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VCE Mathematical Methods ½
Polynomials [1.5]
Workbook

Outline:

Learning Objectives:

☐ **MM12 [1.5.1]** – Identify the properties of polynomials and solve long division.

☐ **MM12 [1.5.2]** – Apply remainder and factor theorem to find remainders and factors.

☐ **MM12 [1.5.3]** – Find factored form of polynomials.

☐ **MM12 [1.5.4]** – Graph factored and unfactored polynomials.

Section A: Algebra of Polynomial Functions

Sub-Section: Terminologies of Polynomials

Degree of Polynomial Functions

$$\textit{Degree} = \textit{Highest Power of the Polynomial}$$

Question 1

State the degree of each polynomial.

a. $x^3 - 4x^2 + 5x + 6$

3

b. $3x + 5x^2 - x^7$

7

c. A Quadratic.

2

Roots of Polynomial Functions

$$\textbf{Roots} = \textbf{x-intercept}$$

$$\underbrace{\text{Poly} = 0 \rightarrow x = \_\_ , x = \_\_ \ldots}_{\text{Roots}}$$

Discussion: Can a quadratic have more than 2 roots? Hence, can there be more roots than the degree?

No!

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Question 2

Find the roots of the following polynomial:

$$\underline{(x-1)}^2\underline{(x+3)}^4 = 0$$

$x = 1, x = -3$

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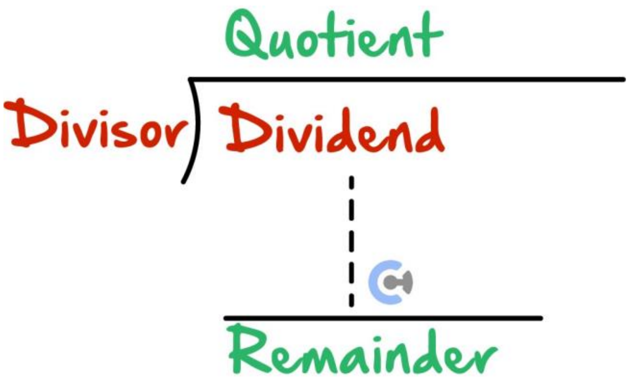
Sub-Section: Long Division

Polynomial Long Division

- Division of polynomials:



$$\frac{\textit{Dividend}}{\textit{Divisor}} = \textit{Quotient} + \frac{\textit{Remainder}}{\textit{Divisor}}$$

Question 3 Walkthrough.

Simplify the following using polynomial long division.

$$\frac{3x^2+10x+20}{2x+4}$$

$$\frac{3}{2}x+2$$

$$2x+4\,\overline{\big)\,3x^2+10x+20}$$

$$-(3x^2+6x)\ \downarrow$$

$$4x+20$$

$$-(4x+8)$$

$$\boxed{12}$$

$$=\frac{3}{2}x+2+\frac{12}{2x+4}$$

TIP: Always focus on the highest degree term first.

Your turn!

Question 4

Simplify the following using polynomial long division.

$$\frac{x^2-3x+5}{x-1}$$

$x-2$

$x-1 \,\overline{)\, x^2-3x+5}$

$-(x^2-x) \downarrow$

$-2x+5$

$-(-2x+2)$

$\textcircled{3}$

$$= x-2+\frac{3}{x-1}$$

Now, a slightly more difficult example!

Question 5

Simplify the following using polynomial long division.

$$\frac{x^3+x^2+2}{x-3}$$

$x^2+4x+12$

$x-3 \,\overline{)\, x^3+x^2+0x+2}$

$-(x^3-3x^2) \downarrow$

$4x^2+0x$

$-(4x^2-12x)$

$12x+2$

$-(12x-36)$

$\boxed{38}$

$$= x^2+4x+12+\frac{38}{x-3}$$

TIP: Always remember to fill in any missing powers of x in the numerator or denominator with "placeholders" that have a coefficient of 0.

Question 6 Extension.

Simplify the following using polynomial long division.

$$\frac{x^4 + 4x^3 + 3x^2 - 2x + 3}{x + 3}$$

$Q: x^3 + x^2 - 2$

$R: 9$

$$= x^3 + x^2 - 2 + \frac{9}{x+3}$$

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Sub-Section: Remainder Theorem

How can we find the remainder without long division?

Exploration: Derivation of the Remainder Theorem

- Consider $\frac{f(x)}{g(x)}$.

$$g(x) \times \frac{f(x)}{g(x)} = \left(q(x) + \frac{R}{g(x)}\right) \times g(x) \quad \text{where } R = \text{Remainder}$$

- Let's multiply everything by $g(x)$.

$$f(x) = \underline{q(x)g(x) + R}$$

- Remember, we are trying to find the remainder R before we do long division.

  What functions do we already have before long division?

$$f(x) = q(x) \cdot g(x) + R$$

- How can we get $f(x)$ to equal to the remainder R?
  - We can substitute a value of x such that, the divisor, $g(x)$ is equal to 0.

$$f(\alpha) = \underline{q(\alpha) \cdot 0 + R}$$

$$f(\alpha) = \underline{R}$$

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Remainder Theorem

- **Definition**: Finds the remainder of long division without the need of long division.

$$\textit{when } P(x) \textit{ is divided by } (x-\alpha), \textit{ the remainder is } P(\alpha)$$

- **Steps:**

1. Find x values which makes the divisor equal to 0.
2. Substitute it into the dividend function.

Discussion: How do we find the remainder of $f(x) \div (x-2)$?

$f(2)$

Discussion: How do we find the remainder of $f(x) \div (2x+1)$?

$f(-\frac{1}{2})$

Question 7 Walkthrough.

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where, $f(x) = x^3 + x^2 - 2x + 5$ and $g(x) = x + 1$.

$x+1=0$
$x=-1$

$R = f(-1)$

$= (-1)^3 + (-1)^2 - 2(-1) + 5$

$= -1 + 1 + 2 + 5$

$= 7$

$\boxed{R=7}$

Your turn!

Active Recall: Remainder Theorem

1. Find x values which makes the divisor equal to 0.
2. Substitute it into the dividend function.

Question 8

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where, $f(x) = x^3 - 2x^2 + 3x + 1$ and $g(x) = 2x + 4$.

$g(x) = 0$

$2x + 4 = 0$

$x = -2$

$R = f(-2)$

$= (-2)^3 - 2(-2)^2 + 3(-2) + 1$

$= -8 - 8 - 6 + 1$

$= -21$

$\boxed{R = -21}$

Question 9 Extension.

For the polynomial $f(x) = 3x^3 - 2x^2 + (7 - 2a)x + 1$, we get a remainder of 14 when $f(x)$ is divided by $g(x) = x - 1$. Find the value of a.

$f(1) = 14$

$\therefore 3 - 2 + (7 - 2a) + 1 = 14$

$9 - 2a = 14$

$2a = -5$

$a = -5/2$

Sub-Section: Factor Theorem

Discussion: What division could $f(2)$ be the remainder of?

$$\frac{f(x)}{x-2}$$

remainder of $\frac{f(x)}{x-2}$ is $\underline{f(2)}$

Discussion: Hence, what does it mean when $f(2) = 0$?

$R = 0$!

$\therefore$ 2 is a root of f

& $x - 2$ is a factor.

This is called the "Factor theorem"

Factor Theorem

- For every x-intercept, there is a factor:

$$\textit{if } P(\alpha) = 0 \textit{ then, } (x - \alpha) \textit{ is a factor of } P(x)$$

Question 10 Walkthrough.

Determine if $x + 4$ is a factor of $P(x) = 3x^3 + 8x^2 - 20x - 16$.

$$P(-4) = 3(-4)^3 + 8(-4)^2 - 20(-4) - 16$$

$$= -192 + 128 + 80 - 16$$

$$= 208 - 208 = 0$$

$\therefore$ $x + 4$ $\underline{\text{is}}$ a factor of $P(x)$.

Your turn!

Question 11

Determine if $x + 2$ is a factor of $P(x) = 2x^3 - 7x^2 + 7x - 2$.

$$P(-2) = 2(-2)^3 - 7(-2)^2 + 7(-2) - 2$$

$$= -16 - 28 - 14 - 2$$

$$= -60$$

$P(-2) \neq 0 \Rightarrow x+2$ is not a factor of $P(x)$.

Question 12 Extension.

Determine if $x - \frac{3}{2}$ is a factor of $P(x) = 6x^3 - x^2 - 20x + 12$.

.

$$P\left(\tfrac{3}{2}\right) = 6\left(\tfrac{3}{2}\right)^3 - \left(\tfrac{3}{2}\right)^2 - 20\left(\tfrac{3}{2}\right) + 12$$

$$= 6\left(\frac{27}{8}\right) - \frac{9}{4} - 30 + 12$$

$$= \frac{81}{4} - \frac{9}{4} - 18$$

$$= 18 - 18 = 0$$

$\therefore x - 3/2$ is a factor

Sub-Section: Factorising Polynomials

Factorising Polynomials

- The steps are:
  - Find a single root by trial and error.

    (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

Question 13 Walkthrough.

Find all the roots of $f(x) = x^3 + 3x^2 - 6x - 8$.

① $\cdot f(1) = 1+3-6-8 \neq 0$

$\cdot f(-1) = -1+3+6-8 = 9-9 = 0 \longrightarrow x+1$ is a factor.

②

$$\begin{array}{r|l} & \quad x^2 + 2x - 8 \\ \hline x+1 & x^3 + 3x^2 - 6x - 8 \\ & -(x^3 + x^2) \downarrow \\ \hline & \quad 2x^2 - 6x \\ & -(2x^2 + 2x) \\ \hline & \qquad -8x - 8 \\ & \qquad -(-8x - 8) \\ \hline & \qquad\qquad 0 \end{array}$$

③ $f(x) = (x+1)(x^2+2x-8)$

$f(x) = (x+1)(x+4)(x-2)$

⚠ Roots are

$x = -1, -4, 2$

NOTE: When the question asks for all roots, you cannot just factorise and end it there!

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Your turn!

Question 14

Find all the roots of $f(x) = x^3 + 12x^2 + 17x - 90$.

$f(2) = 2^3 + 12(2)^2 + 17(2) - 90$

$= 8 + 48 + 34 - 90$

$= 90 - 90 = 0$

$x = 2$ is a root → $x - 2$ is a factor!

$x^2 + 14x + 45$

$x - 2 \,\big|\, x^3 + 12x^2 + 17x - 90$

$-(x^3 - 2x^2)$

$14x^2 + 17x$

$-(14x^2 - 28x)$

$45x - 90$

$-(45x - 90)$

0

$f(x) = (x - 2)(x^2 + 14x + 45)$

$f(x) = (x - 2)(x + 9)(x + 5)$

∴ Roots are $x = 2, -9, -5$

Question 15

Find all the roots of $f(x) = -2x^3 - 13x^2 - 5x + 6$.

$f(-1) = 2 - 13 + 5 + 6$

$= 0$

$x + 1$ is a factor

$-2x^2 - 11x + 6$

$x + 1 \,\big|\, -2x^3 - 13x^2 - 5x + 6$

$-(-2x^3 - 2x^2)$

$-11x^2 - 5x$

$-(-11x^2 - 11x)$

$6x + 6$

$-(6x + 6)$

0

$f(x) = (x + 1)(-2x^2 - 11x + 6)$

$= -(x + 1)(2x^2 + 11x - 6)$

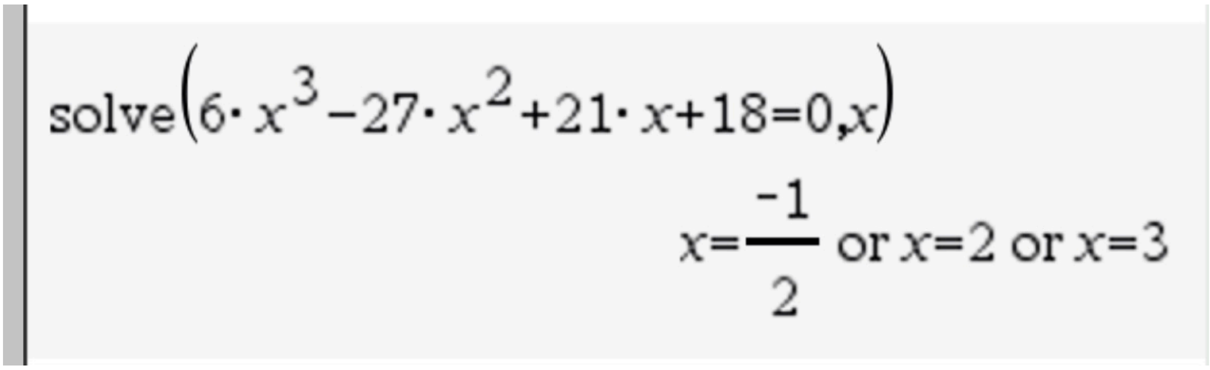
$= -(x + 1)(2x - 1)(x + 6)$

∴ Roots are

$x = \frac{1}{2}, -6, -1$

Question 16

Find all the roots of $f(x) = 6x^3 - 27x^2 + 21x + 18$.



$$\text{solve}\left(6\cdot x^3-27\cdot x^2+21\cdot x+18=0,x\right)$$

$$x=\frac{-1}{2} \text{ or } x=2 \text{ or } x=3$$

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Sub-Section: Rational Root Theorem

$\rightarrow$ roots are $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

Discussion: Consider $(2x-1)(3x-1)(6x-1)$. What are the roots and could we have gotten that from trial and error?

$36x^3 - 36x^2 + 11x - 1$

$x = \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

$\pm \frac{1}{1, 4, 6, 9\ 36, 2, 18}$

So, what should we do?

Rational Root Theorem

- Rational Root Theorem narrows down the possible roots.

$$\text{Potential root} = \pm \frac{\text{Factors of constant term } a_0}{\text{Factors of leading coefficient } a_n}$$

- If the roots are rational numbers, the roots can only be $\pm \frac{\text{factors of constant term } a_0}{\text{factors of leading coefficient } a_n}$.

Question 17 Walkthrough.

Find all the roots of $f(x) = 6x^3 + 13x^2 - 14x + 3$.

$\pm \frac{\{1, 3\}}{\{1, 6, 2, 3\}}$

$f(1) \neq 0$

$f(-1) \neq 0$

$f(3) \neq 0$

$f(-3) = 6(-27) + 13(9) - 14(-3) + 3$

$= -162 + 117 + 42 + 3$

$= -120 + 120 = 0$

$f(-3) = 0 \longrightarrow x + 3$ is a factor

$$\begin{array}{r|l} & 6x^2 - 5x + 1 \\ \hline x+3 & 6x^3 + 13x^2 - 14x + 3 \end{array}$$

$\vdots$

$f(x) = (x+3)(6x^2 - 5x + 1)$

$= (x+3)(3x-1)(2x-1)$

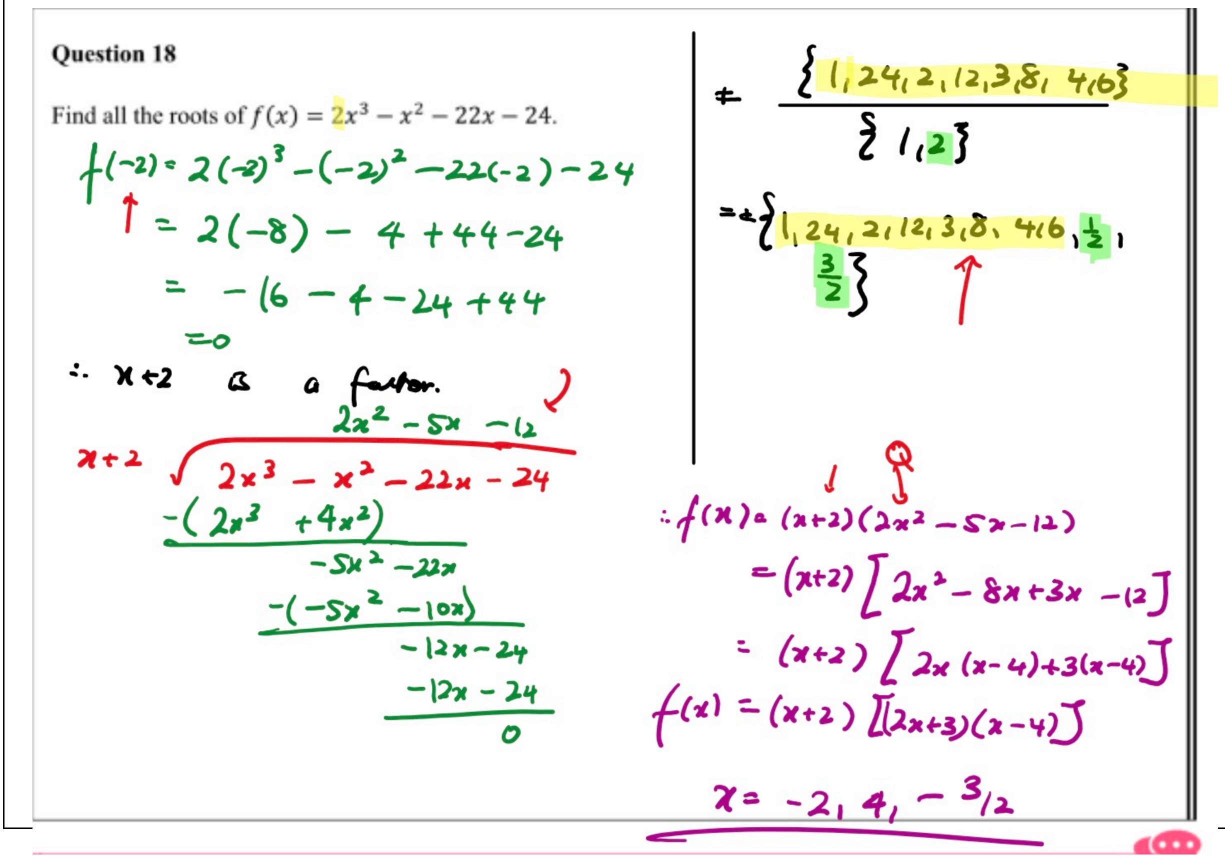
Roots are

$x = -3, \frac{1}{3}, \frac{1}{2}.$

NOTE: All the roots are part of the suggestion given by the rational root theorem.

Question 18

Find all the roots of $f(x) = 2x^3 - x^2 - 22x - 24$.



Question 18

Find all the roots of $f(x) = 2x^3 - x^2 - 22x - 24$.

$f(-2) = 2(-2)^3 - (-2)^2 - 22(-2) - 24$

$= 2(-8) - 4 + 44 - 24$

$= -16 - 4 - 24 + 44$

$= 0$

∴ $x+2$ is a factor.

$$\begin{array}{r|l} & 2x^2 - 5x - 12 \\ x+2 & 2x^3 - x^2 - 22x - 24 \\ & -(2x^3 + 4x^2) \\ & \quad -5x^2 - 22x \\ & \quad -(-5x^2 - 10x) \\ & \qquad -12x - 24 \\ & \qquad -12x - 24 \\ & \qquad 0 \end{array}$$

$\pm \frac{\{1, 24, 2, 12, 3, 8, 4, 6\}}{\{1, 2\}}$

$= \pm\{1, 24, 2, 12, 3, 8, 4, 6, \frac{1}{2}, \frac{3}{2}\}$

$\therefore f(x) = (x+2)(2x^2 - 5x - 12)$

$= (x+2)[2x^2 - 8x + 3x - 12]$

$= (x+2)[2x(x-4) + 3(x-4)]$

$f(x) = (x+2)[(2x+3)(x-4)]$

$x = -2, 4, -3/2$

Discussion: Why is rational root theorem called a rational root theorem?

↑ Uses rational solutions (fractions)

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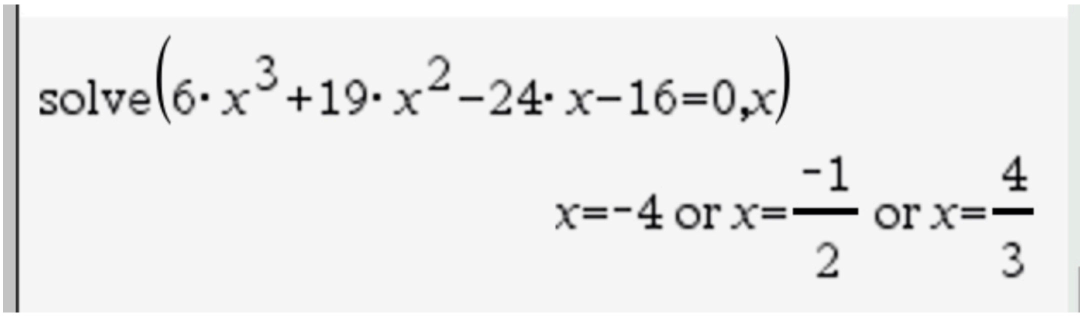
Question 19 Extension.

Find all the roots of $f(x) = 6x^3 + 19x^2 - 24x - 16$.



$$\text{solve}\left(6 \cdot x^3 + 19 \cdot x^2 - 24 \cdot x - 16 = 0, x\right)$$

$$x = -4 \text{ or } x = \frac{-1}{2} \text{ or } x = \frac{4}{3}$$

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Sub-Section: Sum and Difference of Cubes

$a^2-b^2=(a+b)(a-b)$

Sum and Difference of Cubes

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

Question 20 Walkthrough.

Factorise the following polynomial as much as possible.

$$x^3+125$$

$$=x^3+(5)^3$$

$$=(x+5)(x^2-5x+25)$$

Question 21

Factorise the following polynomial as much as possible.

$$8x^3-216$$

①

$$8x^3-216$$

$$=(2x)^3-(6)^3$$

$$=(2x-6)(4x^2+12x+36)$$

②

$$8x^3-216$$

$$=8(x^3-27)$$

$$=8(x^3-(3)^3)$$

$$=8(x-3)(x^2+3x+9)$$

Question 22 Extension.

Factorise the following polynomial as much as possible.

$$32x^3 - 256$$

$$32(x^3 - 8)$$

$$= 32(x^3 - (2)^3)$$

$$= 32(x-2)(x^2+2x+4)$$

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Section B: Graphs of a Polynomial

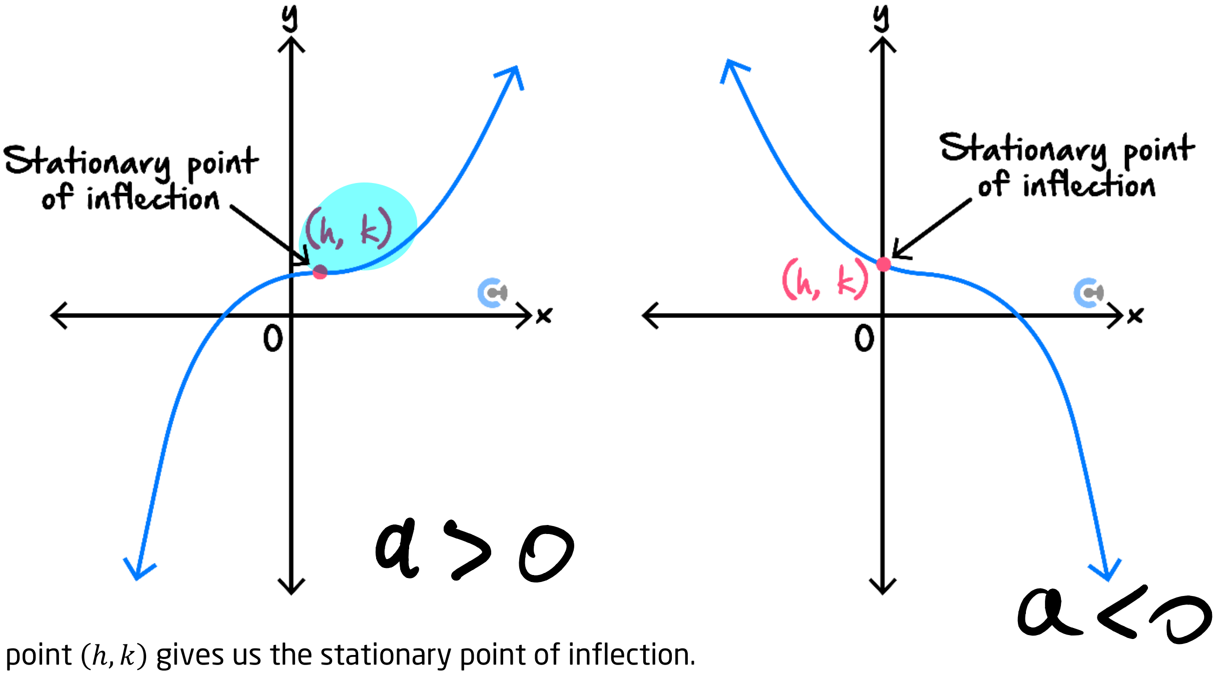
Sub-Section: Graphing Polynomials in the Form of $a(x-h)^n+k$

$y = a(x-h)^n + k$

n odd $(n>1)$

Graphs of $a(x-h)^n+k$, where n is an Odd Positive Integer

- All graphs look like a “cubic”.



- The point (h,k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

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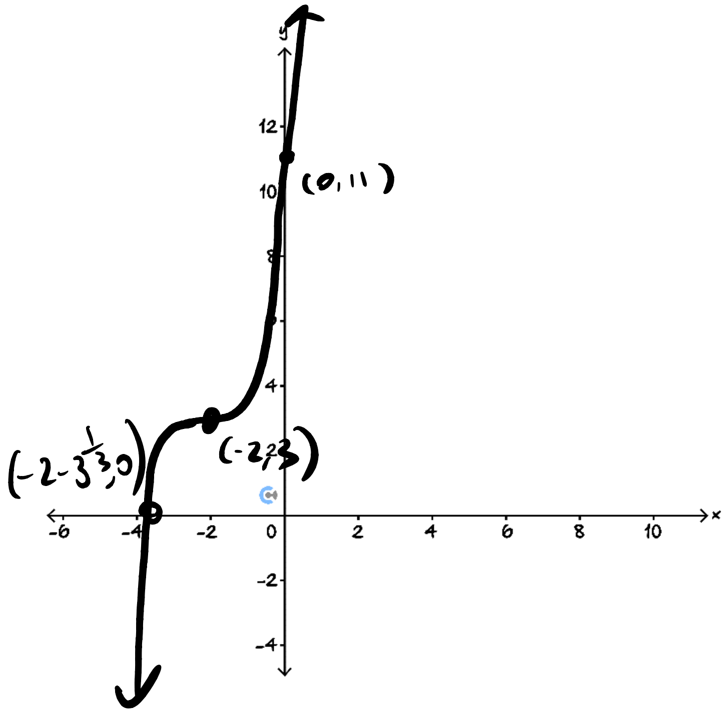
Question 23 Walkthrough.

Point of inflection : $(-2, 3)$

Sketch the graph of $y = (x+2)^3 + 3$ on the axes below.



$y = (2)^3 + 3$

$= 8 + 3$

$= 11$

$(x+2)^3 + 3 = 0$

$(x+2)^3 = -3$

$x + 2 = (-3)^{\frac{1}{3}}$

$x = -2-(3)^{\frac{1}{3}}$

Question 24

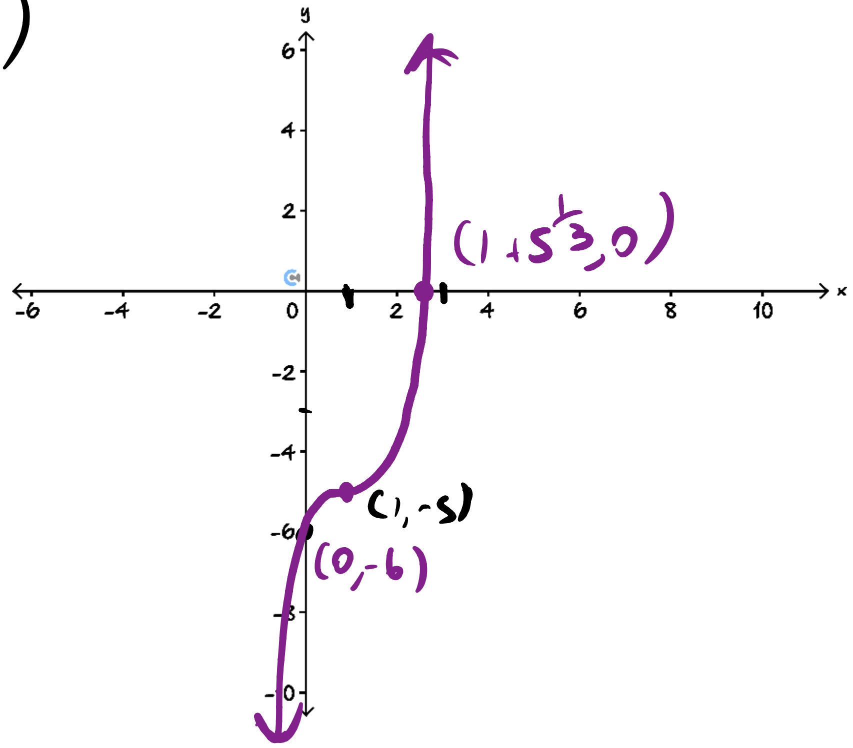
POI : $(1, -5)$

Sketch the graph of $y = (x-1)^3 - 5$ on the axes below.

$y = (-1)^3 - 5 = -6$

$(0, -6)$

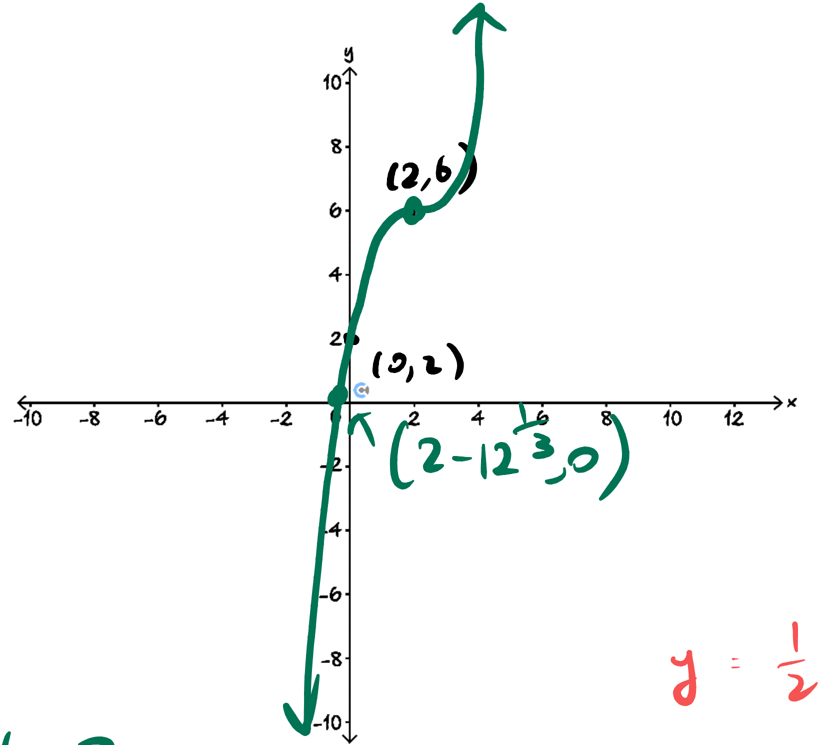
$(1 + 5^{\frac{1}{3}}, 0)$



Question 25

POI : (2,6)

Sketch the graph of $y = \frac{1}{2}(x-2)^3 + 6$ on the axes below.



$y = \frac{1}{2}(-2)^3 + 6$

$= -4 + 6$

$= 2$

$(0,2)$

$\frac{1}{2}(x-2)^3 + 6 = 0$

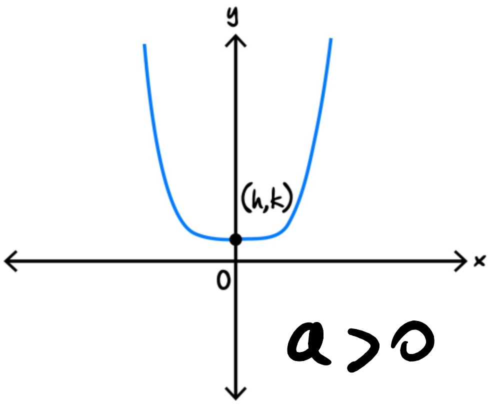
$x = 2 - (12)^{\frac{1}{3}}$

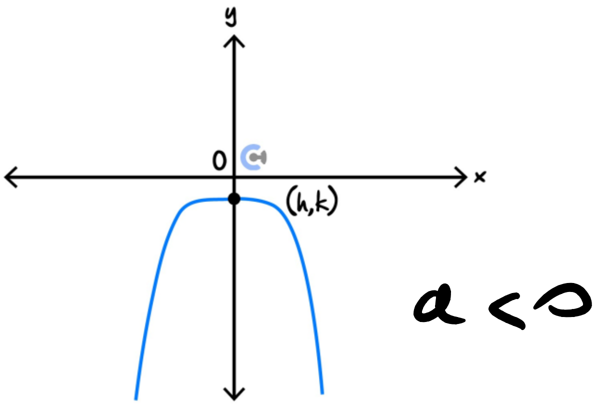
What about even powers?

Graphs of $a(x-h)^n + k$, where n is an Even Positive Integer

n is even

- All graphs look like a "quadratic".





- The point (h,k) gives us the turning point.

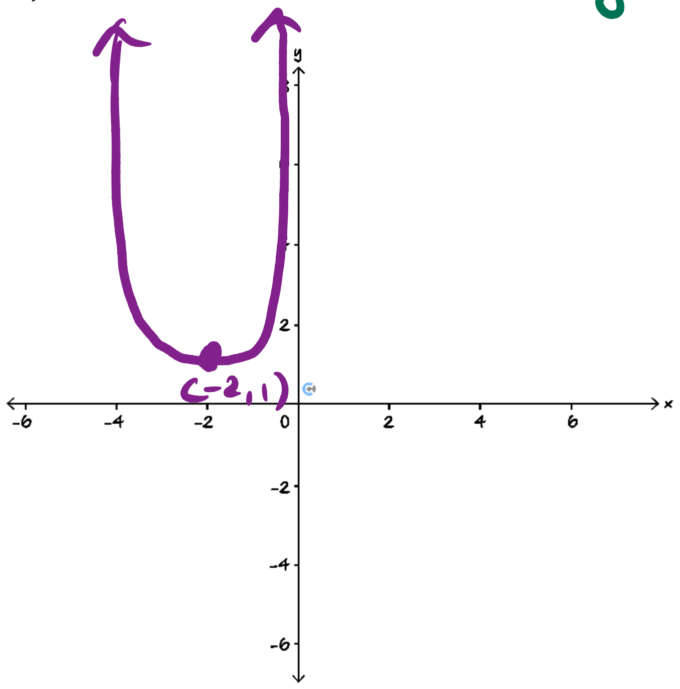
Question 26 Walkthrough.

TP: $(-2, 1)$

Sketch the graph of $y = (x+2)^4 + 1$ on the axes below.

y int: $(2)^4 + 1$

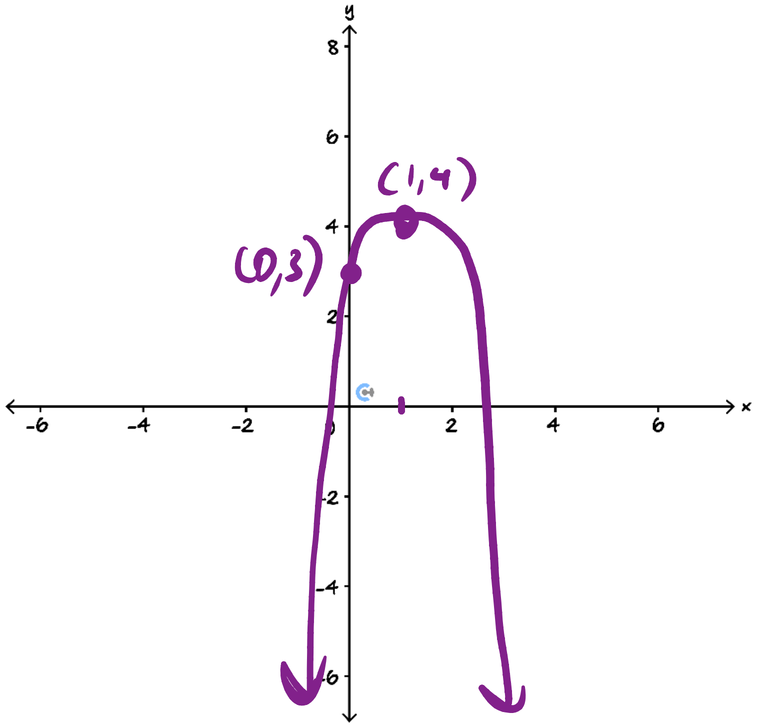
$= (0, 17)$



Question 27

TP: $(1, 4)$

Sketch the graph of $y = -(x-1)^4 + 4$ on the axes below.



Question 28 Extension.

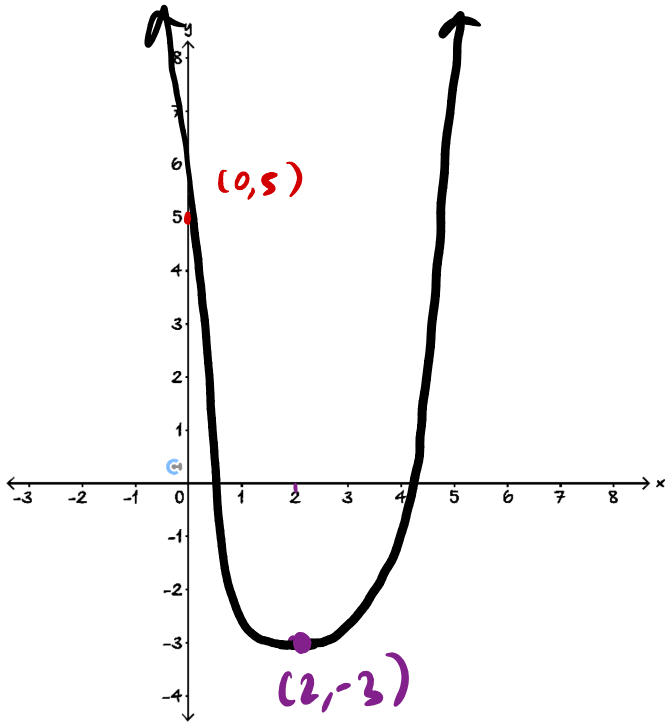
Sketch the graph of $y = \frac{1}{2}(x-2)^4 - 3$ on the axes below.

TP: $(2,-3)$

$y = \frac{1}{2}(-2)^4 - 3$

$= 8 - 3$

$= 5$



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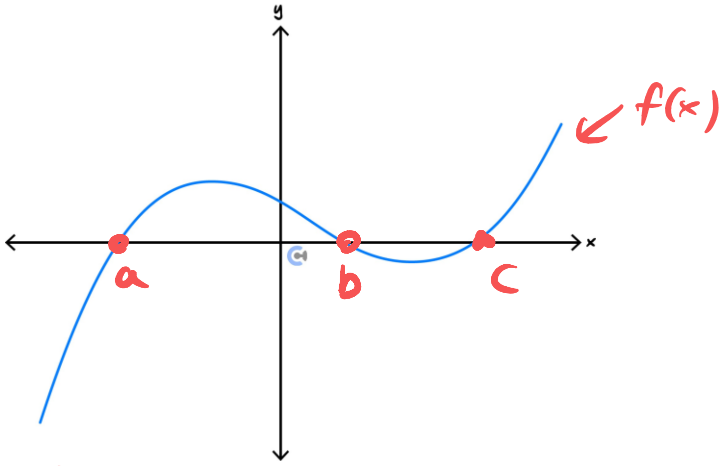
Sub-Section: Graphing Factorised Polynomials

What about the graph of a factorised polynomial?

Exploration: Graphs of Factorised Polynomials

①

All non-repeated *linear factors correspond to* x-intercepts *of the graph.* (cut the x-axis in a 'linear way')

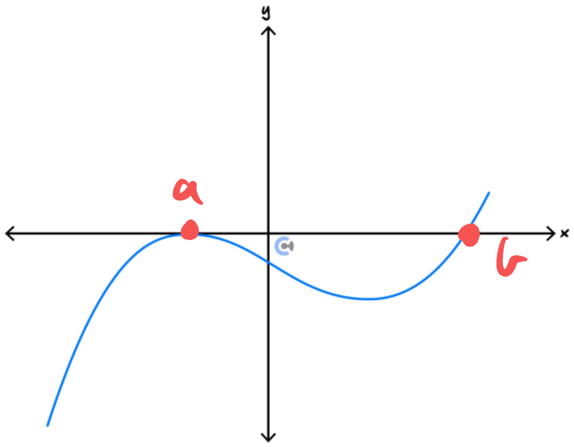


$a < b < c$

- E.g., $f(x) = (x-a)(x-b)(x-c)$ results in x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$.

All squared *linear factors correspond to* x-intercepts & turning points *of the graph.*

(bounce off the x-axis 'like a parabola')

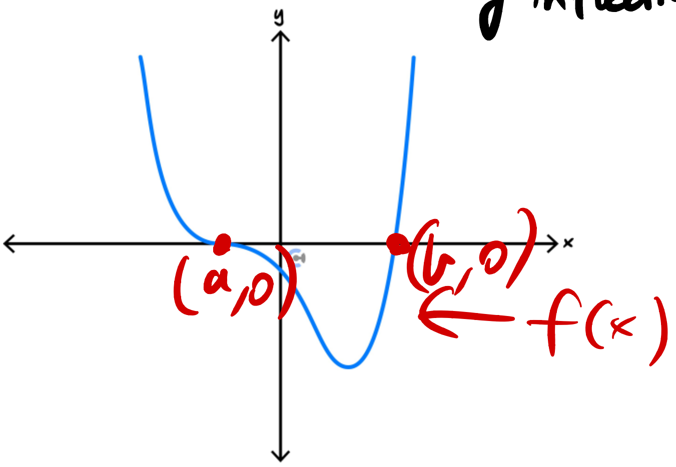


e.g $f(x) = (x-a)^2(x-b)$

- E.g., $f(x) = (x-a)^2(x-b)$ will have an x-intercept $(a, 0)$ which is also a local minimum/maximum.

All cubed *linear factors correspond to* x-intercept & stationary point of inflection *of the graph.*

(cuts the x-axis like a cubic)



- E.g., $f(x) = (x-a)^3(x-b)$ has an x-intercept $(a, 0)$ which is also a stationary point of inflection.

Graphs of Factorised Polynomials

Definition

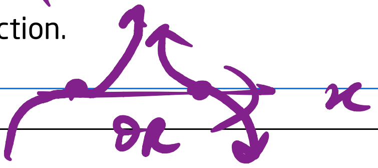
- Steps:

1. Plot x-intercepts.
2. Determine whether the polynomial is positive or negative. (leading coeff >0 or <0)
3. Use the repeated factors to deduce the shape.

Non-Repeated: Only x-intercept.

Even Repeated: x-intercept and a turning point.

Odd Repeated: x-intercept and a stationary point of inflection.



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Question 29 Walkthrough.

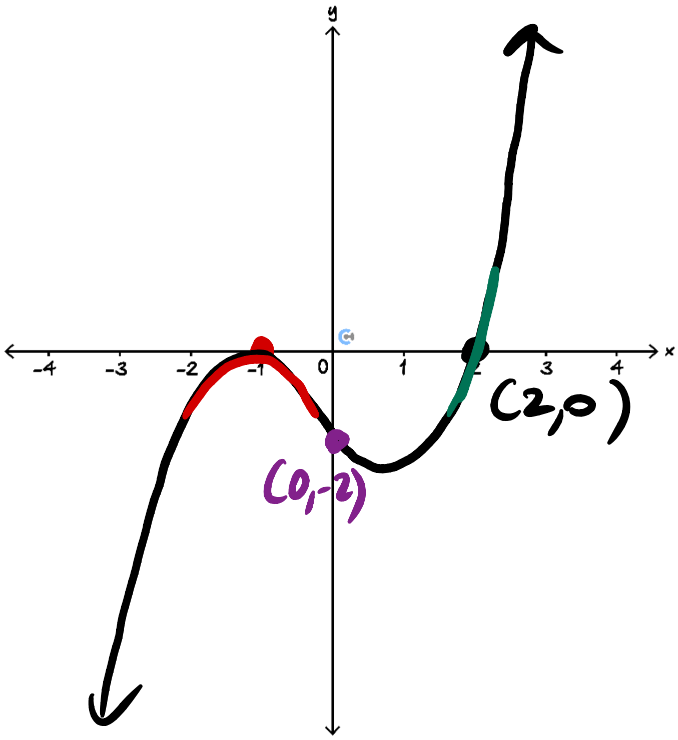
Sketch the graphs of the following functions on the axes provided. Ignore the y-axis scale.

a. $y = (x+1)^2(x-2)$

Pos shape

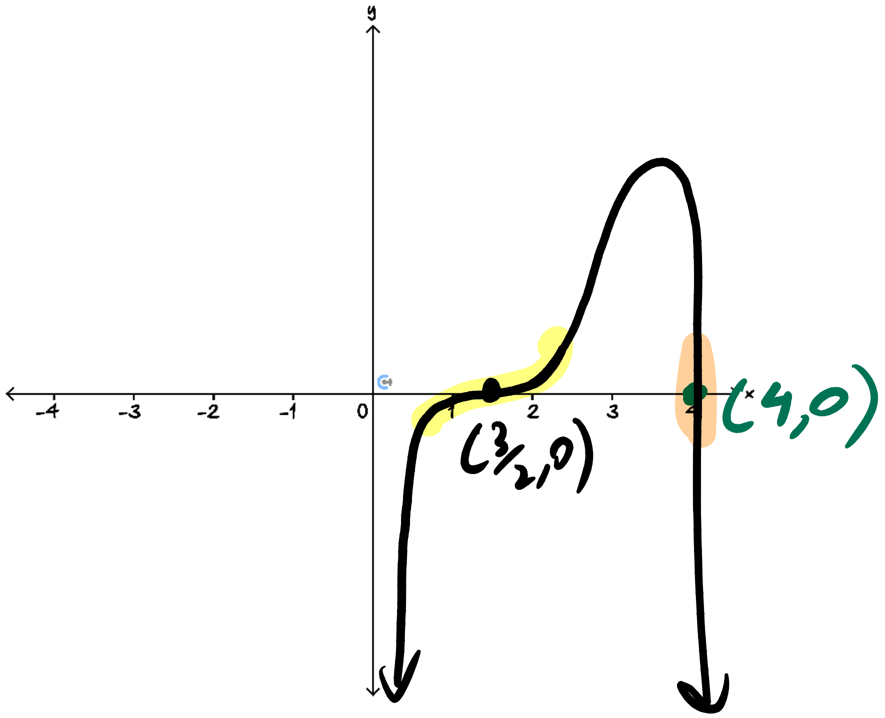
$y = (1)^2(-2)$

$= -2$



b. $y = \left(x - \frac{3}{2}\right)^3 (4 - x)$

Shape < ɔ



Your turn!

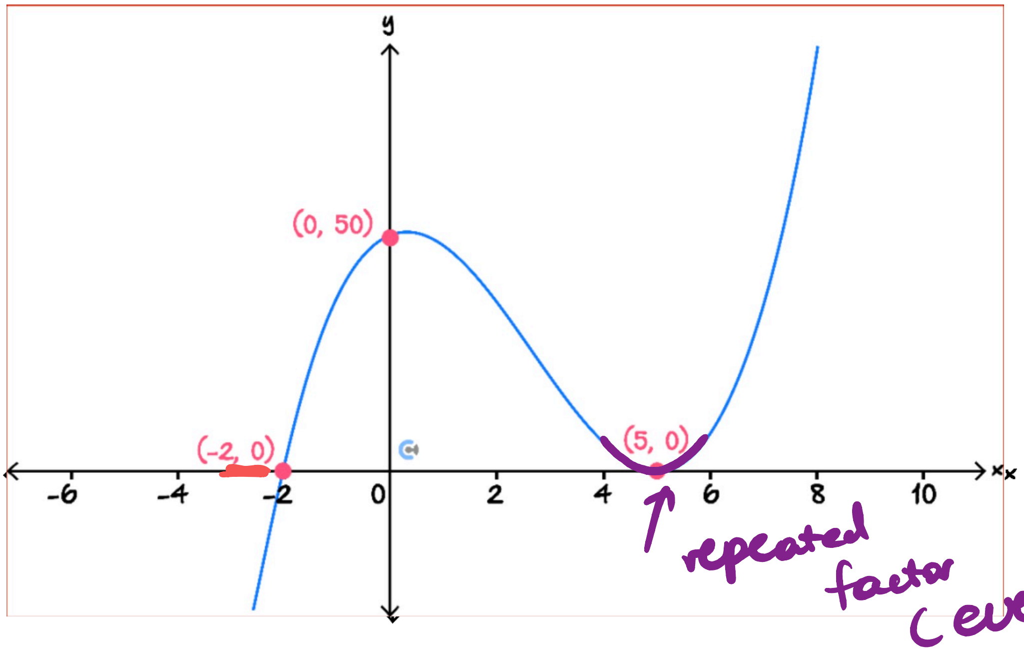
Question 30

Sketch the graphs of the following functions on the axes provided. Ignore the y-axis scale.

a. $y = (2 + x)(5 - x)^2$

> 0

- positive shape
- degree odd (3)



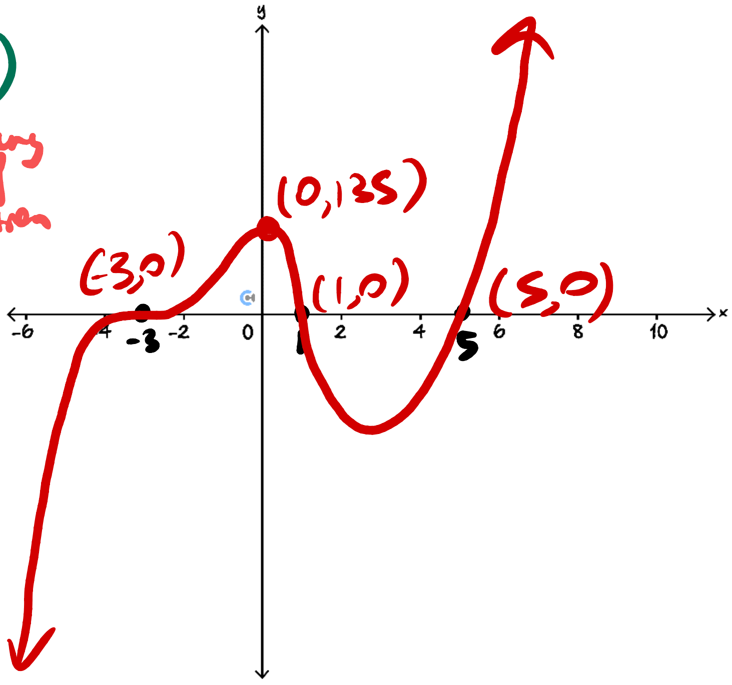
repeated factor (even)

b. $y = (x + 3)^3(x - 1)(x - 5)$

- degree : 5

positive (>0)

$(-3,0)$: Stationary Point of inflection



y-int: $y = (3)^3(-1)(-5)$
$= 27(5) = 135$
$(0,135)$

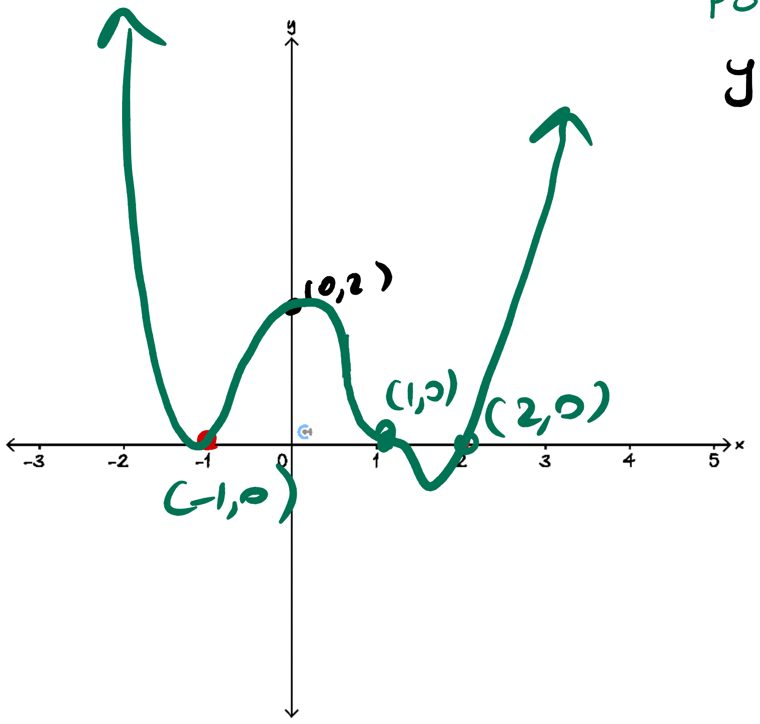
Question 31

Sketch the graph of the following function on the axes provided. Ignore the y-axis scale.

deg: 6
(even)
>0 (pos)

$$y = (x-1)^3\underline{(x+1)^2}(x-2)$$

TP (-1,0)
POI (1, 0)
y-int: (0,2)



Space for Personal Notes

Contour Check

Learning Objective: [1.5.1] – Identify the properties of polynomials and solve long division.

Key Takeaways

- ☐ The degree of a polynomial is the polynomial's highest power.
- ☐ The roots of a polynomial are its x-intercepts.
- ☐ For polynomial long division:

$$\frac{\textit{Dividend}}{\textit{Divisor}} = \textit{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

Learning Objective: [1.5.2] – Apply remainder and factor theorem to find remainders and factors.

Key Takeaways

- ☐ When $P(x)$ is divided by $(x - \alpha)$, the remainder is $P(a)$.
- ☐ If $P(\alpha) = 0$ then $(x - \alpha)$ is a factor of $P(x)$.

⚠

Learning Objective: [1.5.3] - Find factored form of polynomials.

Key Takeaways

☐ Steps to factor a cubic polynomial are:

- ○ Find a single root by trial and error.

  (Factor Theorem: Substitute into the function and see if we get 0.)

- ○ Use long division to find the quadratic factor.

- ○ Factorise the quadratic.

☐ Rational Root Theorem **narrows down** the possible roots. If the roots are rational numbers, it must be that any.

$$\textit{Potential root} = \pm\frac{\textit{Factors of constant term } a_0}{\textit{Factors of leading coefficient } a_n}$$

☐ Sum and difference of cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Learning Objective: [1.5.4] - Graph factored and unfactored polynomials.

Key Takeaways

☐ Graphs of $a(x-h)^n + k$, where n is an Odd Positive Integer that is not equal to 1:

- ○ The point (h, k) gives us the stationary point of inflection.

☐ Graphs of $a(x-h)^n + k$, where n is an Even Positive Integer:

- ○ The point (h, k) gives us the turning point
- ○ These graphs look like a parabola/quadratic

☐ Steps to graphing factorised polynomials:

1. Plot x-intercepts.
2. Determine whether the polynomial is positive or negative.
3. Use the repeated factors to deduce the shape:

$$y = (\underline{x-a})(\underline{x-b})^2(\underline{x-c})^3$$

Non-Repeated: Only x-intercept (cuts x-axis like a line)

Even Repeated: x-intercept and a turning point (bounces off x-axis)

Odd Repeated: x-intercept and a Stationary point of inflection.

VCE Mathematical Methods ½

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