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VCE Mathematical Methods ½
Polynomials Exam Skills [1.6]
Workbook

Outline:

Learning Objectives:

- ☐ MM12 [1.6.1] - Solve polynomial inequalities.
- ☐ MM12 [1.6.2] - Solve number of solution problems.
- ☐ MM12 [1.6.3] - Apply bisection method to approximate x-intercepts.

Section A: Recap

If you were here last week, skip to Section B Warmup Test.

Definition

Degree of Polynomial Functions

$$\text{Degree} = \text{Highest Power of the Polynomial}$$

Question 1

State the degree of each polynomial.

a. $x^3 - 4x^2 + 5x + 6$

b. $3x + 5x^2 - x^7$

Definition

Roots of Polynomial Functions

$$\text{Roots} = x\text{-intercept(s)}$$

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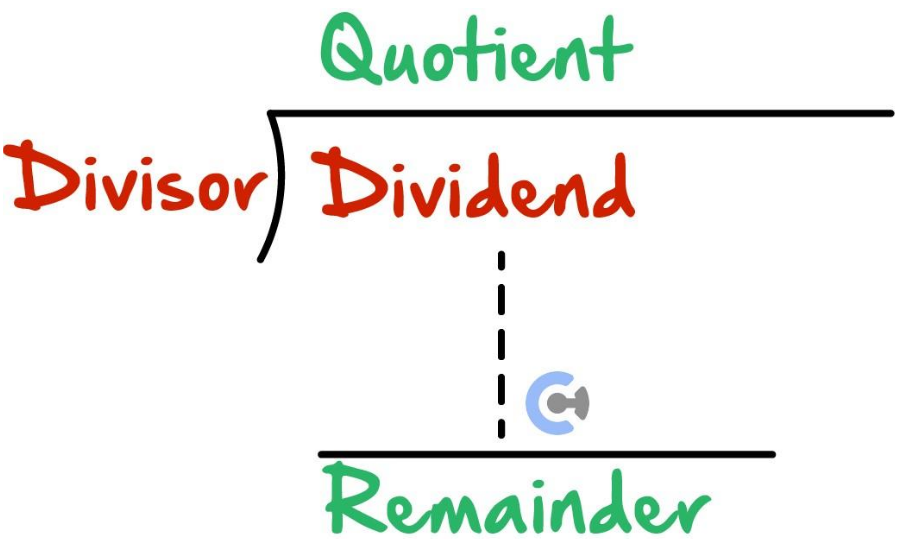
Question 2

Find the roots of the following polynomial:

$$(x-1)^2(x+3)^4$$

<u>Polynomial Long Division</u>

- Division of polynomials:



$$\frac{\textbf{Dividend}}{\textbf{Divisor}} = \textbf{Quotient} + \frac{\textbf{Remainder}}{\textbf{Divisor}}$$

Question 3

Simplify the following using polynomial long division:

$$\frac{x^3 + x^2 + 2}{x - 3}$$

TIP: Always remember to fill in any missing powers of x in the numerator or denominator with "placeholders" that have a coefficient of 0.

Remainder Theorem

- **Definition**: Finds the remainder of long division without the need for long division.

When $P(x)$ is divided by $(x - \alpha)$, the remainder is $P(\alpha)$

- **Steps:**

1. Find x values, which makes the divisor equal to 0.
2. Substitute it into the dividend function.

Question 4

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where, $f(x) = x^3 - 2x^2 + 3x + 1$ and $g(x) = 2x + 4$.

__Factor Theorem__

- For every x-intercept, there is a factor:

if $P(\alpha) = 0$ then, $(x - \alpha)$ is a factor of $P(x)$

Question 5

Determine if $x + 2$ is a factor of $P(x) = 2x^3 - 7x^2 + 7x - 2$.

Factorising Cubic Polynomials

- The steps are:
  - Find a single root by trial and error.

    (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

Question 6

Find all the roots of $f(x) = x^3 + 3x^2 - 6x - 8$.

NOTE: When the question asks for all roots, you cannot just factorise and end it there!

Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$\textit{Potential root} = \pm \frac{\textit{Factors of constant term } a_0}{\textit{Factors of leading coefficient } a_n}$$

- If the roots are rational numbers, the roots can only be $\pm \frac{factors\ of\ constant\ term\ a_0}{factors\ of\ leading\ coefficient\ a_n}$.

NOTE: All the roots are part of the suggestion given by the rational root theorem.

Question 7

Find all the roots of $f(x) = 2x^3 - x^2 - 22x - 24$.

<u>Sum and Difference of Cubes</u>

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

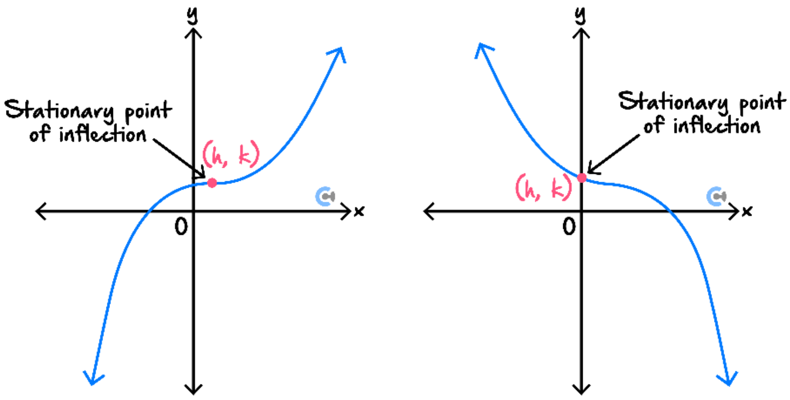
Question 8

Factorise the following polynomial as much as possible:

$$8x^3 - 216$$

Graphs of $a(x-h)^n + k$, Where n is an Odd Positive Integer

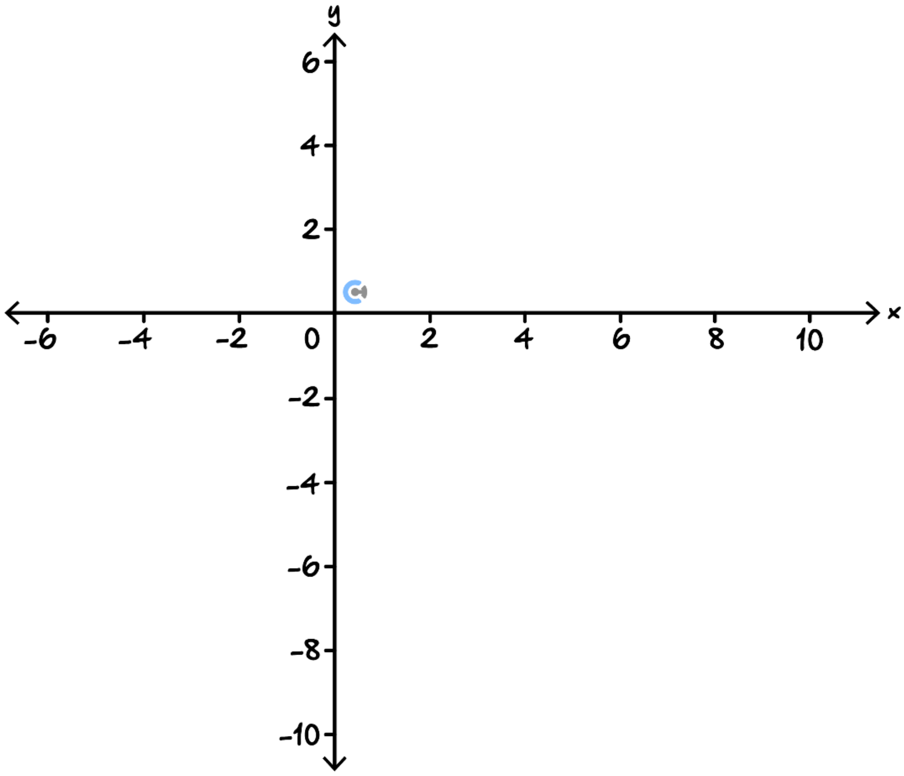
- All graphs look like a "cubic".



- The point (h, k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

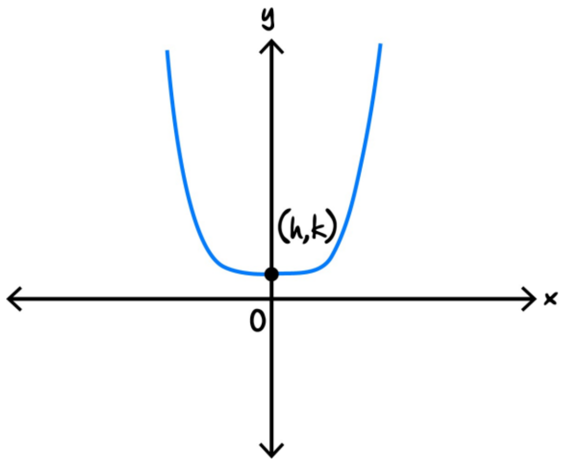
Question 9

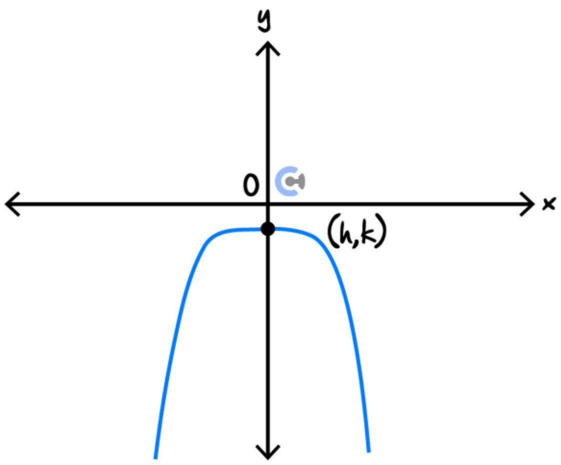
Sketch the graph of $y = (x-1)^3 - 5$ on the axes below. Label all turning point(s) and the y-intercept(s) with coordinates.



Graphs of $a(x-h)^n + k$, Where n is an Even Positive Integer

- All graphs look like a "quadratic".

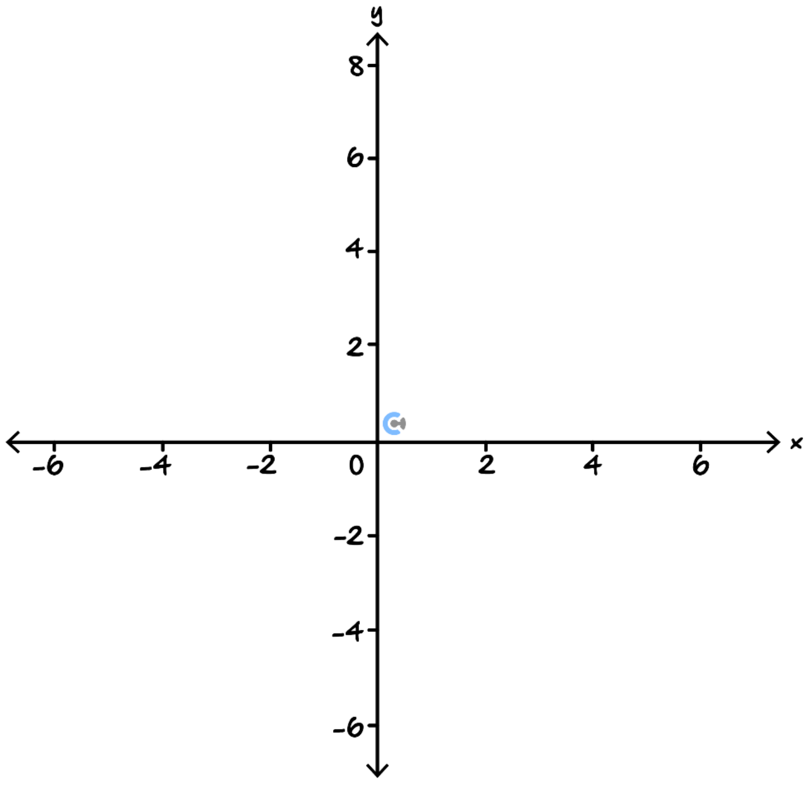




- The point (h, k) gives us the turning point.

Question 10

Sketch the graph of $y = -(x-1)^4 + 4$ on the axes below. Label all turning point(s) and the y-intercept(s) with coordinates.



Definition

Graphs of Factorised Polynomials

- **Steps:**

1. Plot x-intercept(s).
2. Determine whether the polynomial is positive or negative.
3. Use the repeated factors to deduce the shape.

   Non-Repeated: Only x-intercept.

   Even Repeated: x-intercept and a turning point.

   Odd Repeated: x-intercept and a stationary point of inflection.

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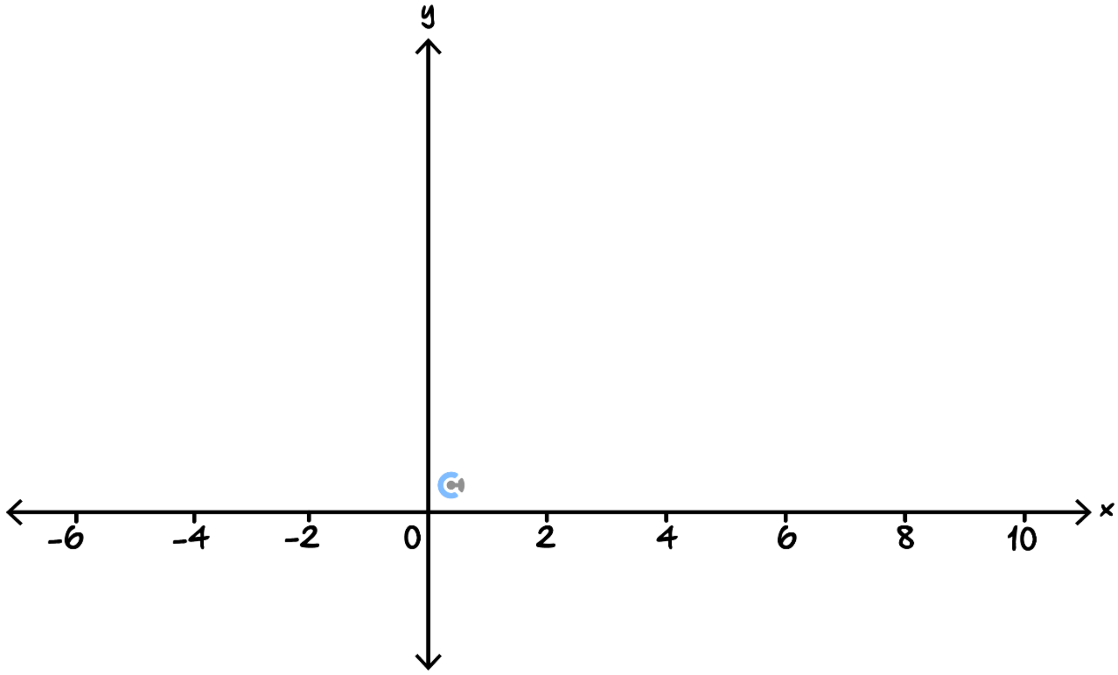
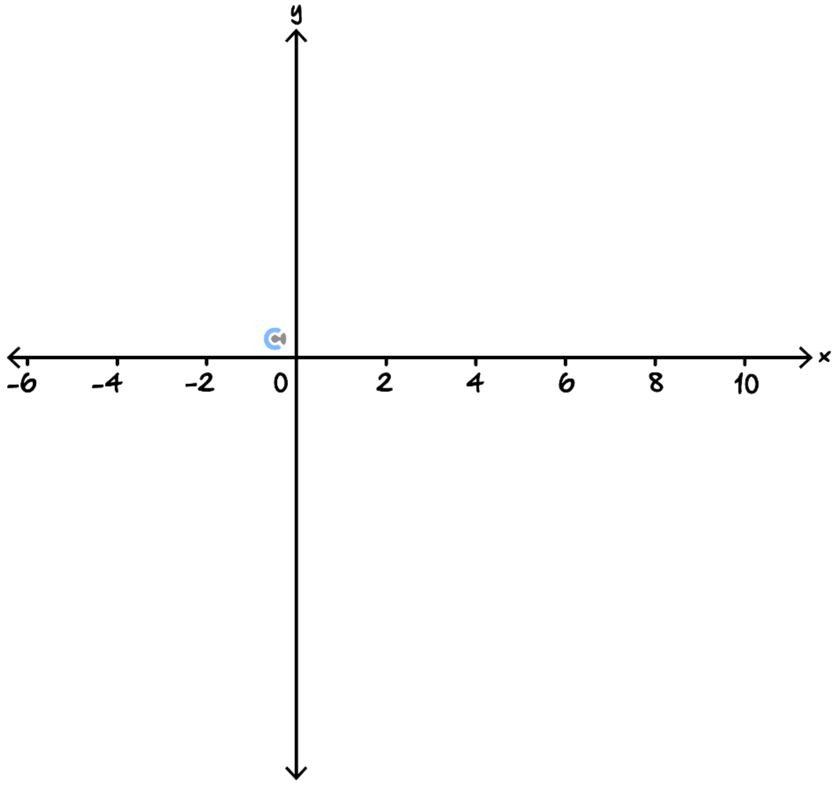
Question 11

Sketch the graphs of the following functions on the axes provided. Ignore the y-axis scale.

a. $y = (2 + x)(5 - x)^2$



b. $y = (x + 3)^3(x - 1)(x - 5)$



Section B: Warmup Test

INSTRUCTION: 15 Marks. 15 Minutes Writing.

Question 12 (3 marks)

Consider the function $f(x) = x^3 + ax^2 + bx - 2$. If $x - 1$ is a factor of $f(x)$ and the remainder of $f(x) \div (x - 2)$ is given by 12, find the value(s) of a and b.

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Question 13 (3 marks)

Solve the following equation for x:

$$2x^3 - 5x^2 = 4x - 3$$

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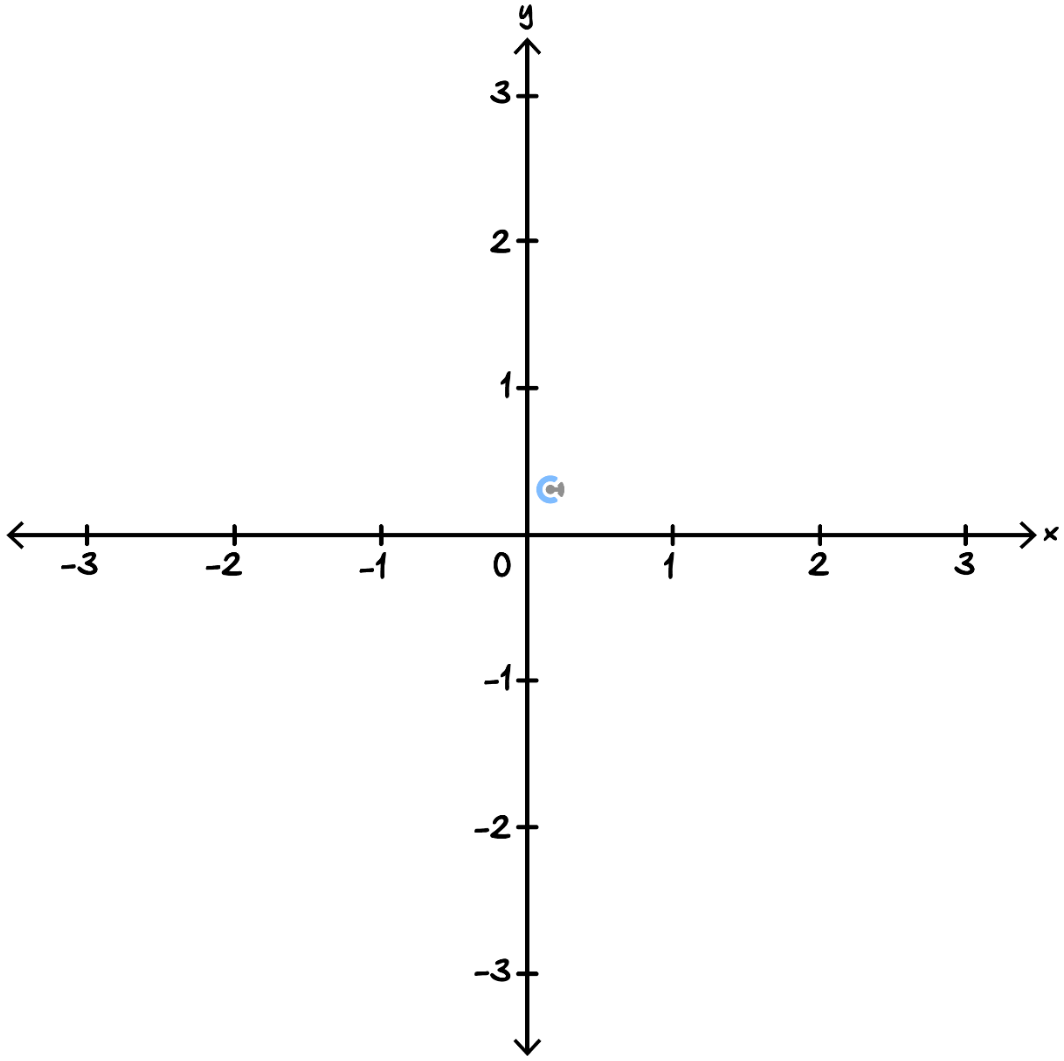
Question 14 (3 marks)

Sketch the graph of the following function on the axes below. Label all axis intercept(s) with their coordinates.

$$y = x^2(2 - x)(x + 1)$$



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Question 15 (6 marks)

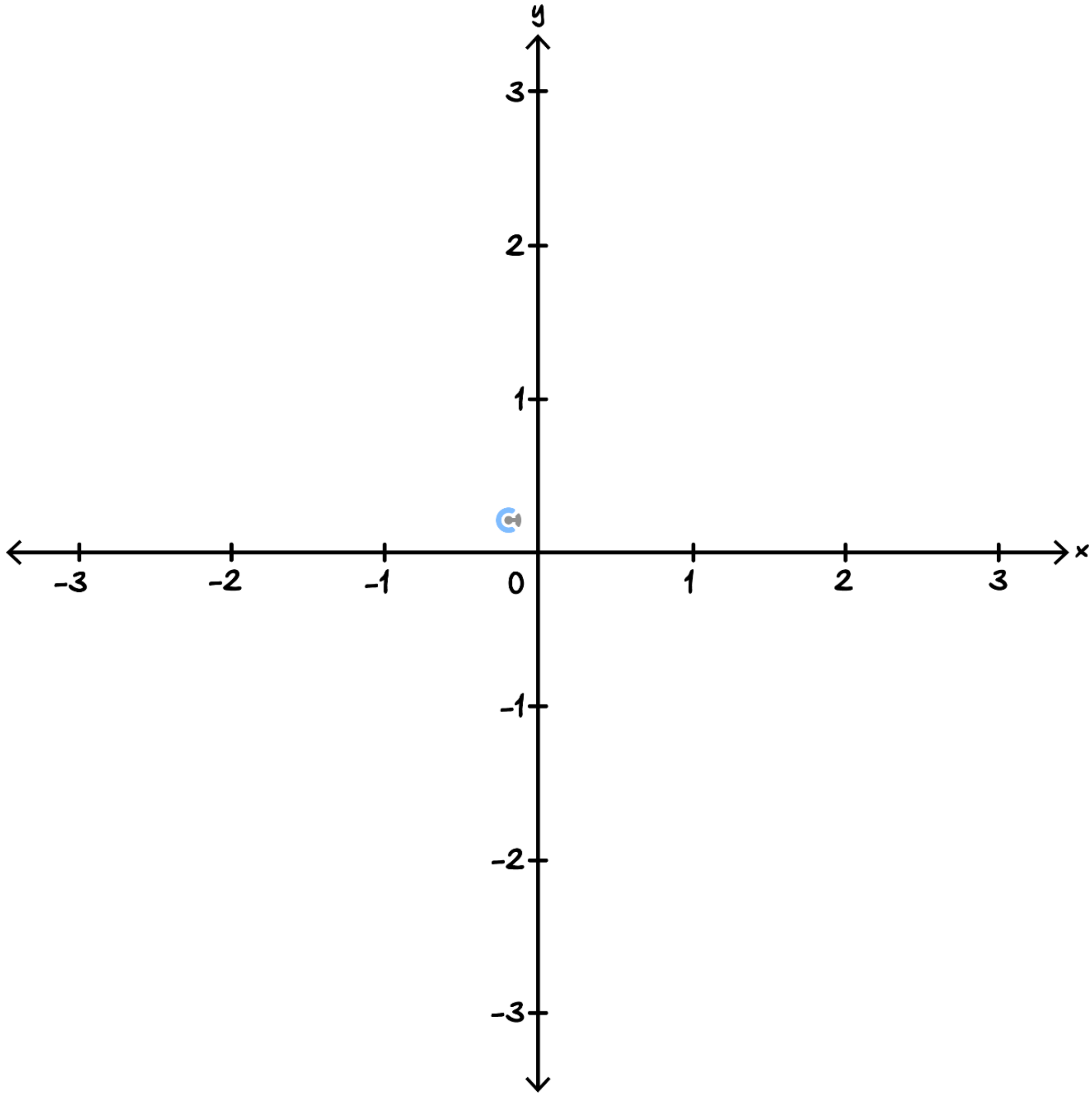
Consider the function $f(x) = 2x^3 - 3x^2 - ax + 2$.

It is known that the remainder, when $f(x)$ is divided by $x - 3$, is 20.

a. Show that $a = 3$. (1 mark)

b. Hence, solve $f(x) = 0$. (3 marks)

c. Sketch the graph of $y = f(x)$ on the axes below. Label all axis intercept(s) with coordinates. (2 marks)



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Section C: Polynomials Exam Skills

Sub-Section: Solve Polynomial Inequalities

Context

- We are used to solving polynomial equations, that is, when we put two polynomials together and put an $=$ sign in between them. But sometimes, instead of an $=$ sign, there is an inequality sign between them instead. What do we do in such a situation?

Exploration: Meaning of a Polynomial Equality

- The 'value' of a polynomial is the y value on the graph.
- Hence, the equation $f(x) > 0$ means find where the y values are positive.

Solving the Polynomial Inequality $f(x) > 0$

- Steps:
  1. Find the x-intercept(s).
  2. Sketch the polynomial.
  3. Shade the places where the y-values are positive.

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Question 16 Walkthrough.

Solve the following inequality for x:

$$(x-1)(x+2)(x+3) > 0$$

Sometimes, we have to factorise or move everything to one side.

Question 17 Walkthrough.

Solve the following inequality for x:

$$2x^3 + x^2 - 5x + 4 > 2$$

Active Recall: The 'value' of $f(x)$ is its ________________ value.

Question 18

Solve the following polynomial inequalities for x.

a. $(3 - x)(x + 4)(x - 2) \geq 0$

b. $2x^3 - 3x^2 - 2x + 6 < 3$

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Sub-Section: Solve Number of Solution Problems

When we can factorise a cubic, we can use the discriminant of the remaining quadratic to figure out a number of solutions.

Definition

When does a cubic have n solutions?

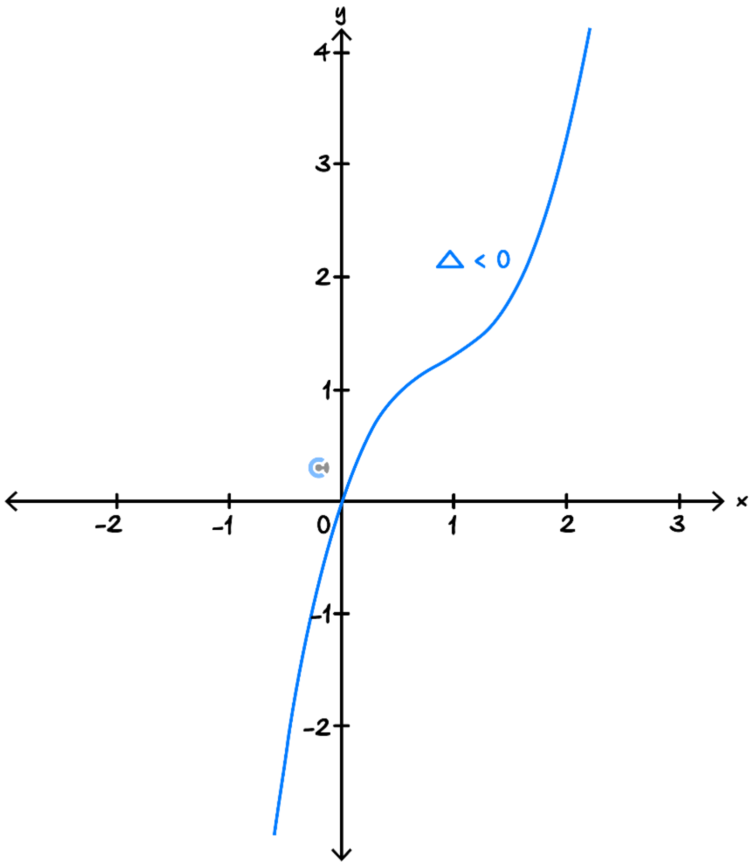
- Steps:

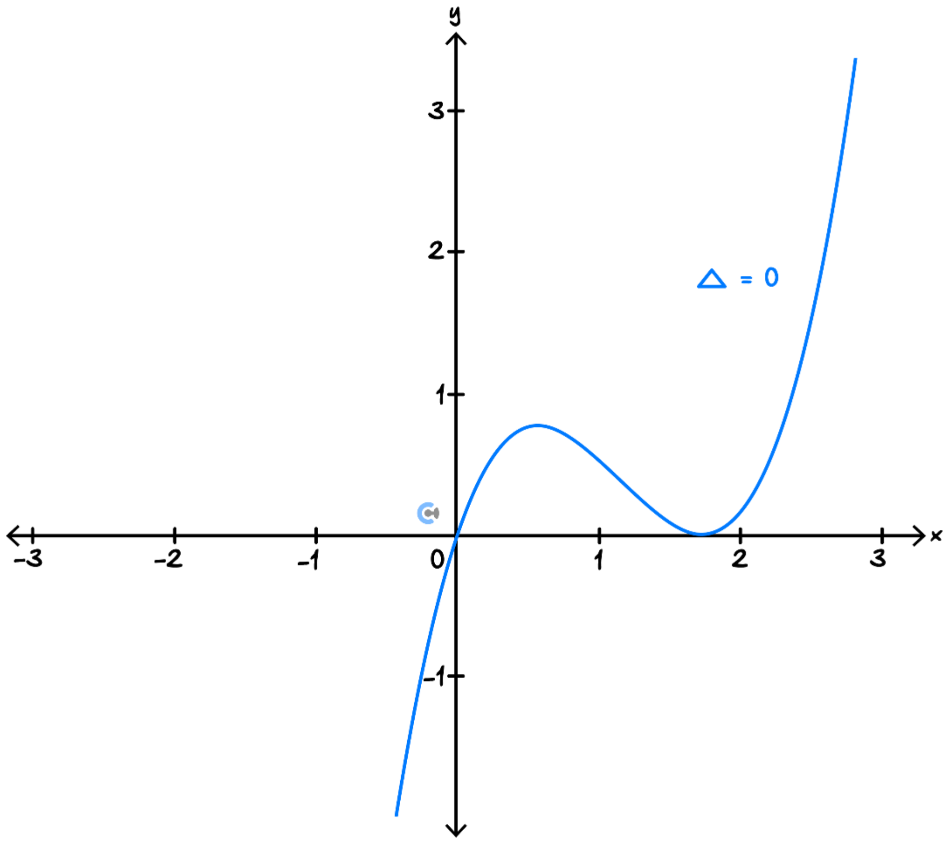
  1. Factorise out the x term.

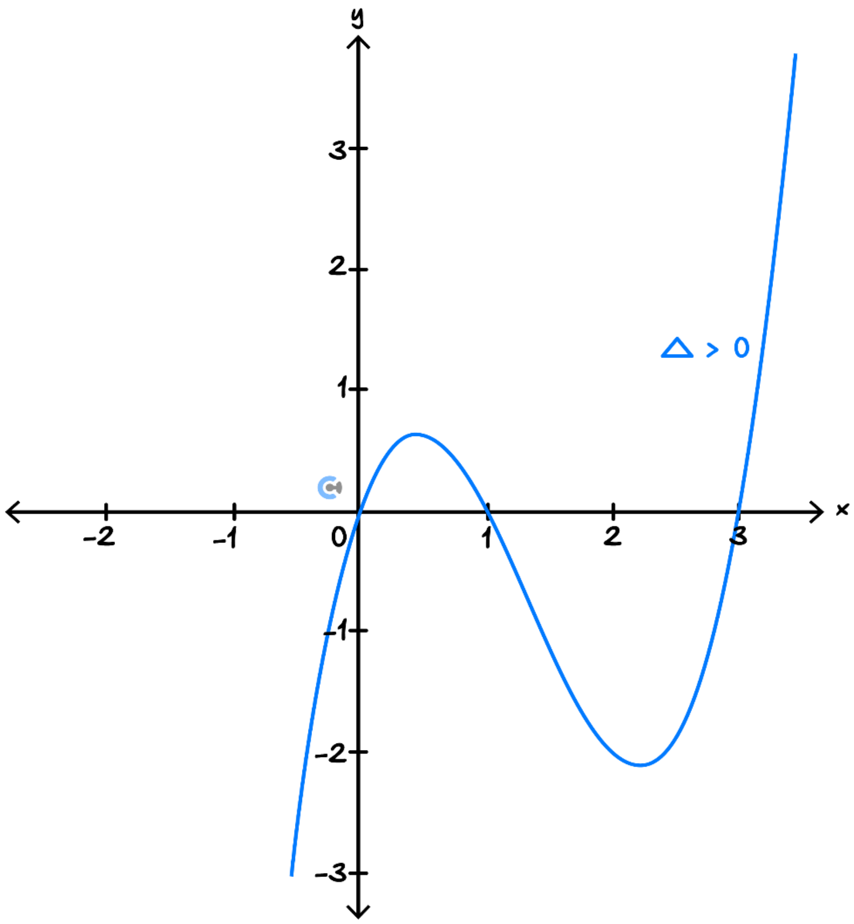
  2. Since the x term gives 1 solution, use discriminant to find when the quadratic has $n-1$ solutions.

Exploration: What Does the Discriminant Control in a Cubic?

- The discriminant controls where the turning point is!




y
3
2
1
0
-1
-3
-2
-1
1
2
3
x
△ = 0


y
3
2
1
0
-1
-2
-3
-2
-1
1
2
3
x
△ > 0

Question 19 Walkthrough.

Consider $f(x) = x^3 - kx^2 + 3x$.

Find the value(s) of k such that $f(x) = 0$ has 2 solutions.

Active Recall: Finding Number of Solutions for a Factorisable Cubic

- Break the cubic down into a ___________ factor and a _________________ factor.
- Use the _______________________ to determine the number of solutions you want the quadratic factor to have.

Question 20 Walkthrough.

Consider $f(x) = x^3 - 3kx^2 + 4x$. Find the values of k such that $f(x) = 0$ has 3 solutions.

Section D: Exam 1 Questions (16 Marks)

Question 21 (3 marks)

Consider the function $f(x) = 8x^3 - 216$.

a. Express $f(x)$ in the form $a(x - b)(x^2 + cx + d)$ for positive real numbers $a, b, c,$ and d. (2 marks)

b. Hence, explain why $x = b$ is the only solution to the equation $f(x) = 0$. (1 mark)

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Question 22 (3 marks)

Solve the inequality $2x^3 - 18x < 3x^2 + 8$ for x.

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Question 23 (5 marks)

Consider $f(x) = 2x^3 + 4kx^2 + 12x$, where k is a real constant.
Find the values of k such that $f(x) = 0$ has:

a. One solution. (3 marks)

b. Two solutions. (1 mark)

c. Three solutions. (1 mark)

Question 24 (5 marks)

Consider the quadratic polynomial:

$$f(x) = x^5 + ax^4 + bx^3 - x^2 + 4x - 4$$

Where a and b are real constants.

- $x - 2$ is a factor of $f(x)$.
- When $f(x)$ is divided by $x - 1$ the remainder is -1.

a. Show that $a = -3$ and $b = 2$. (2 marks)

b. Write the function $g(x) = \frac{f(x)}{x^2-x-2}$ in the form $g(x) = C(x) + \frac{B}{x+d}$.

Where $C(x)$ is a cubic polynomial and B, d are real constants. (3 marks)

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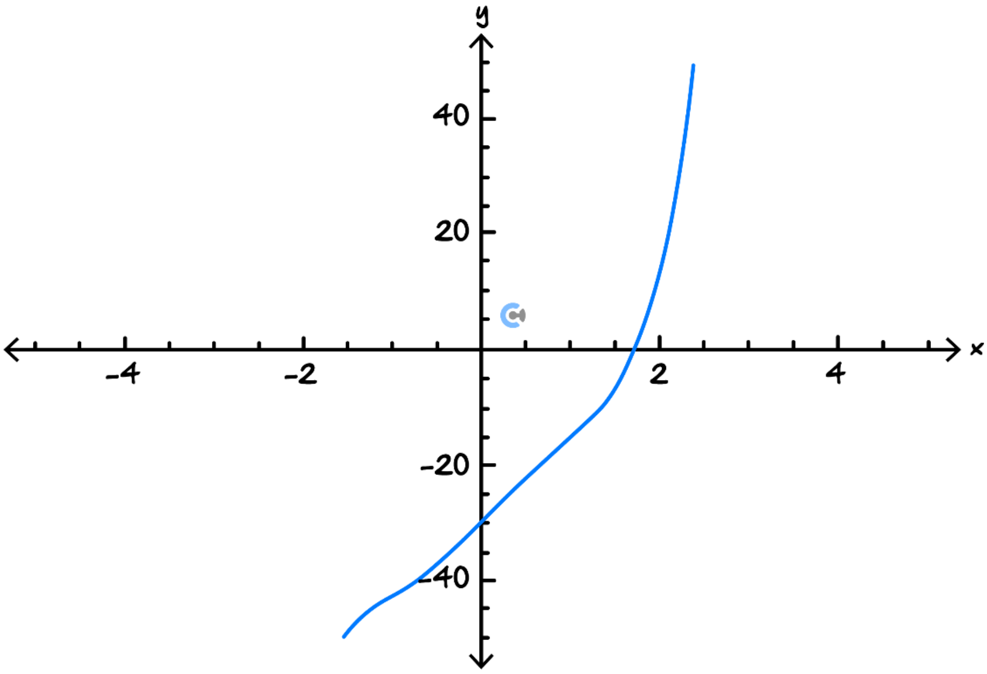
Section E: Tech Active Exam Skills

Sub-Section: Apply Bisection Method to Approximate x-Intercepts

Context: Bisection Method

- We know how to solve the equation $x^2 - 4 = 0$ easily.
- We've also learnt how to solve the cubic equations using factor theorem as well.
- What if the equation is too hard to solve?



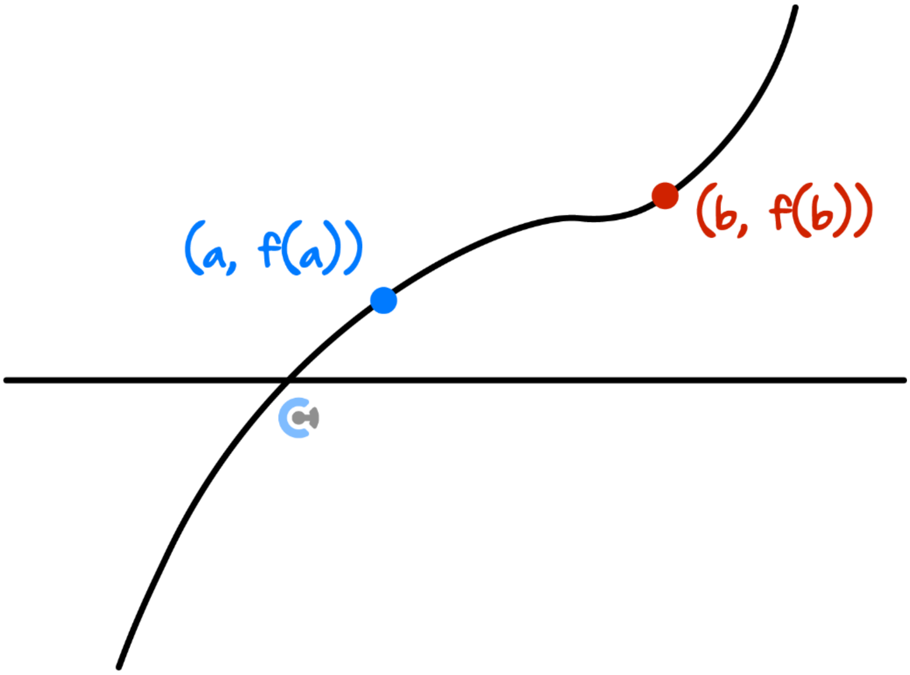
$$x^5 - 3x^3 + x^2 + 16x - 30 = 0$$

- Bisection method can be used to approximate the answer to any polynomial equations.

Discussion: How do we tell if two points are on the opposite side of the x-axis (one below and one above the x-axis)?

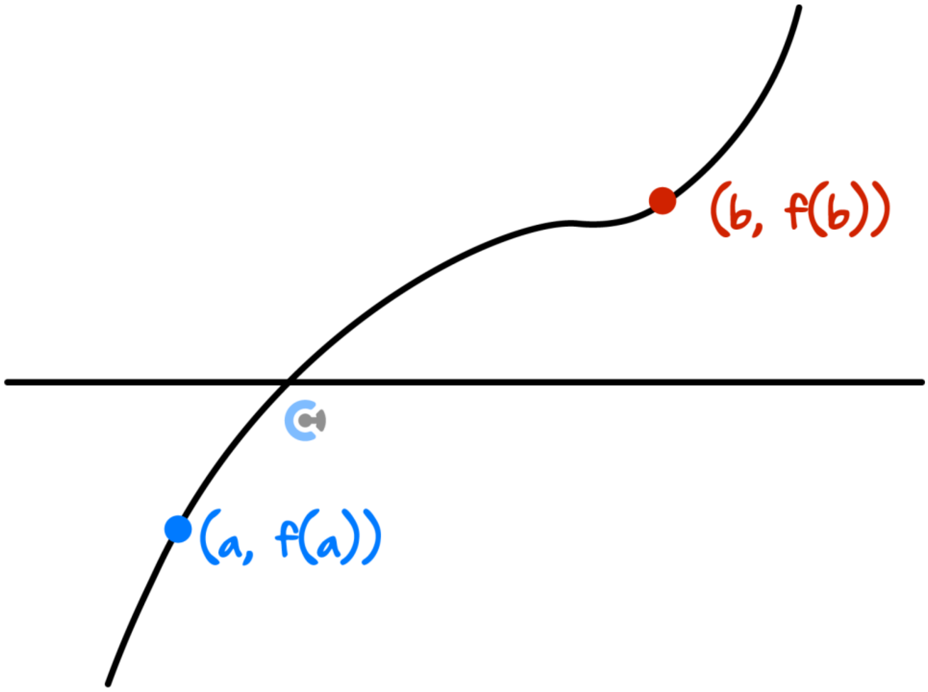
<u>Exploration</u>: **Identifying Whether Two Points are on the Opposite Side of the x-Axis**

- Consider the two points that are on the **same** side of the x-axis.



- What does $f(a) \times f(b)$ give us? **[positive, negative]**

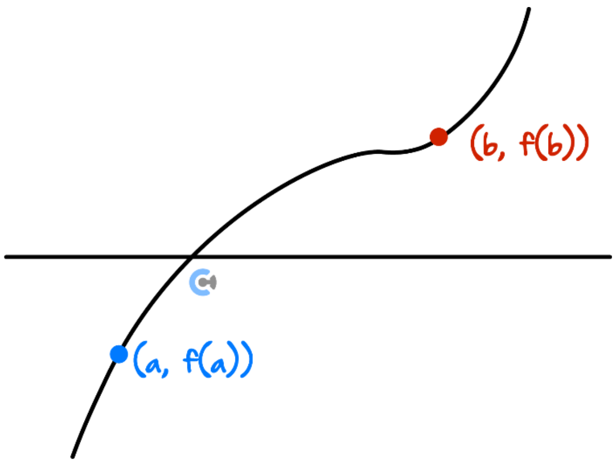
- Now consider the two points that are on the **opposite** side of the x-axis.



- What does $f(a) \times f(b)$ give us? **[positive, negative]**

Definition

Identifying Whether Two Points are on the Opposite Side of the x-Axis



$f(a) \times f(b) =$ Negative

Question 25

Consider the function $f(x) = x^3 - x - 3$.

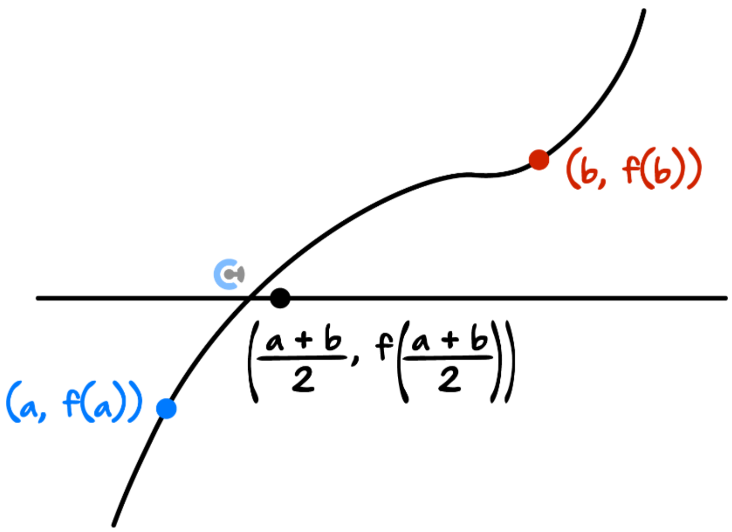
a. Identify whether the function is on the opposite side of the x-axis for $x = -2$ and $x = 2$.

b. Hence, give a possible range of values where the x-intercept could be.

Discussion: What does it tell us when two points are on the other side of the axis?

Discussion: Let's say $f(2) \times f(6) =$ Negative. How could we estimate the x-intercept of $f(x)$?

Estimating x-Intercept



$$\text{if } f(a) \times f(b) = \text{Negative}$$

$$x\text{-}intercept \in (a, b)$$

$$x\text{-}intercept \approx \frac{a+b}{2}$$

- We simply find the average.

Question 26

Consider the function $f(x) = 2x^3 - x + 2$.

a. Identify whether the function is on the opposite side of the x-axis for $x = -2$ and $x = 1$.

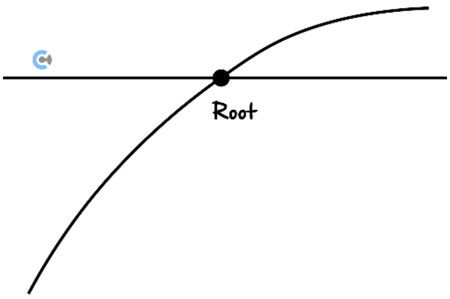
b. Hence, find an estimation of the x-intercept.

<u>Discussion:</u> Is this process perfect? How can we improve it?

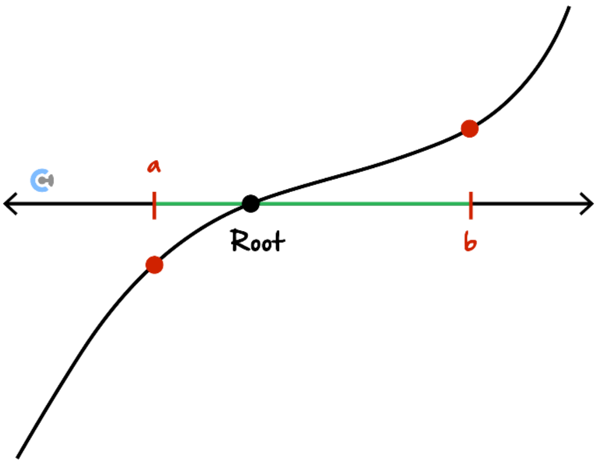
Sub-Section: Iterative Process of Bisection Method

Let's look at how we can do this iteratively!

Exploration: Consider the Function Below

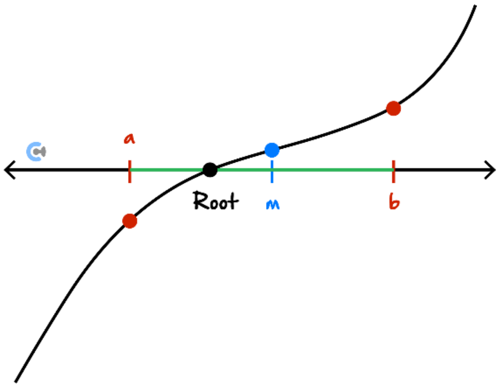


Step 1: Pick a random interval $[a, b]$ where $f(a) \times f(b) =$ Negative.



- We need $f(a) \times f(b) =$ Negative to ensure there is an x-intercept ________________________________.
- We are picking an appropriate range to begin with. It's a _______________.

Step 2: Find a midpoint to estimate the root.



where $m = \frac{a+b}{2}$

- We can say that the estimation of the root is given by the ________________ of a and b.

Step 3: Create a new interval $[a, b]$ by making m either new a or new b.

- How can we algebraically tell?

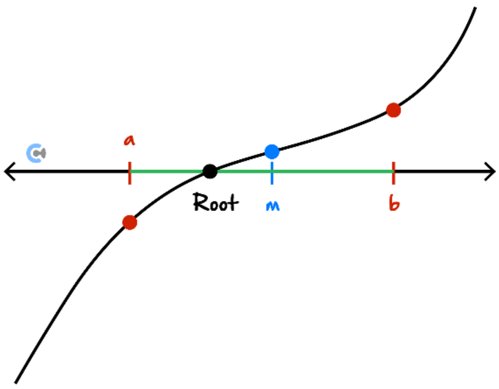
If $f(a) \times f(m) < 0$

New Interval: _____________

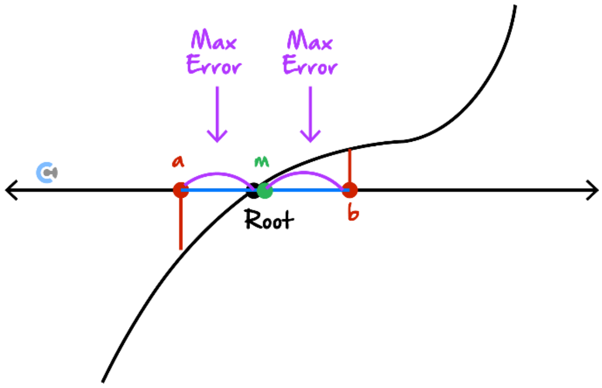
If $f(b) \times f(m) < 0$

New Interval: _____________

- Considering the diagram below, where would our new interval be? $[(a, m), (m, b)]$



Step 4: Repeat until the interval becomes short enough for good accuracy.



$$\text{If } \frac{b-a}{2} < \text{Max Tolerance}$$

We Stop

- The maximum error we can make is the distance between _____ and the _________________________________.
- Maximum error is ____________ of the width of the interval.

$$\text{Max Error} = \frac{b-a}{2}$$

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Definition

Bisection Method

- Step 1: Pick a random interval $[a, b]$ where $f(a) \times f(b) =$ Negative.
- Step 2: Find a midpoint to estimate the root.

$$\text{where } m = \frac{a+b}{2}$$

- Step 3: Create a new interval $[a, b]$ by making m either new a or new b.

$$\text{If } f(a) \times f(m) < 0$$

$$\text{New Interval: } [a, m]$$

$$\text{If } f(b) \times f(m) < 0$$

$$\text{New Interval: } [m, b]$$

- Step 4: Repeat until the interval becomes short enough for good accuracy.
  - The smaller the interval $[a, b]$, more accurate our estimation gets.

$$\text{If } \frac{b-a}{2} < \text{Max Tolerance}$$

$$\text{We stop}$$

  - Maximum error is half of the width of the interval.

$$\text{Max Error} = \frac{b-a}{2}$$

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Question 27 Walkthrough. Tech-Active.

The equation $x^3 + 12x + 12 = 0$ has one real solution, which lies in the interval $[-1,0]$. Approximate the solution using the bisection method with a maximum error of 0.2.

NOTE: We always pick the interval such that $f(a) \times f(b) = \text{Negative} \rightarrow x$-intercept is between a and b.

Your turn!

Question 28 Tech-Active.

The equation $x^3 + 3x + 6 = 0$ has one real solution, which lies in the interval $[-2, -1]$. Approximate the solution using the bisection method with a maximum error of 0.1.

NOTE: Keep going until the length of the interval is less than $2 \times$ maximum error.

TI UDF

Bisection Method

Overview:

Apply the bisection method to a function to approximate x intercepts.

Input:

```
bisection(<function>, <variable>, <lower bound>, <upper bound>)
```

Other notes:

- The program will ask for the threshold type to terminate the algorithm.
- Select None [0] to provide a specific number of iterations
- Select x [1] to provide a threshold for b-a, after which the program will stop if b-a becomes smaller than the threshold.
- Select y[2] to provide a threshold for |f(b)-f(a)|, after which the program will stop if |f(b)-f(a)| becomes smaller than the threshold.

$bisection(x^2-2,x,0,1)$

Number of Iterations: 5

"n"	"a"	"m"	"b"	"f(a)"	"f(m)"	"f(b)"	"b−a"	"|f(b)−f(a)|"
0.	0.	0.5	1.	-2.	-1.75	-1.	1.	1.
1.	0.5	0.75	1.	-1.75	-1.4375	-1.	0.5	0.75
2.	0.75	0.875	1.	-1.4375	-1.23438	-1.	0.25	0.4375
3.	0.875	0.9375	1.	-1.23438	-1.12109	-1.	0.125	0.234375
4.	0.9375	0.96875	1.	-1.12109	-1.06152	-1.	0.0625	0.121094
5.	0.96875	0.984375	1.	-1.06152	-1.03101	-1.	0.03125	0.061523

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Section F: Exam 2 Questions (n Marks)

Question 29 (1 mark)

The equation $5x^3 + 2x - 8 = 0$ has one real solution, which lies in the interval $[-4, 2]$. Approximate the solution using the bisection method with a maximum error of 0.4. What is the approximate solution?

A. $x \approx 0.675$

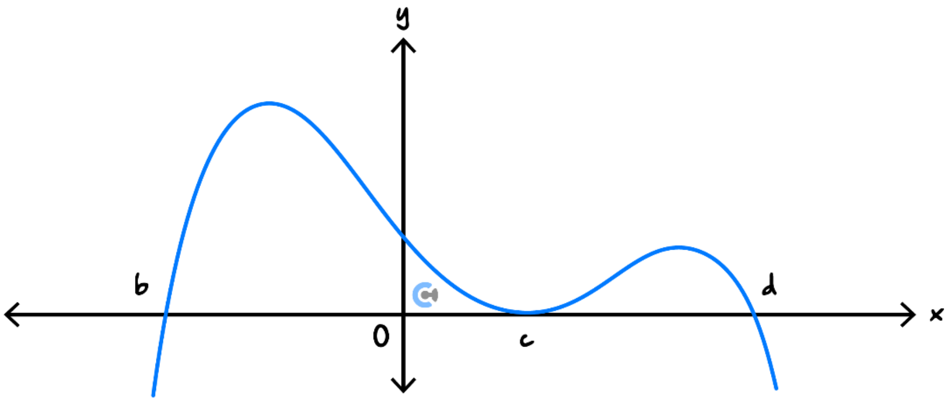
B. $x \approx 1.925$

C. $x \approx 0.875$

D. $x \approx 1.225$

Question 30 (1 mark)

The rule for a function with the graph below could be:



A. $y = -2(x + b)(x - c)^2(x - d)$

B. $y = 2(x + b)(x - c)^2(x - d)$

C. $y = -2(x - b)(x - c)^2(x - d)$

D. $y = 2(x - b)(x - c)\ (x - d)$

Space for Personal Notes

Question 31 (1 mark)

The polynomial $x^3 + ax^2 + bx + 4$ is perfectly divisible by $x - 1$ and has a reminder of 3 when divided by $x + 2$. The values (a, b) are:

A. $(-1, 4)$

B. $\left(-\frac{1}{2}, -\frac{9}{2}\right)$

C. $\left(-\frac{3}{5}, -\frac{5}{2}\right)$

D. $\left(-\frac{7}{2}, \frac{3}{2}\right)$

Question 32 (1 mark)

The equation $x^3 - 5kx^2 + 4x = 0$ has exactly two solutions when:

A. $k = \pm\frac{4}{5}$

B. $-\frac{4}{5} < k < \frac{4}{5}$

C. $k > \frac{4}{5}$

D. $k < -\frac{4}{5}$

Question 33 (1 mark)

A graph with the rule $f(x) = \frac{1}{3}x^3 - x^2 + c$, where c is a real number, has three distinct x-intercepts. All possible values of c are:

A. $c > \frac{4}{3}$

B. $-\frac{4}{3} < c < 0$

C. $0 < c < \frac{4}{3}$

D. $c < \frac{4}{3}$

Question 34 (11 marks)

Consider the cubic polynomial $f(x) = x^3 - 3x^2 - 3x - 4$.

a. Explain why $f(x)$ has a root between $x = 3$ and $x = 5$. (1 mark)

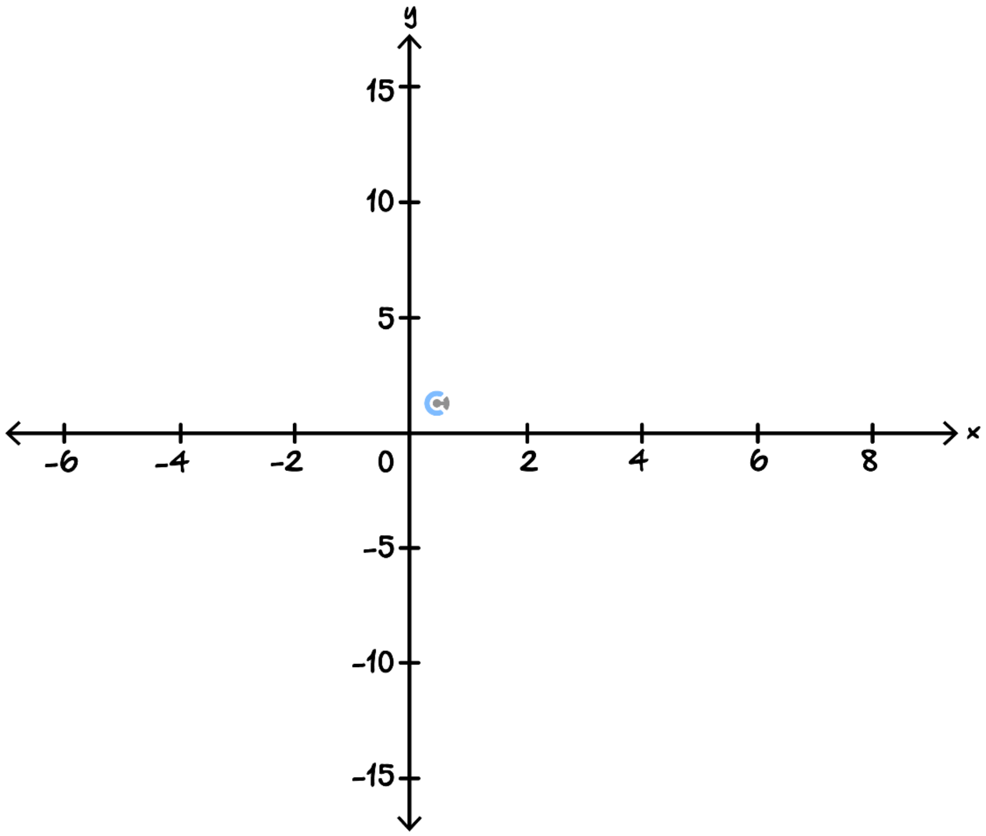
b. Write $f(x)$ in the form $f(x) = (x - a)\,Q(x)$ where $a > 0$ and $Q(x)$ is a quadratic function. (1 mark)

c. Find the values of k for which the equation $x^3 - 3x^2 + kx - 4x - 4k = 0$ has three solutions. (2 marks)

Consider the function $g(x) = \frac{1}{4}(x^3 - 3x^2 - 24x + 32)$.

d. Solve the equation $g(x) = 0$. Give your answer correct to two decimal places. (1 mark)

e. Sketch the graph of $y = g(x)$ on the axes below. Label all turning point(s) and the y-intercept(s) with coordinates. (3 marks)



f. Find the values of k such that $g(x) + k = 0$ has:

i. One solution. (2 marks)

ii. Three solutions. (1 mark)

Contour Check

Learning Objective: [1.6.1] – Solve Polynomial Inequalities

Key Takeaways

- [ ] The 'value' of $f(x)$ is its ________________ value.
- [ ] $f(x) > 0$ means find the x values for which the y values are ________________.

Learning Objective: [1.6.2] – Solve Number of Solution Problems

Key Takeaways

- [ ] When a cubic has n roots, the quadratic factor has $n - 1$ roots.

Learning Objective: [1.6.3] - Apply Bisection Method to Approximate x-Intercepts

Key Takeaways

- [ ] When two points are on the opposite of the axis, there is an x-intercept ____________________ the two points.
- [ ] When applying the bisection method over $[a, b]$, if $f(a) \times f(m) < 0$ then m becomes the new __________________ bound.
- [ ] If $f(b) \times f(m) < 0$ then f becomes the new ______________ bound.

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