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# VCE Mathematical Methods ½
## Polynomials Exam Skills [1.6]
## Homework Solutions

### Homework Outline:

| | |
|---|---|
| Compulsory Questions | Pg 2 – Pg 25 |
| Supplementary Questions | Pg 26 – Pg 47 |

## Section A: Compulsory Questions

### Sub-Section [1.6.1]: Solve Polynomial Inequalities

**Question 1**

Solve the following inequalities for $x$:

**a.** $(x-5)(x+2)(x-1) > 0$

$-2 < x < 1$ or $x > 5$

**b.** $(x-1)(2-x)(x+3) < 0$

$-3 < x < 1$ or $x > 2$

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**Question 2**

Solve the following inequalities for $x$:

**a.** $x(x^2 - 4x + 6) > 0$

$x > 0$

**b.** $(3 - x)(x^2 - 5x + 4) < 0$

$1 < x < 3$ or $x > 4$

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**Question 3**

Solve the following inequalities for $x$:

**a.** $x^3 - x^2 - 14x + 24 \leq 0$

Factor as $(x-2)(x-3)(x+4)$.
Therefore $x \leq -4$ or $2 \leq x \leq 3$

**b.** $2x^3 - 7x^2 - 33x + 18 > 0$

Factor as $2\left(x - \frac{1}{2}\right)(x+3)(x-6)$

Therefore $-3 < x < \frac{1}{2}$ or $x > 6$.

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## Sub-Section [1.6.2]: Solve Number of Solution Problems

**Question 4**

Find the values of $k$, for which the equation $x^3 + 3kx^2 + 9x = 0$ has:

**a.** 1 solution.

Only solution is $x = 0$ if the discriminant of the quadratic $x^2 + 3kx + 9$ is less than 0. Therefore $-2 < k < 2$

**b.** 2 solutions.

$k = \pm 2$

**c.** 3 solutions.

$k < -2$ or $k > 2$

**Question 5**

Find the values of $k$, for which the equation $x^3 + 3x^2 - 4kx = 0$ has:

**a.** 1 solution.

Only solution is $x = 0$ if the discriminant of the quadratic $x^2 + 3x - 4k$ is less than 0.
Therefore $k < -\frac{9}{16}$

**b.** 2 solutions.

$k = -\frac{9}{16}$ or $k = 0$

**c.** 3 solutions.

$-\frac{9}{16} < k < 0$ or $k > 0$

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**Question 6**

Find the values of $k$, for which the equation $(x^2 - 4kx + 8)(x^2 - 4x + 4k) = 0$ has:

**a.** 4 solutions.

We require the discriminant for both quadratics to be greater than zero.

$$\Delta_1 = 16k^2 - 32$$
$$\Delta_2 = 16 - 16k$$

Therefore $k < -\sqrt{2}$ or $k > \sqrt{2}$ **and** $k < 1$.
So 4 solutions if $k < -\sqrt{2}$

**b.** 3 solutions.

One discriminant is zero and the other is greater than zero.
$k = \pm\sqrt{2}$ **and** $k < 1 \implies k = -\sqrt{2}$
$-\sqrt{2} < k$ or $k > \sqrt{2}$ **and** $k = 1$ cannot be satisfied.
Therefore $k = -\sqrt{2}$

**c.** 2 solutions.

One discriminant is less than zero and the other discriminant is greater than zero.
$-\sqrt{2} < k < \sqrt{2}$ **and** $k < 1 \implies -\sqrt{2} < k < 1$
$k < -\sqrt{2}$ or $k > \sqrt{2}$ **and** $k > 1 \implies k > \sqrt{2}$
Therefore $-\sqrt{2} < k < 1$ or $k > \sqrt{2}$

**d.** 1 solution.

One discriminant equals zero and the other is less than zero.
$k = \pm\sqrt{2}$ **and** $k > 1 \implies k = \sqrt{2}$
$-\sqrt{2} < k < \sqrt{2}$ **and** $k = 1 \implies k = 1$
Therefore $k = \sqrt{2}$ or $k = 1$

**e.** No solutions.

Both discriminants are less than zero.
Therefore $1 < k < \sqrt{2}$

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## Sub-Section [1.6.3]: Apply Bisection Method to Approximate $x$-Intercepts

**Question 7 CAS-Active.**

Use the bisection method to find the approximate real solution to the equation $x^3 - 3x^2 + 3x + 2 = 0$. Use the interval $[-1, 1]$ for the first iteration and a maximum error of 0.1. Give your approximation correct to two decimal places.

Our answer is the midpoint of the first interval that has width $< 0.2$

$x \approx -0.44$

```
In[28]:= ResourceFunction["BisectionMethodFindRoot"][x^3 - 3 x^2 + 3 x + 2,
  {x, -1, 1}, 3, 6, "Steps"]
```

Out[28]=

| steps | a | f[a] | b | f[b] |
|---|---|---|---|---|
| 1 | -1.00 | -5. | 1.00 | 3. |
| 2 | -1.00 | -5. | $0. \times 10^{-3}$ | 2. |
| 3 | -0.50 | -0.375 | $0. \times 10^{-3}$ | 2. |
| 4 | -0.50 | -0.375 | -0.25 | 1.04688 |
| 5 | -0.50 | -0.375 | -0.38 | 0.400391 |
| 6 | -0.50 | -0.375 | -0.44 | 0.029541 |

**Question 8 CAS-Active.**

Use the bisection method to find the approximate real solution to the equation $x^2 \log_2(x) - 3x - 2 = 0$. Use the interval $[1, 4]$ for the first iteration and a maximum error of 0.1. Give your approximation correct to two decimal places.

Our answer is the midpoint of the first interval that has width $< 0.2$

$x \approx 2.59$

```
In[47]:= ResourceFunction["BisectionMethodFindRoot"][
  x^2 Log[2, x] - 3 x - 2, {x, 1, 4}, 8, 6, "Steps"]
```

Out[47]=

| steps | a | f[a] | b | f[b] |
|---|---|---|---|---|
| 1 | 1.0000000 | -5. | 4.0000000 | 18. |
| 2 | 2.5000000 | -1.23795 | 4.0000000 | 18. |
| 3 | 2.5000000 | -1.23795 | 3.2500000 | 6.21089 |
| 4 | 2.5000000 | -1.23795 | 2.8750000 | 1.96819 |
| 5 | 2.5000000 | -1.23795 | 2.6875000 | 0.23892 |
| 6 | 2.5937500 | -0.530619 | 2.6875000 | 0.23892 |

**Question 9 CAS-Active.**

Use the bisection method to approximate $\sqrt[3]{5}$ correct to two decimal places.

Since $1^3 < 5 < 2^3$ we will choose our interval to be $[1, 2]$ and we will solve the equation $x^3 - 5 = 0$.

Our answer is the midpoint of the first interval that has width $< 0.01$ rounded to two decimal places.

$x \approx 1.71$

```
ResourceFunction["BisectionMethodFindRoot"][x^3 - 5, {x, 1, 2}, 4, 9, "Steps"]
```

| steps | a | f[a] | b | f[b] |
|---|---|---|---|---|
| 1 | 1.000 | -4. | 2.000 | 3. |
| 2 | 1.500 | -1.625 | 2.000 | 3. |
| 3 | 1.500 | -1.625 | 1.750 | 0.359375 |
| 4 | 1.625 | -0.708984 | 1.750 | 0.359375 |
| 5 | 1.688 | -0.19458 | 1.750 | 0.359375 |
| 6 | 1.688 | -0.19458 | 1.719 | 0.0773621 |
| 7 | 1.703 | -0.0598564 | 1.719 | 0.0773621 |
| 8 | 1.703 | -0.0598564 | 1.711 | 0.00843954 |
| 9 | 1.707 | -0.0257866 | 1.711 | 0.00843954 |

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## Sub-Section: Exam 1 Questions

**Question 10**

Consider the polynomial $f(x) = x^3 + ax^2 + bx + 4$. It is known that $x - 1$ is a factor of $f$ and when $f$ is divided by $x - 2$ the remainder is 6. Find the values of $a$ and $b$.

We have that $f(1) = 0$ and $f(2) = 6$ this yields the equations

$$5 + a + b = 0$$
$$12 + 4a + 2b = 6$$

which we solve to find $a = 2$ and $b = -7$

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**Question 11**

Solve the equation $2x^3 - 4x^2 - 22x + 24 = 0$.

Let $f(x) = 2x^3 - 4x^2 - 22x + 24$. Then note that $f(1) = 0 \implies x - 1$ is a factor of $f(x)$. We then factorise $f(x)$ as

$$f(x) = 2(x-1)(x-4)(x+3)$$

and so $f(x) = 0 \implies x = -3, 1, 4$

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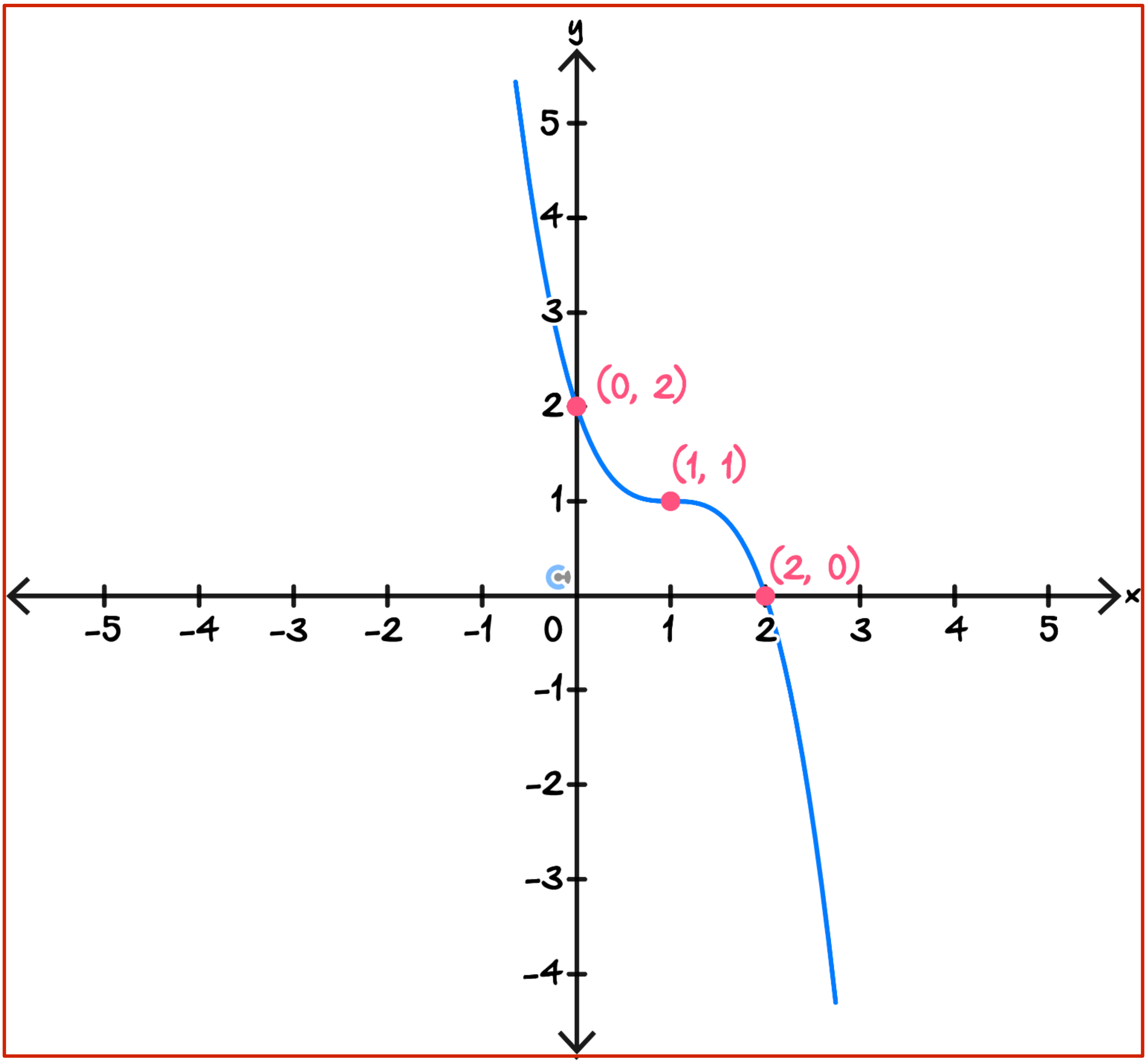
**Question 12**

Sketch the graph of $y = -(x - 1)^3 + 1$ on the axes below. Label all axis intercepts and the inflection point with coordinates.

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**Question 13**

Consider the function $f(x) = x^4 + x^3 - 3x^2 - x + 2$.

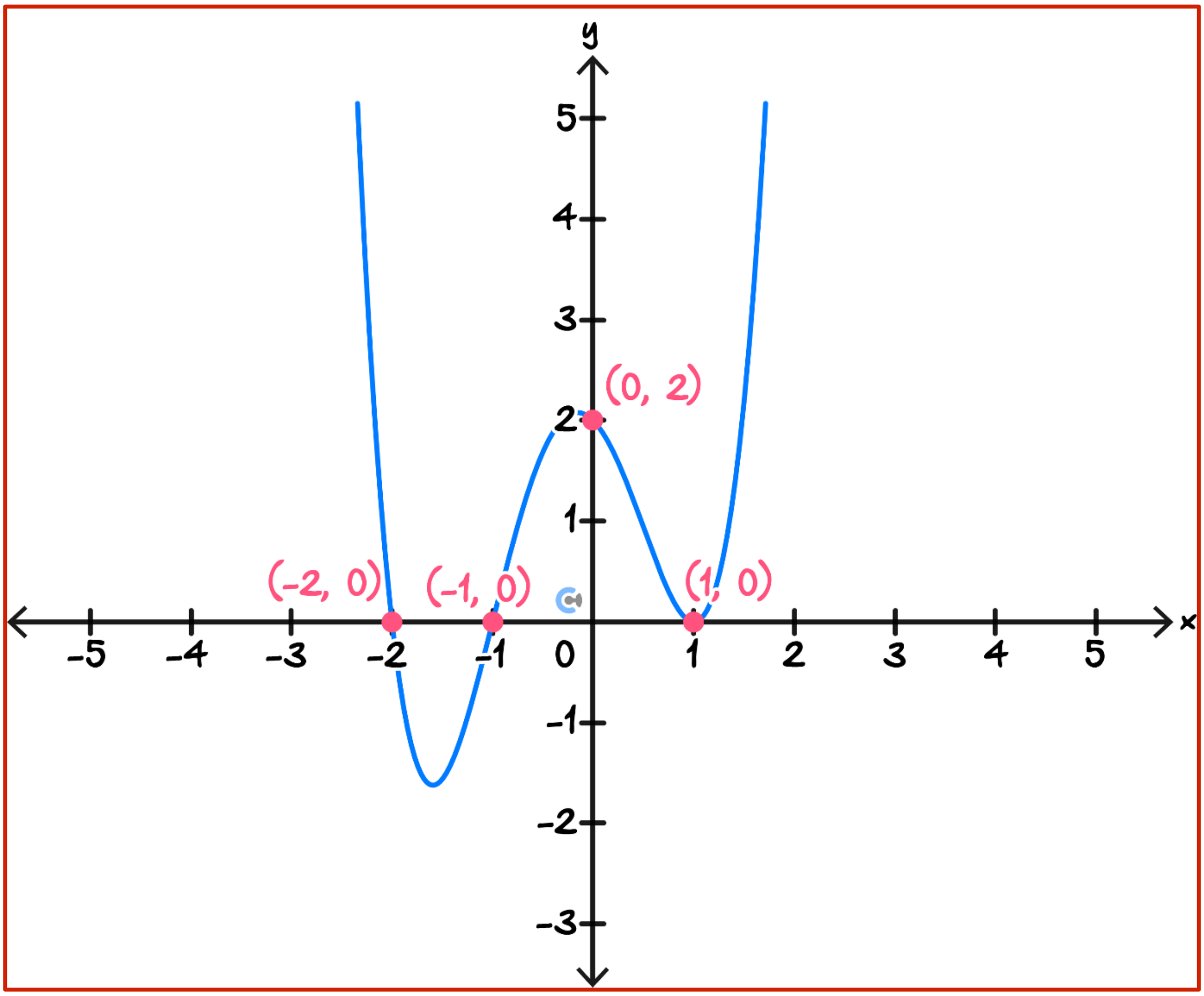
**a.** Show that $x + 2$ is a factor of $f(x)$.

$f(-2) = 16 - 8 - 12 + 2 + 2 = 0$
Therefore $x + 2$ is a factor of $f(x)$

**b.** Fully factorise $f(x)$.

$$\begin{aligned} f(x) &= (x+2)(x^3 - x^2 - x + 1) \\ &= (x+2)(x-1)(x^2-1) \\ &= (x-1)^2(x+1)(x+2) \end{aligned}$$

**c.** Hence, sketch the graph of $y = f(x)$. Label all axis intercepts with coordinates. Note that some turning points occur at approximately $(-1.59, -1.63)$ and $(-0.16, 2.08)$.




**d.** Solve the inequality $f(x) \leq 0$.

$-2 \leq x \leq -1$ or $x = 1$.

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**Question 14**

Consider $f(x) = 2x^3 + 2kx^2 + 5x$, where $k$ is a real constant.

Find the values of $k$, such that $f(x) = 0$ has:

**a.** One solution.

We have that $f(x) = x(2x^2 + 2kx + 5)$.
The only solution will be $x = 0$ if $\Delta = 4k^2 - 40 < 0$ and therefore

$$-\sqrt{10} < k < \sqrt{10}$$

**b.** Two solutions.

$k = \pm\sqrt{10}$

**c.** Three solutions.

$k < -\sqrt{10}$ or $k > \sqrt{10}$

## Sub-Section: Exam 2 Questions

**Question 15**

The equation $3x^2 + 2x - 8 = 0$ has one real solution, which lies in the interval $[0,2]$. Approximate the solution using the bisection method with a maximum error of $0.1$. The approximate solution correct to two decimal places is:

**A.** $x \approx 1.25$

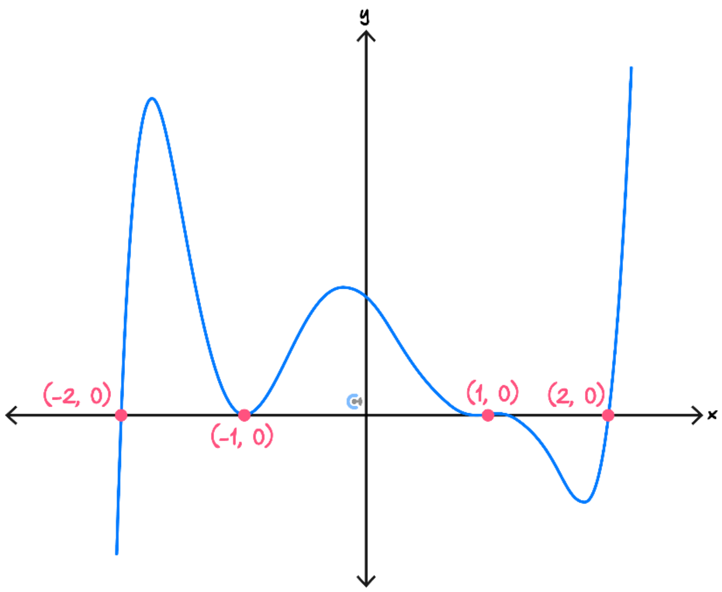
**B.** $x \approx 1.13$

**C.** $x \approx 1.19$

**D.** $x \approx 1.15$

**Question 16**

The minimum degree of the polynomial sketched below is:




**A.** 5

**B.** 6

**C.** 7

**D.** 8

**Question 17**

The polynomial $ax^3 + 3x^2 + bx + 5$ is perfectly divisible by $x - 1$ and has a remainder of 6 when divided by $x + 2$. The values $(a, b)$ are:

**A.** $(8, -12)$

**B.** $\left(-\frac{1}{2}, -\frac{9}{2}\right)$

**C.** $\left(-\frac{3}{2}, -\frac{5}{2}\right)$

**D.** $\left(\frac{9}{2}, -\frac{25}{2}\right)$

**Question 18**

The equation $x^3 - 5kx^2 + 9x = 0$ has exactly one solution when:

**A.** $k = \pm\frac{6}{5}$

**B.** $-\frac{6}{5} < k < \frac{6}{5}$

**C.** $k > \frac{6}{5}$

**D.** $k < -\frac{6}{5}$

**Question 19**

A graph with rule $f(x) = x^3 - 3x^2 - 4c$, where $c$ is a real number, has three distinct $x$-intercepts. All possible values of $c$ are:

**A.** $c > 1$

**B.** $-1 < c < 0$

**C.** $0 < c < 1$

**D.** $c < 1$

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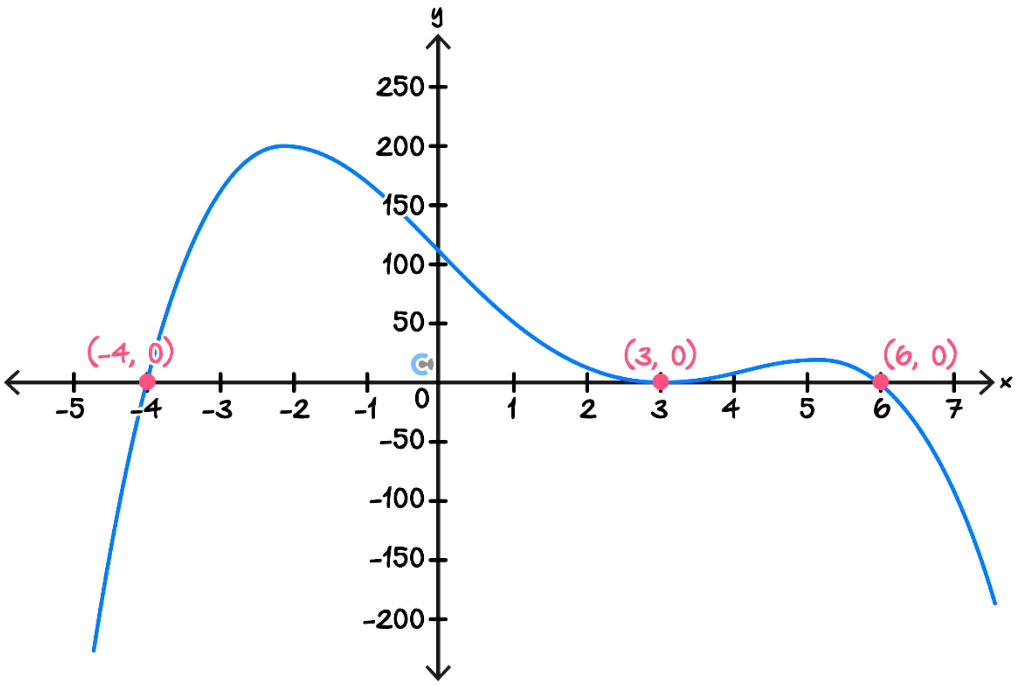
**Question 20**

Consider the function $f$ that is sketched on the axes below. It is given that the point $(2,12)$ lies on the graph.




**a.**

**i.** State the degree of $f$.

4

**ii.** Find a rule for $f(x)$.

From the shape of the graph we see that we must have

$$f(x) = a(x+4)(x-3)^2(x-6)$$

Now we use $f(2) = 12$ to find that $a = -\frac{1}{2}$. Therefore

$$f(x) = -\frac{1}{2}(x-3)^2(x+4)(x-6)$$

**b.** Consider the function $g(x) = f(x) + 10(k^2 - 4k + 3)$, where $k$ is a real constant.

**i.** Find the values of $k$ such that $g(x) = f(x)$.

$g(x) = f(x)$ if the quadratic term $10(k^2 - 4k + 3) = 0$.
Therefore $k = 1$ or $k = 3$

**ii.** Find the values of $k$, such that $g(x) > f(x)$.

$g(x) > f(x)$ if the quadratic $10(k^2 - 4k + 3) > 0$.
Therefore $k < 1$ or $k > 3$

It is known that the function $f(x)$ has a turning point when $x = \frac{3\pm\sqrt{51}}{2}$.

Let $h(x) = -\frac{4}{3}(x-3)^2(x+4)(x-6)$.

**c.** Find all values of $k$, such that $h(x) = k$ has two solutions.

Note that $h(x) = \frac{8}{3}f(x)$.

Therefore $h(x)$ has turning points when $x = \frac{3 \pm \sqrt{51}}{2}$.
The coordinates of these turnings points are therefore

$$\left(\frac{3-\sqrt{51}}{2}, 291 + 34\sqrt{51}\right) \quad \text{and} \quad \left(\frac{3+\sqrt{51}}{2}, 291 - 34\sqrt{51}\right)$$

Inspecting the shape of the graph of $y = f(x)$ we conclude that $h(x) = k$ will have two solutions when

$$k < 0 \text{ or } 291 - 34\sqrt{51} < k < 291 + 34\sqrt{51}$$

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**Question 21**

Consider the cubic polynomial $f(x) = x^3 + 2x^2 - 7x - 2$.

**a.**

**i.** Explain why $f(x)$ has a root between $x = -1$ and $x = 0$.

$f(-1) = 6$ and $f(0) = -2$ so it must cross the $x$-axis between these two points.

**ii.** Approximate the root in the interval $[-1,0]$ using the bisection method with a maximum error of 0.05. Give your answer correct to two decimal places.

The first interval that has width $< 0.1$ is $[-0.25, -0.3125]$ so our approximate solution is the midpoint of this interval.
Thus $x \approx -0.28$

```
In[332]:= ResourceFunction["BisectionMethodFindRoot"][f[x],
  {x, -1, 0}, 6, 10, "Steps"]
```

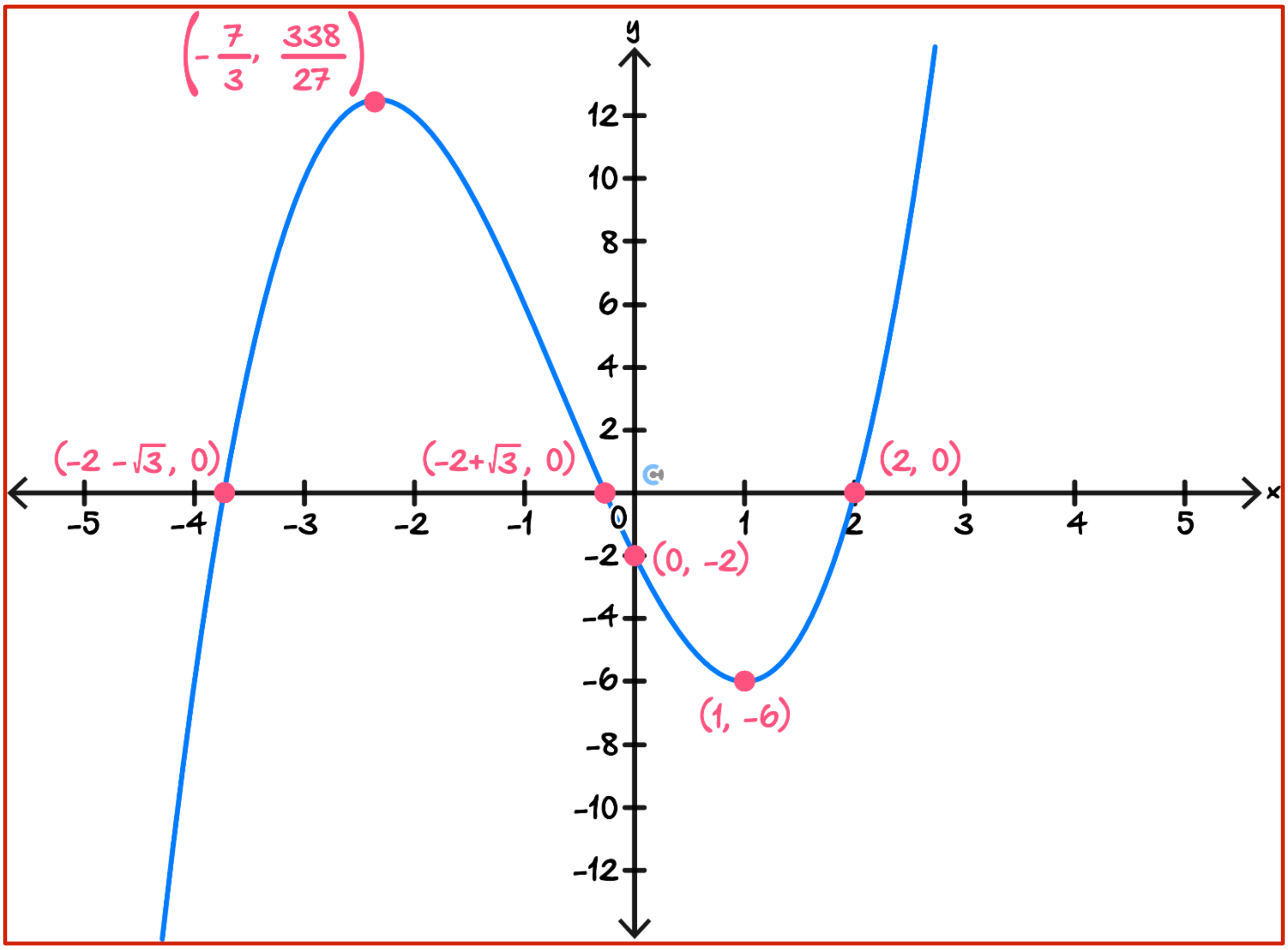
Out[332]=

| steps | a | f[a] | b | f[b] |
|---|---|---|---|---|
| 1 | 0 | -2. | -1.00000 | 6. |
| 2 | 0 | -2. | -0.500000 | 1.875 |
| 3 | -0.250000 | -0.140625 | -0.500000 | 1.875 |
| 4 | -0.250000 | -0.140625 | -0.375000 | 0.853516 |
| 5 | -0.250000 | -0.140625 | -0.312500 | 0.352295 |
| 6 | -0.250000 | -0.140625 | -0.281250 | 0.104706 |

**iii.** Find the distance between our approximate root and the actual root that lies in the interval $[-1,0]$. Give your answer correct to two decimal places.

Actual root is $x = \sqrt{3} - 2 \approx -0.2674849$
So our distance is $-0.2674849 + 0.281250 \approx 0.01$

**b.** Sketch the graph of $y = f(x)$ on the axes below. Label all turning points and axis intercepts with coordinates.




**c.** Find the values of $k$, such that $f(x) + k = 0$, where $k$ is a positive constant, has one solution.

By looking at the graph and noting that $k > 0$ we conclude that $k > 6$

**d.** Let $a$ be a real constant.

Find the values of $a$ such that the equation $x^3 - (4a + 2)x^2 + (8a + 3)x - 6 = 0$ has three real solutions.

We can factor the equation as

$$(x-2)(x^2-4ax+3)$$

this function will have three roots if the quadratic has two roots that are not equal to $x = 2$.
Consider the discriminant of the quadratic

$$\Delta = 16a^2 - 12 > 0 \implies a < -\frac{\sqrt{3}}{2} \text{ or } a > \frac{\sqrt{3}}{2}$$

But what if the quadratic also has a solution $x = 2$?
Sub in $x = 2$ into the quadratic $7 - 8a = 0 \implies a = \frac{7}{8} > \frac{\sqrt{3}}{2}$.
When $a = \frac{7}{8}$ our function becomes $\frac{1}{2}(x-2)^2(2x-3)$ so there are only two solutions.
Therefore, three solutions when

$$a < -\frac{\sqrt{3}}{2} \text{ or } \frac{\sqrt{3}}{2} < a < \frac{7}{8} \text{ or } a > \frac{7}{8}.$$

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## Section B: Supplementary Questions

### Sub-Section [1.6.1]: Solve Polynomial Inequalities

**Question 22**

Solve the following inequalities for $x$:

**a.** $x(x-1)(x+2) \leq 0$

$x \leq -2$ or $0 \leq x \leq 1$

**b.** $(x-2)(x+1)(x+3) > 0$

$-3 < x < -1$ or $x > 2$

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**Question 23**

Solve the following inequalities for $x$:

**a.** $(x-5)(x^2 + x - 2) \leq 0$

$x \leq 2$ or $1 \leq x \leq 5$

**b.** $(1-x)(x^2 - 4x + 4) \geq 0$

$x \leq 1$ *or* $x = 2$

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**Question 24**

Solve the following inequalities for $x$:

**a.** $x^3 - 5x^2 - 8x + 12 > 0$

Factor as $(x - 1)(x + 2)(x - 6)$.
Therefore $-2 < x < 1$ or $x > 6$

**b.** $-x^3 + 4x^2 + x - 4 \leq 0$

Factor as $-(x-1)(x+1)(x-4)$
Therefore $-1 \leq x \leq 1$ or $x \geq 4$.

**Question 25**

Solve the inequality $4x^5 - 16x^4 + 13x^3 - 3x^2 > 4x^3 - 16x^2 + 13x - 3$.

We rewrite our inequality as $4x^5 - 16x^4 + 9x^3 + 13x^2 - 13x + 3 > 0$ and realise that $x^2 - 1$ is a factor of the left hand side.
Hence we factor the left side as $(2x-1)^2(x-3)(x^2-1)$.
Therefore $-1 < x < \frac{1}{2}$ or $\frac{1}{2} < x < 1$ or $x > 3$.

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## Sub-Section [1.6.2]: Solve Number of Solution Problems

**Question 26**

Find the values of $k$, for which the equation $x(x^2 + 4) = 4kx^2$ has:

**a.** 1 solution.

Only solution is $x = 0$ if the discriminant of the quadratic $x^2 - 4kx + 4$ is less than 0.
Therefore $-1 < k < 1$

**b.** 2 solutions.

$k = \pm 1$

**c.** 3 solutions.

$k < -1$ or $k > 1$

**Question 27**

Find the values of $k$, for which the equation $kx^9 + 2x^6 + x^3 = 0$ has:

**a.** 1 solution.

We observe that $x^3$ is a one to one and onto function.
Thus the number of solutions to $kx^9 + 2x^6 + x^3 = 0$ is simply the number of solutions to $kx^3 + 2x^2 + x = x(kx^2 + 2x + 1) = 0$
Only solution is $x = 0$ if the discriminant of the quadratic $kx^2 + 2x + 1$ is less than 0.
Therefore $k > 1$

**b.** 2 solutions.

$k = 1$ or $k = 0$

**c.** 3 solutions.

$k < 0$ or $0 < k < 1$

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**Question 28**

Find the values of $k$, for which the equation $x(x-2k-2)(x^2+kx+4) = -x^2-kx-4$ has:

**a.** 4 solutions.

We can simplify our equation to be $(x^2+2x(k-1)+1)(x^2+kx+4)=0$
We require the discriminant for both quadratics to be greater than zero.

$$\Delta 1 = 4(k+1)^2 - 4$$
$$\Delta 2 = k^2 - 16$$

Therefore $k < -2$ or $k > 0$ and $k < -4$ or $k > 4$.
So 4 solutions if $k < -4$ or $k > 4$.

**b.** 3 solutions.

One discriminant is zero and the other is greater than zero.
Therefore $k = -4, 4$

**c.** 2 solutions.

One discriminant is less than zero and the other discriminant is greater than zero.
Therefore $-4 < k < 0$ or $2 < k < 4$

**d.** 1 solution.

One discriminant equals zero and the other is less than zero.
Therefore $k = 0$ or $k = -2$

**e.** No solutions.

Both discriminants are less than zero.
Therefore $-2 < k < 0$

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**Question 29**

Consider the polynomial $P(x) = x^3 + ax + b$.

Show that if $\Delta = -4a^3 - 27b^2 = 0$, that $P(x) = 0$ has less than 3 solutions.

Hint: If $r_1, r_2, r_3$ are the roots of $P(x)$, show that $\Delta = (r_1 - r_2)^2(r_2 - r_3)^2(r_3 - r_1)^2$.

*Please use a calculator.*

If $P(x) = 0$ has 3 distinct solutions, we can factorise $P(x) = (x - r_1)(x - r_2)(x - r_3)$, where $r_1, r_2, r_3$ are all different.
By expanding our factorised form, we can compare $x^2$ coefficients to see that $r_1 + r_2 + r_3 = 0$, hence $P(x) = (x - r_1)(x - r_2)(x + r_1 + r_2)$.
Since $r_3 = -r_2 - r_1$, we can express $(r_1 - r_2)^2(r_2 - r_3)^2(r_3 - r_1)^2$ in terms of $r_2$ and $r_1$. Similarly we can express $a$ and $b$, and thus $\Delta$ in terms of $r_2$ and $r_1$. The following equations show that our expressions are equal.

$$\begin{aligned}
\Delta &= -4a^3 - 27b^2 \\
&= -4(-r_1^2 - r_1 r_2 - r_2^2)^3 - 27(r_1^2 r_2 + r_1 r_2^2)^2 \\
&= 4r_1^6 + 12r_2 r_1^5 - 3r_2^2 r_1^4 - 26r_2^3 r_1^3 - 3r_2^4 r_1^2 + 12r_2^5 r_1 + 4r_2^6 \\
&= (r_1 - r_2)^2(2r_1 + r_2)^2(r_1 + 2r_2)^2 \\
&= (r_1 - r_2)^2(r_2 - r_3)^2(r_3 - r_1)^2
\end{aligned}$$

Thus if $\Delta = 0$ we see that either $r_1 = r_2$ or $r_2 = r_3$ or $r_3 = r_1$, hence $P(x) = 0$ has less than 3 solutions.

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## Sub-Section [1.6.3]: Apply Bisection Method to Approximate $x$-Intercepts

**Question 30 CAS-Active.**

Use the bisection method to find the approximate real solution to the equation $x^3 + 2x^2 - 5x + 3 = 0$. Use the interval $[-4, -3]$ for the first iteration and a maximum error of 0.1. Give your approximation correct to two decimal places.

Our answer is the midpoint of the first interval that has width $< 0.2$

$x \approx -3.61$

```
ResourceFunction["BisectionMethodFindRoot"][x^3 + 2 x^2 - 5 x + 3,
 {x, -4, -3}, 4, 8, "Steps"]
```

| steps | a | f[a] | b | f[b] |
|---|---|---|---|---|
| 1 | -4.000 | -9. | -3.000 | 9. |
| 2 | -4.000 | -9. | -3.500 | 2.125 |
| 3 | -3.750 | -2.85938 | -3.500 | 2.125 |
| 4 | -3.625 | -0.228516 | -3.500 | 2.125 |
| 5 | -3.625 | -0.228516 | -3.563 | 0.982178 |
| 6 | -3.625 | -0.228516 | -3.594 | 0.385406 |
| 7 | -3.625 | -0.228516 | -3.609 | 0.0806007 |
| 8 | -3.617 | -0.0734172 | -3.609 | 0.0806007 |

**Question 31 CAS-Active.**

Use the bisection method to find the approximate real solution to the equation $x\log_2(x) + 3x = 4$. Use the interval $[0.1, 2]$ for the first iteration and a maximum error of 0.01. Give your approximation correct to two decimal places.

Our answer is the midpoint of the first interval that has width $< 0.02$

$x \approx 1.22$

```
In[38]:= ResourceFunction["BisectionMethodFindRoot"][x * Log[2, x] + 3 x - 4,
 {x, 0.1, 2}, 8, 9, "Steps"]
```

Out[38]=

| steps | a | f[a] | b | f[b] |
|---|---|---|---|---|
| 1 | 0.10000000 | -4.03219 | 2.0000000 | 4. |
| 2 | 1.0500000 | -0.776091 | 2.0000000 | 4. |
| 3 | 1.0500000 | -0.776091 | 1.5250000 | 1.50343 |
| 4 | 1.0500000 | -0.776091 | 1.2875000 | 0.331887 |
| 5 | 1.1687500 | -0.230821 | 1.2875000 | 0.331887 |
| 6 | 1.1687500 | -0.230821 | 1.2281250 | 0.0484618 |
| 7 | 1.1984375 | -0.0917099 | 1.2281250 | 0.0484618 |
| 8 | 1.2132813 | -0.0217551 | 1.2281250 | 0.0484618 |
| 9 | 1.2132813 | -0.0217551 | 1.2207031 | 0.0133208 |

**Question 32 CAS-Active.**

Use the bisection method to approximate $\pi$ correct to three decimal places.

We choose a function $f$ such that $f(0) = \pi$.
Such a function can be $f(x) = \sin(x)$
Our answer is the midpoint of the first interval that has width $< 0.001$
Since $3 < \pi < 4$ we can choose an interval of $[3, 4]$.
$x \approx 3.141$

```
ResourceFunction["BisectionMethodFindRoot"][Sin[x], {x, 3, 4}, 6, 12, "Steps"]
```

| steps | a | f[a] | b | f[b] |
|---|---|---|---|---|
| 1 | 4.00000 | -0.756802 | 3.00000 | 0.14112 |
| 2 | 3.50000 | -0.350783 | 3.00000 | 0.14112 |
| 3 | 3.25000 | -0.108195 | 3.00000 | 0.14112 |
| 4 | 3.25000 | -0.108195 | 3.12500 | 0.0165919 |
| 5 | 3.18750 | -0.0458912 | 3.12500 | 0.0165919 |
| 6 | 3.15625 | -0.0146568 | 3.12500 | 0.0165919 |
| 7 | 3.15625 | -0.0146568 | 3.14063 | 0.000967653 |
| 8 | 3.14844 | -0.00684479 | 3.14063 | 0.000967653 |
| 9 | 3.14453 | -0.00293859 | 3.14063 | 0.000967653 |
| 10 | 3.14258 | -0.000985471 | 3.14063 | 0.000967653 |
| 11 | 3.14160 | $-8.90891 \times 10^{-6}$ | 3.14063 | 0.000967653 |
| 12 | 3.14160 | $-8.90891 \times 10^{-6}$ | 3.14111 | 0.000479372 |

**Question 33**

Explain why you cannot use the bisection method to approximate the solution to the equation $x^4 - 2x^2 + 1 = 0$.

Because $f(x) = x^4 - 2x^2 + 1 = (x^2 - 1)^2 > 0$ we will not be able to find an initial interval $[a, b]$ such that $f(a)f(b) < 0$.

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## Sub-Section: Exam 1 Questions

**Question 34**

Consider the polynomial $f(x) = x^3 - 7x + 6$.

**a.** Show that $f(1) = 0$.

$f(1) = 1^3 - 7 \times 1 + 6 = 0$

**b.** Solve $f(x) = 0$ for $x$.

Since $f(1) = 0$ we know that $x - 1$ is a factor of $f$.
We can then factorise $f$ as such:

$$f(x) = x^3 - 7x + 6 = (x-1)(x^2 + x - 6) = (x-1)(x-2)(x-3)$$

Hence $x = -3, 1, 2$.

**c.** Hence, solve $f(x) \geq 0$ for $x$.

$-3 \leq x \leq 1$ or $x \geq 2$.

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**Question 35**

For what values of $k$ does the equation $k(x^3 + x^2) = x$ have exactly one solution.

We can rewrite our equation to be $x(kx^2 + kx - 1) = 0$.
We see that if $k \neq 0$ we have a product of a quadratic and $x$, hence we require the quadratic to have no solution.
Hence the discriminant of the quadratic $\Delta = k^2 + 4k < 0$.
From the graph of $k^2 + 4k$ we see that it is less than 0 if $-4 < k < 0$.
Now if $k = 0$ our equation turns to $x = 0$ which obviously has one solution.
Thus our equation has exactly one solution of $-4 < k \leq 0$.

**Question 36**

Consider the polynomial $f(x) = x^3 - 3x^2 + x + 1$.

**a.** Fully factorise $f(x)$ into linear factors.

Since $f(1) = 0$ we know that $x - 1$ is a factor of $f$.
Hence $f(x) = (x-1)(x^2 - 2x - 1)$.
We can solve $x^2 - 2x - 1 = 0$ to extract the other linear factors.
The other linear factors are, $1 \pm \sqrt{2}$.
Hence $f(x) = (x-1)(x-1-\sqrt{2})(x-1+\sqrt{2})$

**b.** A bisection method is used to solve $f(x) = 0$ with the first interval being [2,3].
Use the fact that $\sqrt{2} \approx 1.4$ to write down the next 3 intervals.

The only root we can approximate with this method is $1 + \sqrt{2} \approx 2.4$.
Thus all of our intervals will have to contain 2.4.
Hence our first interval is $[2, 2.5]$, our second interval is $[2.25, 2.5]$ and our third interval is $[2.375, 2.5]$.

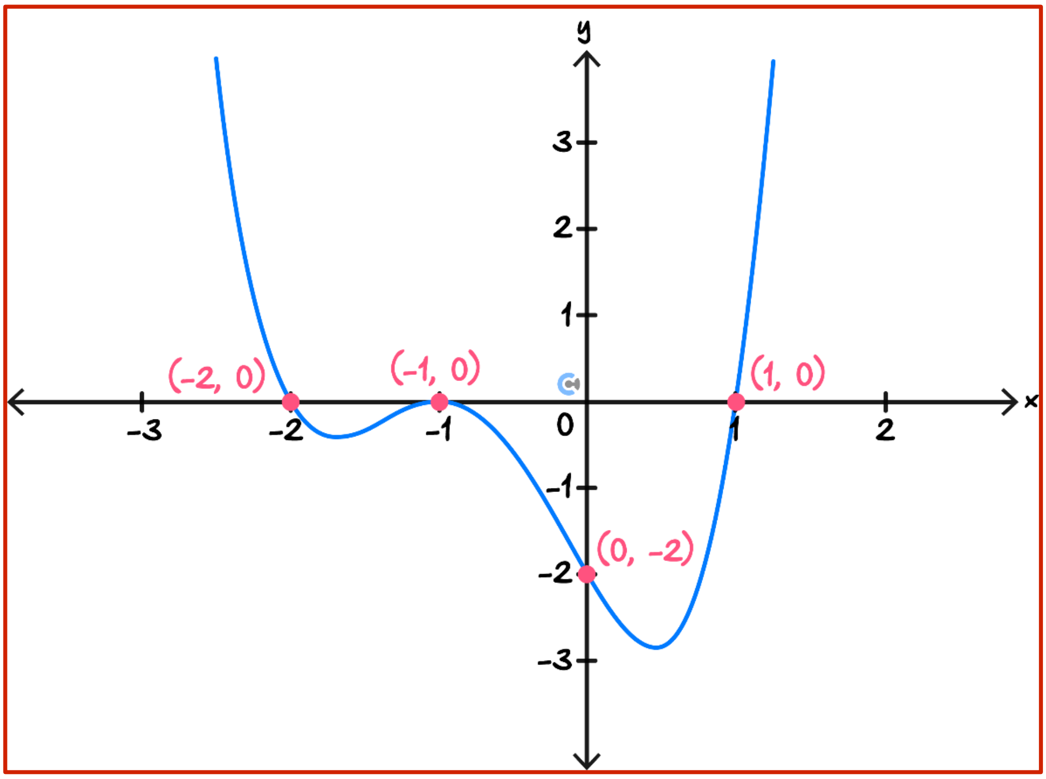
**Question 37**

Let $f(x) = x^4 + 3x^3 + x^2 - 3x - 2$.

**a.** Show that $x^2 - 1$ is a factor of $f$.

Since $f(1) = 1 + 3 + 1 - 3 - 2 = 0$ we know that $x - 1$ is a factor of $f$.
Since $f(-1) = 1 - 3 + 1 + 3 - 2 = 0$ we know that $x + 1$ is a factor of $f$.
Hence $x^2 - 1 = (x - 1)(x + 1)$ must be a factor of $f$.

**b.** Sketch the graph of $y = f(x)$ on the axis below. Label all axis intercepts with their coordinates. Note that some turning points occur at $(-1.69, -0.40)$ and $(0.44, -2.83)$.

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## Sub-Section: Exam 2 Questions

**Question 38**

The equation $x^2(x - 2k) = -2x$ has exactly two solutions when,

**A.** $k < -\sqrt{2}$ or $k > \sqrt{2}$.

**B.** $k = \pm\sqrt{2}$

**C.** $-\sqrt{2} < k < 0$ or $0 < k < \sqrt{2}$.

**D.** $-\sqrt{2} < k < \sqrt{2}$

**Question 39**

The polynomial $x^3 + ax^2 - 2x + b$ has a factor of $x + 1$, and has a remainder of $12$ when divided by $x - 2$. The values of $a$ and $b$ are:

**A.** $a = 3$ and $b = -4$.

**B.** $a = \frac{7}{3}$ and $b = -\frac{4}{3}$.

**C.** $a = \frac{17}{3}$ and $b = -\frac{20}{3}$.

**D.** $a = 5$ and $b = -4$.

**Question 40**

A bisection method is used to solve the equation $x^3 = 7$. The initial interval is $[1,2]$. The bisection reduces this interval down four times, and then takes the midpoint of the final interval. The result of this method is closest to:

**A.** 1.94

**B.** 1.92

**C.** 1.91

**D.** 1.88

**Question 41**

The equation $kx^3 - 3kx = 1$ has exactly one solution.

The possible values of $k$ are:

**A.** $k < -2$ or $k > 2$.

**B.** $-2 < k < 2$

**C.** $k < -\frac{1}{2}$ or $k > \frac{1}{2}$.

**D.** $-\frac{1}{2} < k < \frac{1}{2}$

**Question 42**

The maximum number of $x$-intercepts a quartic can have is:

**A.** 2

**B.** 3

**C.** 4

**D.** 5

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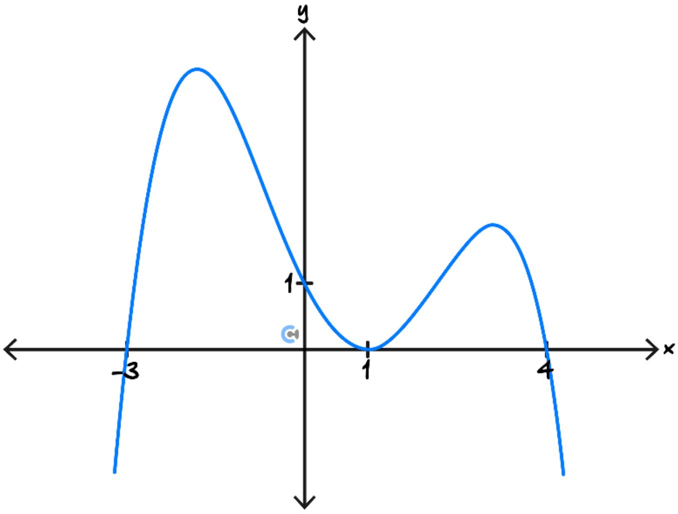
**Question 43**

The graph of $f(x) = ax^4 + bx^3 + cx^2 + dx + 1$ is drawn below.




**a.** Find the values of $a, b, c$ and $d$.

$f(x) = (x-1)^2(x+3)(x-4) = -\frac{1}{12}x^4 + \frac{1}{4}x^3 + \frac{3}{4}x^2 - \frac{23}{12}x + 1.$

Thus $a = -\frac{1}{12}, b = \frac{1}{4}, c = \frac{3}{4}$ and $d = -\frac{23}{12}$

**b.** Hence or otherwise, solve $f(x) > 1$. Give your answers correct to 2 decimal places.

$-2.88 < x < 0$ or $2.12 < x < 3.77$

**c.** Find all values of $a$ correct to 3 decimal places such that $f(x) = a$ has exactly three solutions.

$a = 0$ or $a = 2.018$

**d.** Consider the polynomial $g(x) = (x - a)^2(x + 3)(x - 4)$.

**i.** For what values of $a$ is the solution to $g(x) \leq 0$ an interval.

$-3 \leq a \leq 4$

**ii.** For what values of $a$ is the solution to $g(x) \geq 0$ an interval.

$a \leq -3$ or $a \geq 4$

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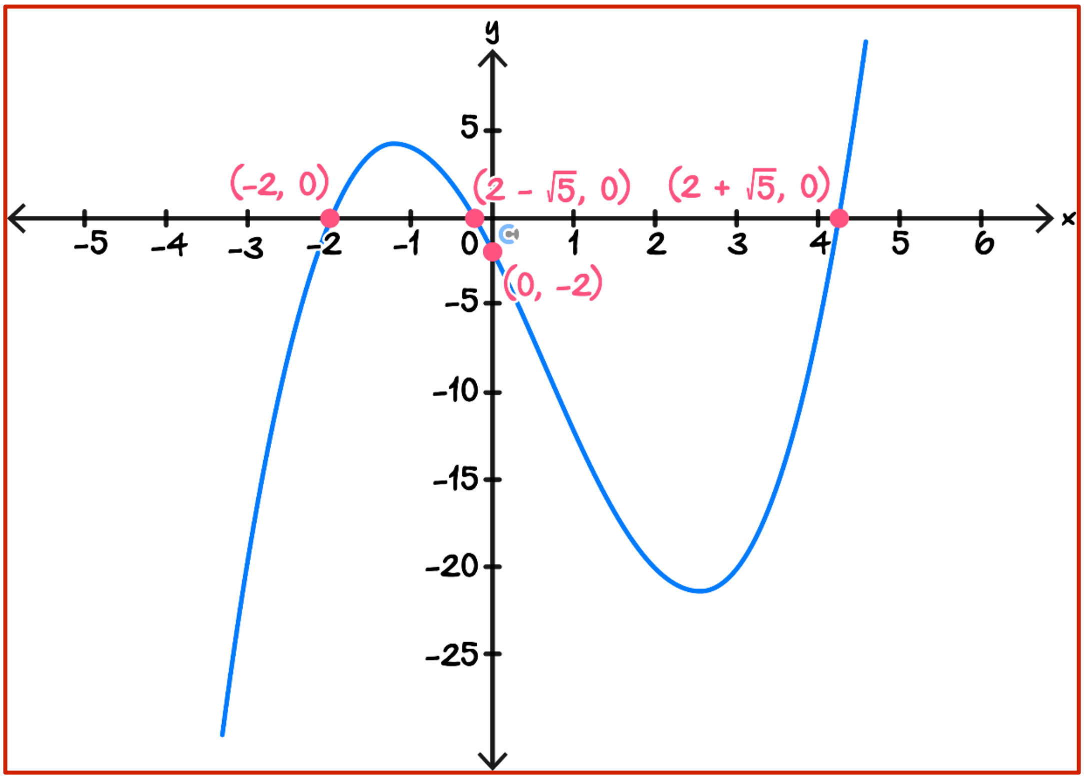
**Question 44**

Consider the polynomial $f(x) = x^3 - 2x^2 - 9x - 2$.

**a.** State the co-ordinates of the axis intercepts of $f$.

$(-2, 0), \left(2 - \sqrt{5}, 0\right), \left(2 + \sqrt{5}, 0\right)$, and $(0, -2)$.

**b.** Hence, sketch the graph of $f$, labelling all axis intercepts with their co-ordinates.



**c.** A bisection method with an initial interval of [3,5] is used to approximate the solution to $f(x) = 0$.

First, the interval is refined $n$ times, before the midpoint of the last interval is taken as an answer.

**i.** If $n = 3$, what answer will this approach yield?

We refine the interval to $[4, 5]$ then to $[4, 4.5]$ and lastly to $[4, 4.25]$. Thus our answer is $4.125$.

**ii.** What is the smallest value of $n > 2$ which gives a better approximation to the actual solution than $n = 2$ does?

$n = 6$.

**d.** If the bisection method is instead applied with an initial interval of $[-11,5]$, what root will be approximated?

Justify your answer.

The interval will be refined to $[-3, 5]$ since $f(-3)f(-11) > 0$ and then to $[1, 5]$ since $f(1)f(-3) > 0$.
The only root remaining in $[1, 5]$ is $2 + \sqrt{5}$, which is the root our method will approximate.

**e.** Use the rational root theorem to show that $\sqrt{7}$ cannot be rational.

We know that $\sqrt{7}$ is a solution to the polynomial equation $x^2 - 7$.
By the rational root theorem, the only possible rational solutions to this equation are $\pm 1, \pm 7$.
However as $(\pm 1)^2 - 7 = -6$ and $(\pm 7)^2 - 7 = 42$ we see $x^2 - 7$ has no rational roots.
Hence $\sqrt{7}$ is not rational.

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