-Section: Solve Polynomial Inequalities

Context

- We are used to solving polynomial equations, that is, when we put two polynomials together and put an $=$ sign in between them. But sometimes, instead of an $=$ sign, there is an inequality sign between them instead. What do we do in such a situation?

Exploration: Meaning of a Polynomial Equality

- The 'value' of a polynomial is the y value on the graph.
- Hence, the equation $f(x) > 0$ means find where the y values are positive.

Solving the Polynomial Inequality $f(x) > 0$

$f(x) = y$

$f(x) > 0$

$y > 0$

- Steps:

1. Find the x-intercepts.
2. Sketch the polynomial.
3. Shade the places where the y-values are positive.

Space for Personal Notes

Question 16 Walkthrough.

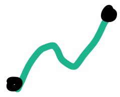
Solve the following inequality for x:

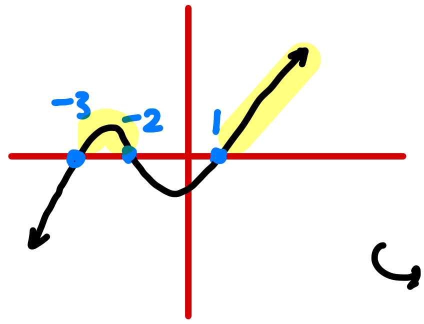
$x = 1, -2, -3$

$$(x-1)(x+2)(x+3) > 0$$

$x \cdot x \cdot x = +x^3$

Shape $\Rightarrow$ +ve cubic





$x > 1$ or $-3 < x < -2$

$\therefore x \in (-3, -2) \cup (1, \infty)$

element of

Sometimes, we have to factorise or move everything to one side.

Question 17 Walkthrough.

Solve the following inequality for x:

$$2x^3 + x^2 - 5x + 4 > 2$$

$2x^3 + x^2 - 5x + 2 > 0$

1. Finding a Single Root (T&E):

Sub $x = 1$:

$f(1) = 2 + 1 - 5 + 2 = 0$

$\therefore (x-1)$ is a factor

2. Synthetic Division:

1	2	1	-5	2
	↓	2	3	-2
	2	3	-2	0

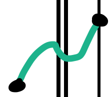
3. Factorise:

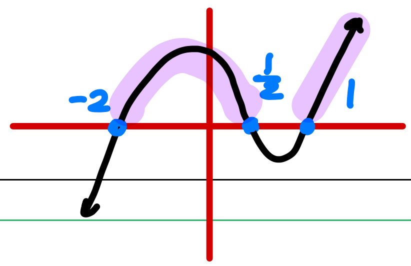
$= (x-1)(2x^2 + 3x - 2)$

$= (x-1)(2x-1)(x+2)$

$\therefore x = 1, \frac{1}{2}, -2$

$2x^3 \Rightarrow$ +ve cubic





$\therefore x \in (-2, \frac{1}{2}) \cup (1, \infty)$

or $-2 < x < \frac{1}{2}$ or $x > 1$

Active Recall: The 'value' of $f(x)$ is its ____ y ____ value.

Question 18

Solve the following polynomial inequalities for x.

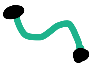
a. $(3-x)(x+4)(x-2) \geq 0$

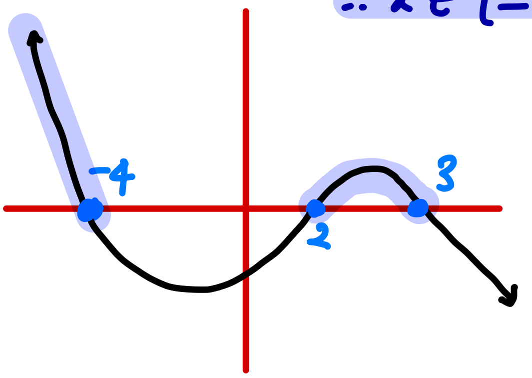
$\therefore x \in (-\infty, 4] \cup [2, 3]$

$x = 3, -4, 2$

$(-x) \cdot x \cdot x = -x^3$

Shape = -ve cubic





b. $2x^3 - 3x^2 - 2x + 6 < 3$

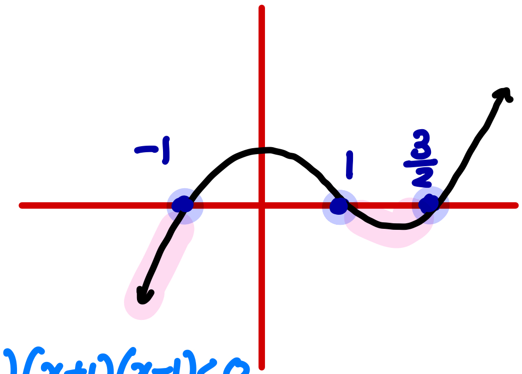
$2x^3 - 3x^2 - 2x + 3 < 0$

$x^2(2x-3) - 1(2x-3) < 0$

a c -b c

$(2x-3)(x^2-1) < 0$

$ac - bc = c(a-b)$



$(2x-3)(x+1)(x-1) < 0$

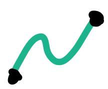
$\therefore x \in (-\infty, -1) \cup (1, \frac{3}{2})$

$\therefore x = \frac{3}{2}, -1, 1$

Space for Personal Notes

$+2x^3$

Shape => +ve cubic



Sub-Section: Solve Number of Solution Problems

When we can factorise a cubic, we can use the discriminant of the remaining quadratic to figure out a number of solutions.

Definition

When does a cubic have n solutions?

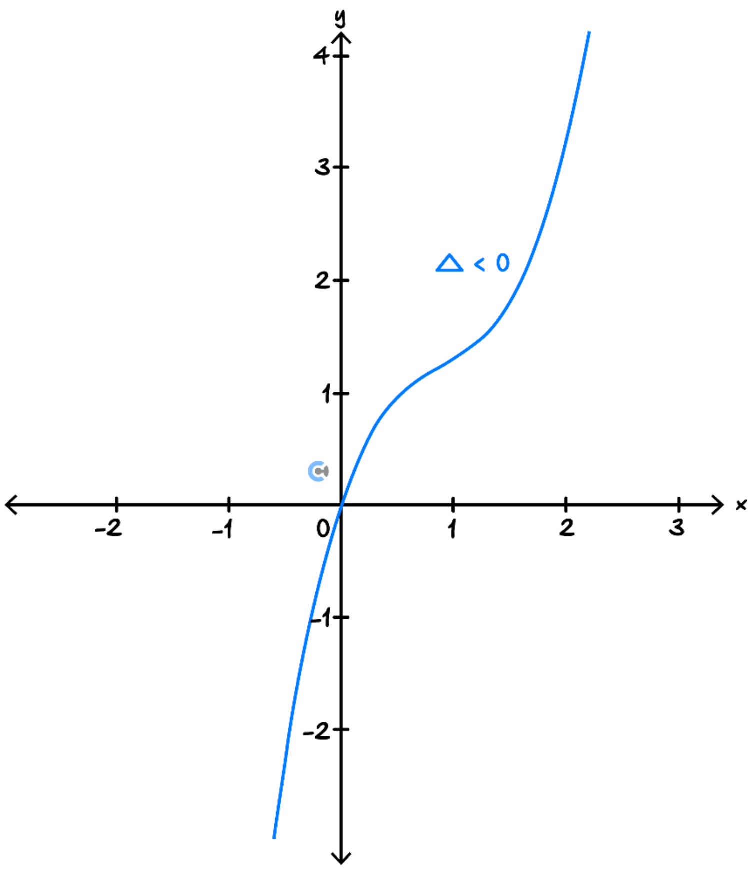
- Steps:

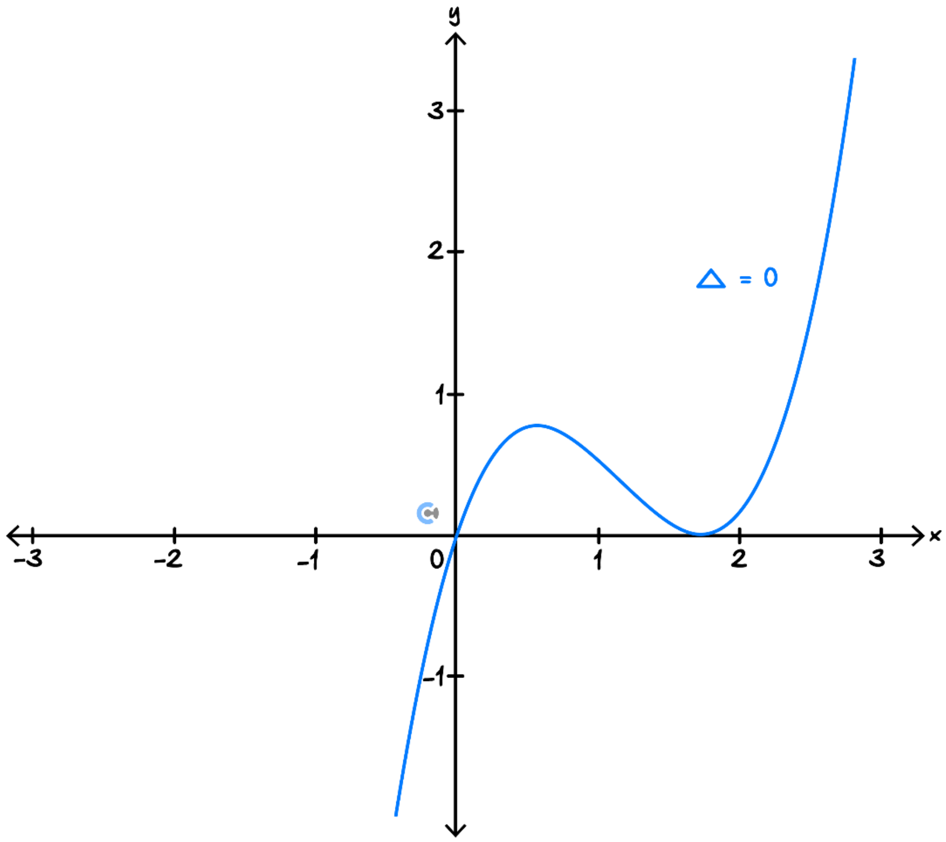
  1. Factorise out the x term.

  2. Since the x term gives 1 solution, use discriminant to find when the quadratic has $n-1$ solutions.

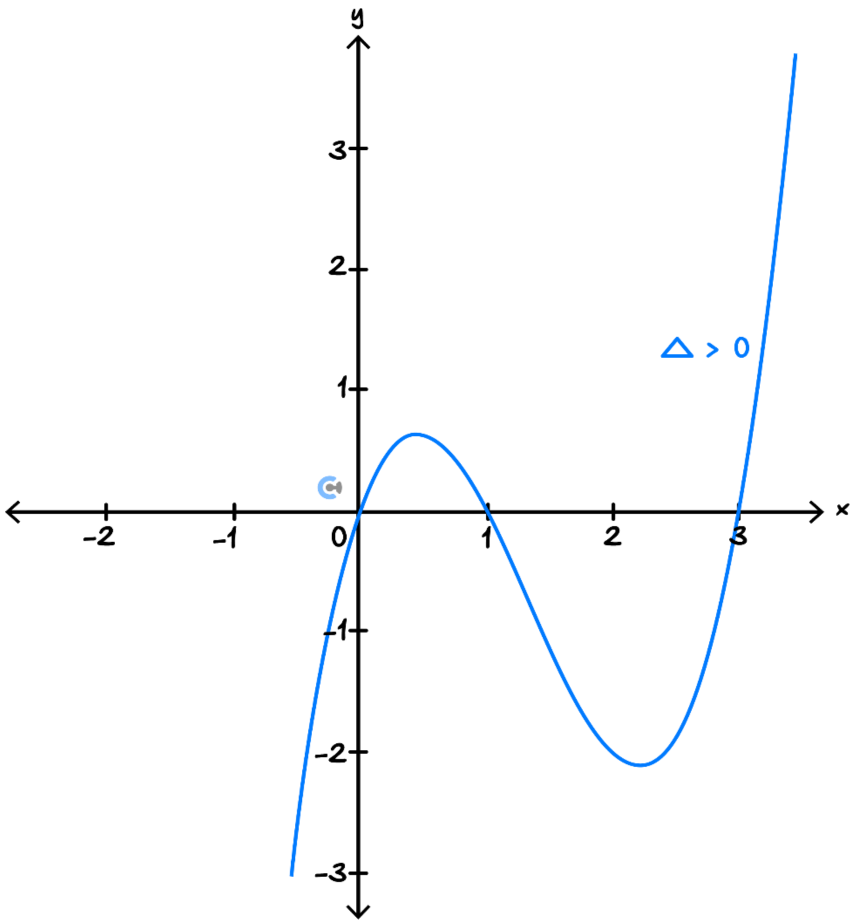
Exploration: What Does the Discriminant Control in a Cubic?

- The discriminant controls where the turning point is!





y
x
3
2
1
-1
-3
-2
0
$\Delta = 0$



y
x
3
2
1
-1
-2
-3
0
$\Delta > 0$

Question 19 Walkthrough.

Consider $f(x) = x^3 - kx^2 + 3x$.

Find the value(s) of k such that $f(x) = 0$ has 2 solutions.

$f(x) = 0$:

$x^3 - kx^2 + 3x = 0$

$x\underbrace{(x^2 - kx + 3)}_{\Delta = 0} = 0$

$x = 0$

(1) (1) = 2

$\Delta = (-k)^2 - 4(1)(3) = 0$

$k^2 - 12 = 0$

$k^2 = 12$

$k = \pm\sqrt{12} = \pm 2\sqrt{3}$

Active Recall: Finding Number of Solutions for a Factorisable Cubic

- Break the cubic down into a linear factor and a quadratic factor.
- Use the discriminant to determine the number of solutions you want the quadratic factor to have.

Question 20 ~~Walkthrough.~~

Consider $f(x) = x^3 - 3kx^2 + 4x$. Find the values of k such that $f(x) = 0$ has 3 solutions.

$f(x) = 0$:

$x^3 - 3kx^2 + 4x = 0$

$x\underbrace{(x^2 - 3kx + 4)}_{\Delta > 0} = 0$

(1) + (2) = 3

$\Delta > 0$:

$(-3k)^2 - 4(1)(4) > 0$

$9k^2 - 16 > 0$

$k^2 > \frac{16}{9}$

$\therefore k > \frac{4}{3}$ or $k < -\frac{4}{3}$

$a^2 < 4$

$-2 < a < 2$

$a^2 > 4$

$a > 2$ or $a < -2$

Section D: Exam 1 Questions (16 Marks)

Question 21 (3 marks)

Consider the function $f(x) = 8x^3 - 216$.

a. Express $f(x)$ in the form $a(x-b)(x^2+cx+d)$ for positive real numbers a, b, c, and d. (2 marks)

$f(x) = 8(x^3 - 27)$
$= 8(x-3)(x^2+3x+9)$

b. Hence, explain why $x = b$ is the only solution to the equation $f(x) = 0$. (1 mark)

$f(x) = 8\underbrace{(x-3)}_{x=3}\underline{(x^2+3x+9)}$

$\hookrightarrow \Delta = (3)^2 - 4(1)(9)$
$= 9 - 36$
$= -27$

$\therefore$ As $\Delta < 0$, $x = 3$ is the only solution for $f(x) = 0$.

Space for Personal Notes

Question 22 (3 marks)

Solve the inequality $2x^3 - 18x < 3x^2 + 8$ for x.

$2x^3 - 3x^2 - 18x - 8 < 0$

1. Finding a Single Root (T&E):

Sub $x = -2$:

$f(-2) = 2(-8) - 3(4) + 36 - 8$
$= -16 - 12 + 28 = 8$

$\therefore (x+2)$ is a factor

2. Synthetic Division:

-2	2	-3	-18	-8
	$\downarrow$	-4	14	8
	2	-7	-4	0

3. Factorise:

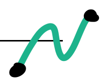
$= (x+2)(2x^2 - 7x - 4)$

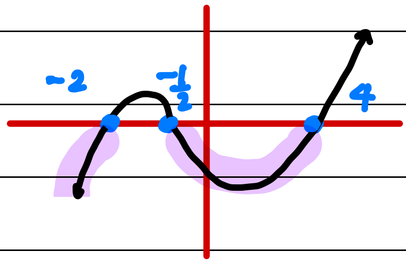
$= (x+2)(2x+1)(x-4)$

$\therefore x = -2, -\frac{1}{2}, 4$

$+2x^3$

Shape: +ve cubic





$\therefore x \in (-\infty, -2) \cup (-\frac{1}{2}, 4)$

Space for Personal Notes

Question 23 (5 marks)

Consider $f(x) = 2x^3 + 4kx^2 + 12x$, where k is a real constant.
Find the values of k such that $f(x) = 0$ has:

a. One solution. (3 marks)

$f(x) = 0$:

$2x^3 + 4kx^2 + 12x = 0$

$2x\underbrace{(x^2 + 2kx + 6)}_{\Delta < 0} = 0$

$(1) + (0) = (1)$

$\Delta = (2k)^2 - 4(1)(6) < 0$

$4k^2 - 24 < 0$

$k^2 - 6 < 0$

$k^2 < 6$

$-\sqrt{6} < k < \sqrt{6}$

b. Two solutions. (1 mark)

$\Delta = 0$

$k^2 = 6 \Rightarrow \therefore k = \pm\sqrt{6}$

c. Three solutions. (1 mark)

$\Delta > 0$

$k^2 > 6 \Rightarrow \therefore k > \sqrt{6}$ or $k < -\sqrt{6}$

Question 24 (5 marks)

Consider the quadratic polynomial:

$$f(x) = x^5 + ax^4 + bx^3 - x^2 + 4x - 4$$

Where a and b are real constants.

- $x - 2$ is a factor of $f(x)$.
- When $f(x)$ is divided by $x - 1$ the remainder is -1.

a. Show that $a = -3$ and $b = 2$. (2 marks)

$R = f(2) = 0 \Rightarrow f(2) = 32 + 16a + 8b - 4 + 8 - 4 = 0$

$R = f(1) = -1$

$32 + 16a + 8b = 0$

$16a + 8b = -32$

$f(1) = 1 + a + b - 1 + 4 - 4 = -1$

$\boxed{2a + b = -4}$ ..①

$\boxed{a + b = -1}$ ..②

$2a + b = -4$

$-(a + b = -1)$ ① − ②

$\therefore a = -3$

$-3 + b = -1$

$\therefore b = 2$

b. Write the function $g(x) = \frac{f(x)}{x^2-x-2}$ in the form $g(x) = C(x) + \frac{B}{x+d}$.

Where $C(x)$ is a cubic polynomial and B, d are real constants. (3 marks)

$$\frac{f(x)}{x^2-x-2} = \frac{x^5-3x^4+2x^3-x^2+4x-4}{(x-2)(x+1)}$$

$$\begin{array}{c|cccccc} 2 & 1 & -3 & 2 & -1 & 4 & -4 \\ & & 2 & -2 & 0 & -2 & 4 \\ \hline & 1 & -1 & 0 & -1 & 2 & 0 \end{array}$$

$$= \frac{\cancel{(x-2)}(x^4-x^3-x+2)}{\cancel{(x-2)}(x+1)} = \frac{x^4-x^3-x+2}{x+1}$$

$$\begin{array}{c|ccccc} -1 & 1 & -1 & 0 & -1 & 2 \\ & & -1 & 2 & -2 & 3 \\ \hline & 1 & -2 & 2 & -3 & 5 \end{array}$$

$$= x^3-2x^2+2x-3+\frac{5}{x+1}$$

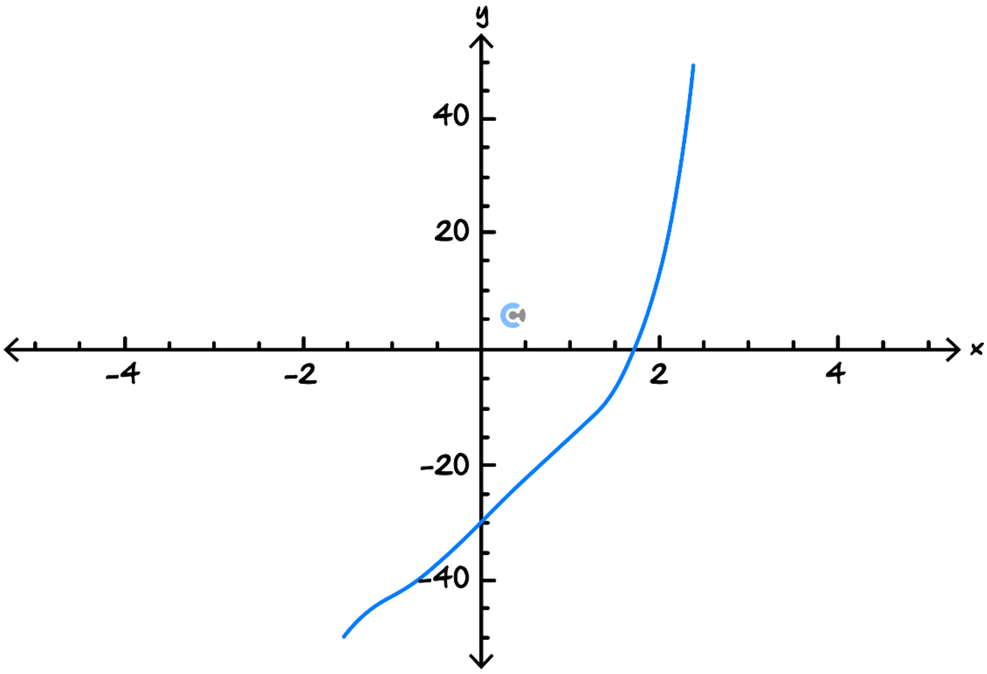
Space for Personal Notes

Section E: Tech Active Exam Skills

Sub-Section: Apply Bisection Method to Approximate x-Intercepts

Context: Bisection Method

- We know how to solve the equation $x^2 - 4 = 0$ easily.
- We've also learnt how to solve the cubic equations using factor theorem as well.
- What if the equation is too hard to solve?



$$x^5 - 3x^3 + x^2 + 16x - 30 = 0$$

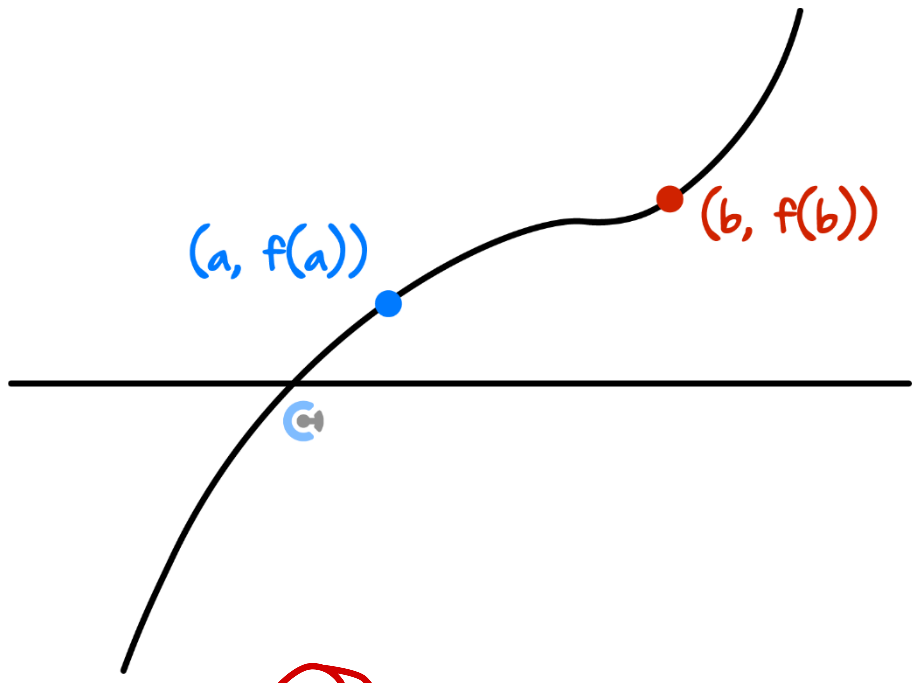
- Bisection method can be used to approximate the answer to any polynomial equations.

Discussion: How do we tell if two points are on the opposite side of the x-axis (one below and one above the x-axis)?

above x-axis: $+y$
below x-axis: $-y$

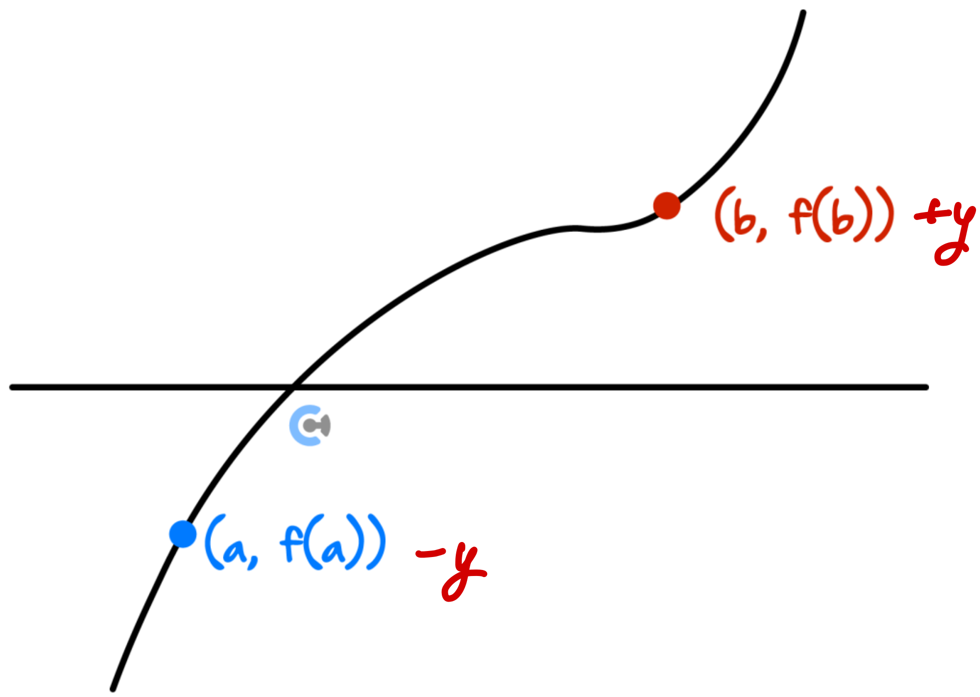
<u>Exploration</u>: **Identifying Whether Two Points are on the Opposite Side of the x-Axis**

- Consider the two points that are on the **same** side of the x-axis.



- What does $f(a) \times f(b)$ give us? **[positive, negative]**

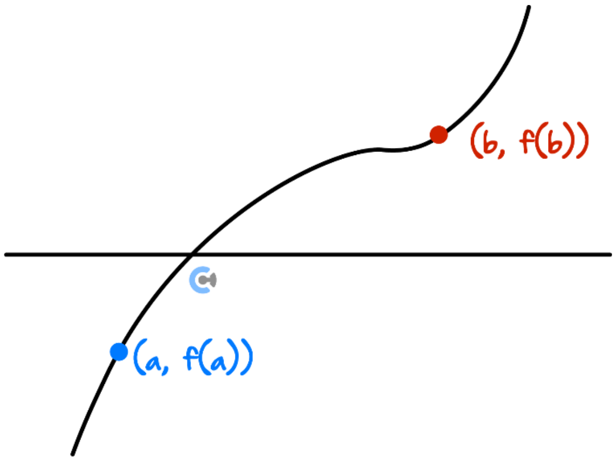
- Now consider the two points that are on the **opposite** side of the x-axis.



- What does $f(a) \times f(b)$ give us? **[positive, negative]**

Definition

Identifying Whether Two Points are on the Opposite Side of the x-Axis



$f(a) \times f(b) =$ Negative

Question 25

Consider the function $f(x) = x^3 - x - 3$.

a. Identify whether the function is on the opposite side of the x-axis for $x = -2$ and $x = 2$.

$f(-2) = -8+2-3 = -9$

$f(2) = 8-2-3 = 3$

$f(-2) \cdot f(2) = -27$

$\therefore x=-2$ and $x=2$ are on opposite sides of the x-axis.

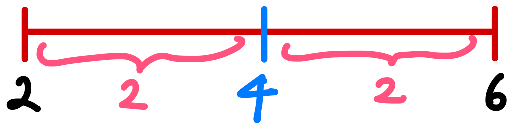
b. Hence, give a possible range of values where the x-intercept could be.

x-intercept $\in [-2, 2]$

<u>Discussion:</u> What does it tell us when two points are on the other side of the axis?

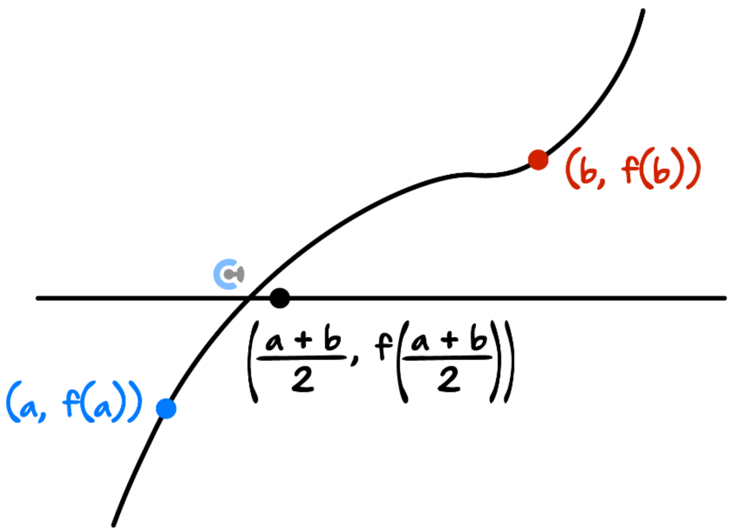
↳ x-intercept MUST be inbetween!

<u>Discussion:</u> Let's say $f(2) \times f(6) =$ Negative. How could we estimate the x-intercept of $f(x)$?



Max Error = 2

<u>Estimating x-Intercept</u>



$$\text{if } f(a) \times f(b) = \text{Negative}$$

$$x\text{-intercept} \in (a, b)$$

$$x\text{-intercept} \approx \frac{a+b}{2}$$

- We simply find the average.

~~**Question 26**~~

Consider the function $f(x) = x^3 - x - 3$.

a. Identify whether the function is on the opposite side of the x-axis for $x = -2$ and $x = 2$.

b. Hence, find an estimation of the x-intercept.

<u>Discussion:</u> **Is this process perfect? How can we improve it?**

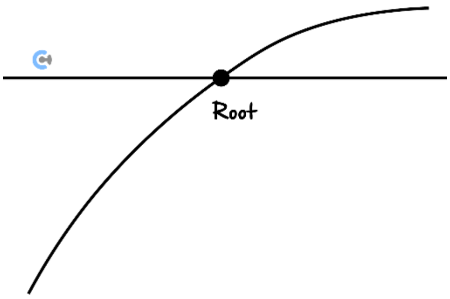
↳ No!

↳ Repeating it!

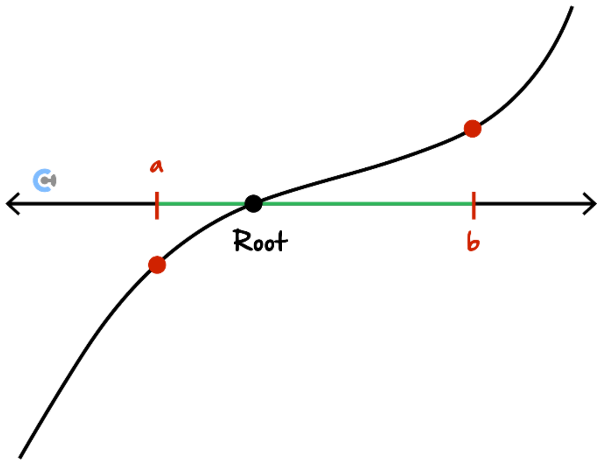
Sub-Section: Iterative Process of Bisection Method

Let's look at how we can do this iteratively!

Exploration: Consider the Function Below

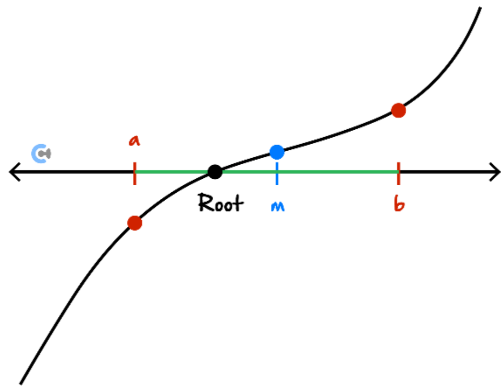


Step 1: Pick a random interval $[a, b]$ where $f(a) \times f(b) =$ Negative.



- We need $f(a) \times f(b) =$ Negative to ensure there is an x-intercept in-between a & b.
- We are picking an appropriate range to begin with. It's a guess.

Step 2: Find a midpoint to estimate the root.



$$\text{where } m = \frac{a+b}{2}$$

- We can say that the estimation of the root is given by the average of a and b.

Step 3: Create a new interval $[a, b]$ by making m either new a or new b.

- How can we algebraically tell?

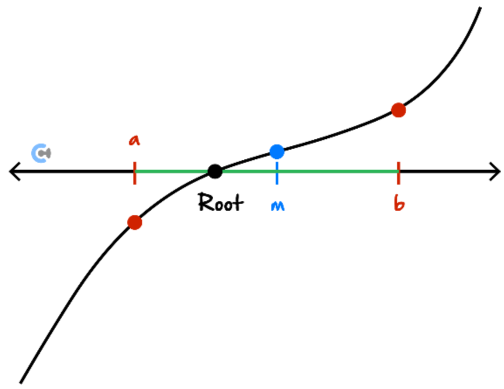
$$\text{If } f(a) \times f(m) < 0$$

New Interval: (a, m)

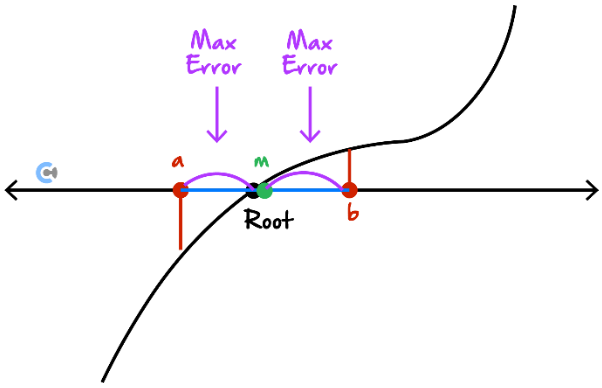
$$\text{If } f(b) \times f(m) < 0$$

New Interval: (b, m)

- Considering the diagram below, where would our new interval be? $[(a, m), (m, b)]$



Step 4: Repeat until the interval becomes short enough for good accuracy.



If $\frac{b-a}{2} <$ Max Tolerance

We Stop

- The maximum error we can make is the distance between m and the endpoints (a,b).
- Maximum error is half of the width of the interval.

Max Error $= \frac{b-a}{2}$

Space for Personal Notes

Definition

Bisection Method

- Step 1: Pick a random interval $[a, b]$ where $f(a) \times f(b) =$ Negative.
- Step 2: Find a midpoint to estimate the root.

$$\text{where } m = \frac{a+b}{2}$$

- Step 3: Create a new interval $[a, b]$ by making m either new a or new b.

$$\text{If } f(a) \times f(m) < 0$$

$$\text{New Interval: } [a, m]$$

$$\text{If } f(b) \times f(m) < 0$$

$$\text{New Interval: } [m, b]$$

- Step 4: Repeat until the interval becomes short enough for good accuracy.
  - The smaller the interval $[a, b]$, more accurate our estimation gets.

$$\text{If } \frac{b-a}{2} < \text{Max Tolerance}$$

$$\text{We stop}$$

  - Maximum error is half of the width of the interval.

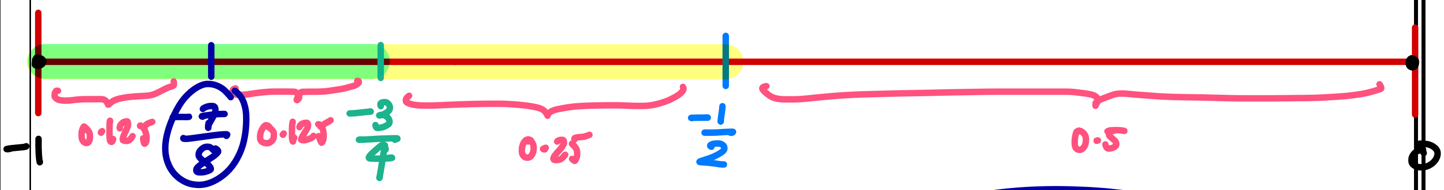
$$\text{Max Error} = \frac{b-a}{2}$$

Space for Personal Notes

Question 27 Walkthrough. Tech-Active.

The equation $x^3 + 12x + 12 = 0$ has one real solution, which lies in the interval $[-1,0]$. Approximate the solution using the bisection method with a maximum error of 0.2.

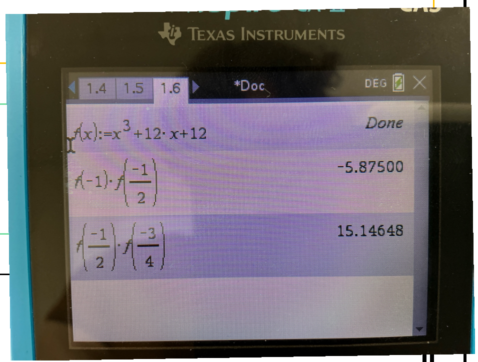
Max Error ≤ 0.2 Tolerance



① Midpoint
② Max Error < Tolerance? ⇒ End!
③ New Interval

x-int estimate = $-\frac{7}{8}$

NOTE: We always pick the interval such that $f(a) \times f(b) =$ Negative $\rightarrow$ x-intercept is between a and b.

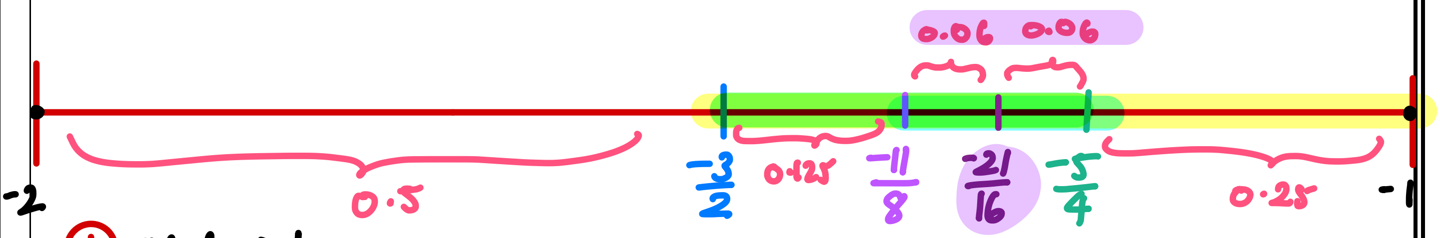


Your turn!

Question 28 Tech-Active.

The equation $x^3 + 3x + 6 = 0$ has one real solution, which lies in the interval $[-2,-1]$. Approximate the solution using the bisection method with a maximum error of 0.1.

Max Error ≤ 0.1



① Midpoint
② Max Error < Tolerance? ⇒ End!
③ New Interval

$\therefore$ x-int estimate = $-\frac{21}{16}$

NOTE: Keep going until the length of the interval is less than $2 \times$ maximum error.

TI UDF

Bisection Method

Overview:

Apply the bisection method to a function to approximate x intercepts.

Input:

```
bisection(<function>, <variable>, <lower bound>, <upper bound>)
```

Other notes:

- The program will ask for the threshold type to terminate the algorithm.
- Select None [0] to provide a specific number of iterations
- Select x [1] to provide a threshold for b-a, after which the program will stop if b-a becomes smaller than the threshold.
- Select y[2] to provide a threshold for |f(b)-f(a)|, after which the program will stop if |f(b)-f(a)| becomes smaller than the threshold.

$bisection(x^2-2,x,0,1)$

Number of Iterations: 5

"n"	"a"	"m"	"b"	"f(a)"	"f(m)"	"f(b)"	"b−a"	"|f(b)−f(a)|"
0.	0.	0.5	1.	-2.	-1.75	-1.	1.	1.
1.	0.5	0.75	1.	-1.75	-1.4375	-1.	0.5	0.75
2.	0.75	0.875	1.	-1.4375	-1.23438	-1.	0.25	0.4375
3.	0.875	0.9375	1.	-1.23438	-1.12109	-1.	0.125	0.234375
4.	0.9375	0.96875	1.	-1.12109	-1.06152	-1.	0.0625	0.121094
5.	0.96875	0.984375	1.	-1.06152	-1.03101	-1.	0.03125	0.061523

Space for Personal Notes

Section F: Exam 2 Questions (16 Marks) (5min MCQ) (9min ERQ)

Question 29 (1 mark)

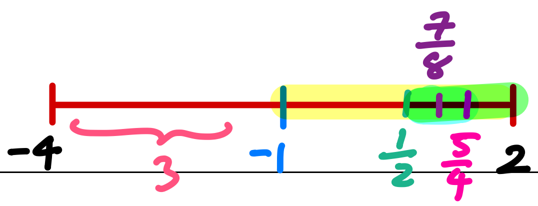
The equation $5x^3 + 2x - 8 = 0$ has one real solution, which lies in the interval $[-4, 2]$. Approximate the solution using the bisection method with a maximum error of 0.4. What is the approximate solution?

A. $x \approx 0.675$

B. $x \approx 1.925$

C. $x \approx 0.875$

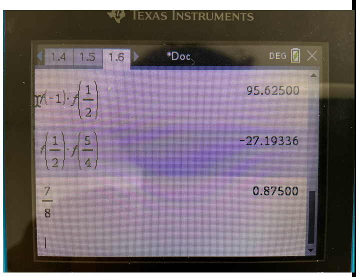
D. $x \approx 1.225$



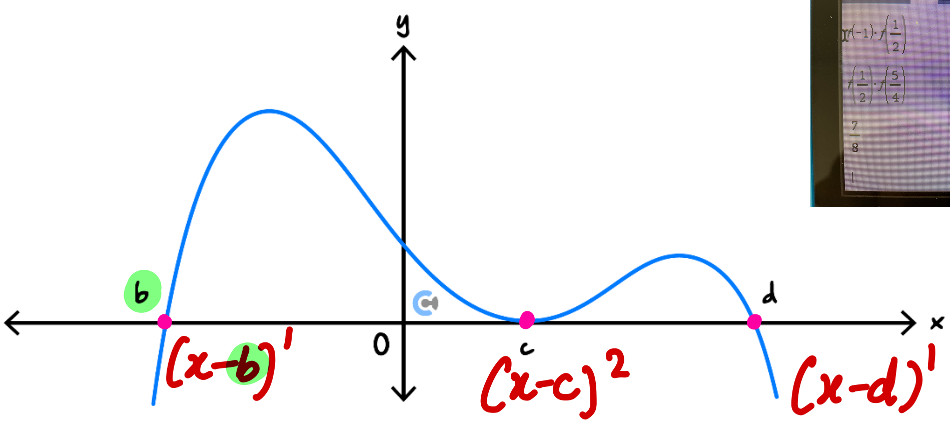
Estimate	Max Error
-1	3
$\frac{1}{2}$	1.5
$\frac{5}{4}$	0.75
$\frac{7}{8}$	0.375

$\frac{7}{8} \approx 0.875$

Question 30 (1 mark)



The rule for a function with the graph below could be:



A. $y = -2(x + b)(x - c)^2(x - d)$

B. $y = 2(x + b)(x - c)^2(x - d)$

C. $y = -2(x - b)(x - c)^2(x - d)$

D. $y = 2(x - b)(x - c)\ (x - d)$

$-3 \Rightarrow (x-(-3))$

$= (x+3)$

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Question 31 (1 mark)

The polynomial $x^3 + ax^2 + bx + 4$ is perfectly divisible by $x - 1$ and has a reminder of 3 when divided by $x + 2$. The values (a, b) are:

$f(x) = x^3 + ax^2 + bx + 4$

A. $(-1, 4)$

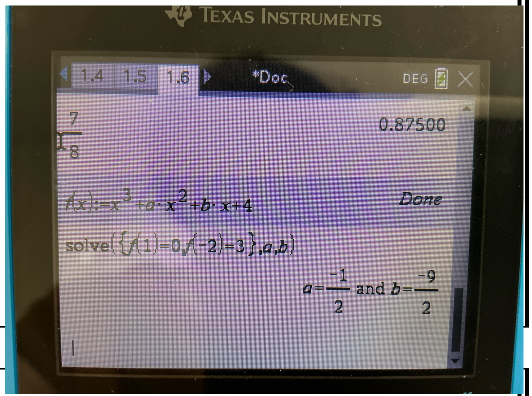
B. $\left(-\frac{1}{2}, -\frac{9}{2}\right)$

C. $\left(-\frac{3}{5}, -\frac{5}{2}\right)$

D. $\left(-\frac{7}{2}, \frac{3}{2}\right)$

$f(1) = 0$

$f(-2) = 3$



Question 32 (1 mark)

The equation $x^3 - 5kx^2 + 4x = 0$ has exactly two solutions when:

A. $k = \pm\frac{4}{5}$

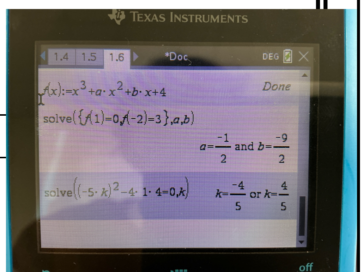
B. $-\frac{4}{5} < k < \frac{4}{5}$

C. $k > \frac{4}{5}$

D. $k < -\frac{4}{5}$

$x\underbrace{(x^2 - 5kx + 4)}_{\Delta = 0} = 0$

$(1) + (1) = (2)$



Question 33 (1 mark)

A graph with the rule $f(x) = \frac{1}{3}x^3 - x^2 + c$, where c is a real number, has three distinct x-intercepts. All possible values of c are:

~~**A.**~~ $c > \frac{4}{3}$ → Try $c = 2$?

~~**B.**~~ $-\frac{4}{3} < c < 0$ → Try $c = -1$?

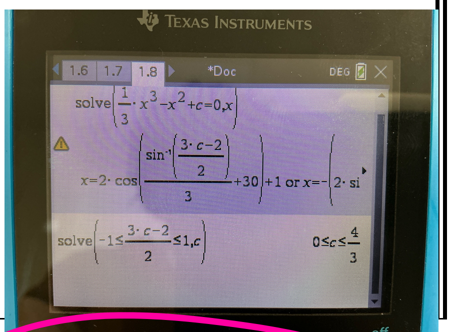
C. $0 < c < \frac{4}{3}$ → Try $c = 1$?

~~**D.**~~ $c < \frac{4}{3}$

Slider:

T & E

Spesh Method:



Dom $\sin^{-1}(x) \in [-1, 1]$

Question 34 (11 marks)

Consider the cubic polynomial $f(x) = x^3 - 3x^2 - 3x - 4$.

a. Explain why $f(x)$ has a root between $x = 3$ and $x = 5$. (1 mark)

$f(3) =$

$f(5) =$

b. Write $f(x)$ in the form $f(x) = (x - a)\,Q(x)$ where $a > 0$ and $Q(x)$ is a quadratic function. (1 mark)

→ Use "Factor" on CAS

→ Menu → 3 → 2

c. Find the values of k for which the equation $x^3 - 3x^2 + kx - 4x - 4k = 0$ has three solutions. (2 marks)

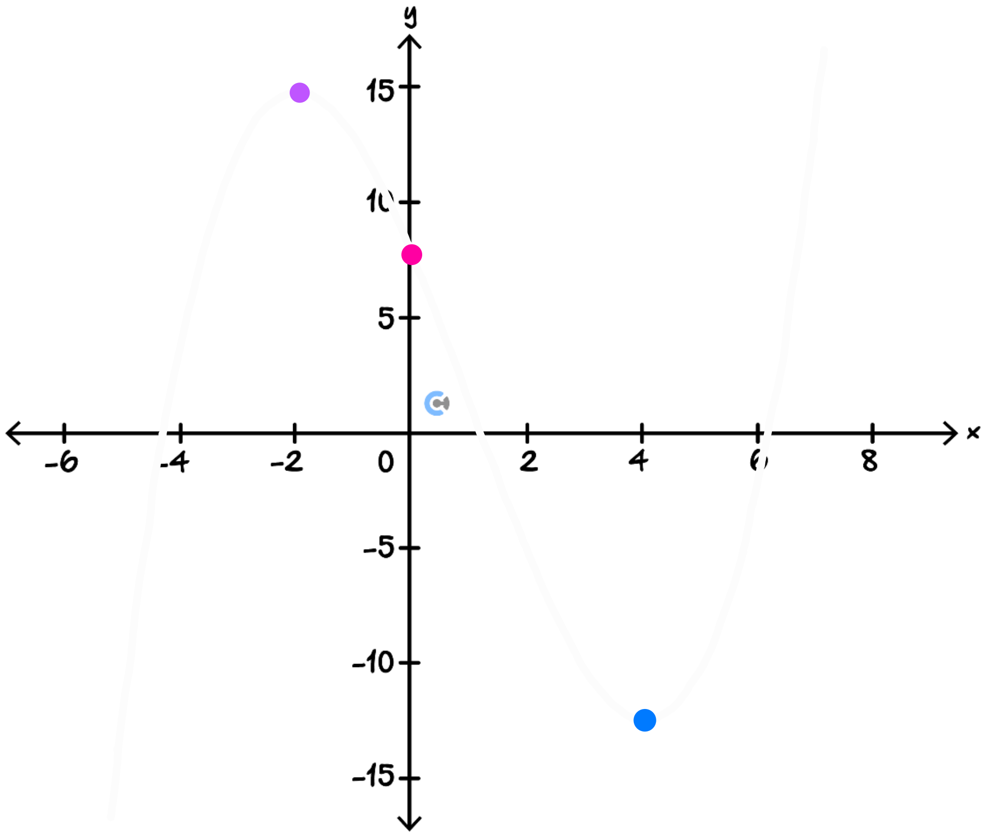
① Use Factor on CAS:

② Solve $f(x) = 0$:

Consider the function $g(x) = \frac{1}{4}(x^3 - 3x^2 - 24x + 32)$.

d. Solve the equation $g(x) = 0$. Give your answer correct to two decimal places. (1 mark)

e. Sketch the graph of $y = g(x)$ on the axes below. Label all turning points and the y-intercepts with coordinates. (3 marks)



f. Find the values of k such that $g(x) + k = 0$ has:

i. One solution. (2 marks)

① $\underbrace{g(x)}_{y=g(x)} = \underbrace{-k}_{y=-k}$

② $g(x) + k = 0$ (x-int)

ii. Three solutions. (1 mark)

① $\underbrace{g(x)}_{y=g(x)} = \underbrace{-k}_{y=-k}$

② $g(x) + k = 0$ (x-int)

Contour Check

Learning Objective: [1.6.1] – Solve Polynomial Inequalities

Key Takeaways

- The 'value' of $f(x)$ is its ______________ value.
- $f(x) > 0$ means find the x values for which the y values are ______________.

Learning Objective: [1.6.2] – Solve Number of Solution Problems

Key Takeaways

- When a cubic has n roots, the quadratic factor has $n - 1$ roots.

Learning Objective: [1.6.3] - Apply Bisection Method to Approximate x-Intercepts

Key Takeaways

- When two points are on the opposite of the axis, there is an x-intercept ______________ the two points.
- When applying the bisection method over $[a, b]$, if $f(a) \times f(m) < 0$ then m becomes the new ______________ bound.
- If $f(b) \times f(m) < 0$ then f becomes the new ______________ bound.

VCE Mathematical Methods ½

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