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# VCE Mathematical Methods ½
# Polynomials [1.5]
# Workbook

## Outline:



## Learning Objectives:

- ☐ MM12 [1.5.1] - Identify the properties of polynomials and solve long division.
- ☐ MM12 [1.5.2] - Apply remainder and factor theorem to find remainders and factors.
- ☐ MM12 [1.5.3] - Find factored form of polynomials.
- ☐ MM12 [1.5.4] - Graph factored and unfactored polynomials.

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## Section A: Algebra of Polynomial Functions

### Sub-Section: Terminologies of Polynomials

$x^3 \Leftarrow$ cubic
$x^4 \Leftarrow$ quartic

Definition

**Degree of Polynomial Functions**

$$\textit{Degree} = \textit{Highest Power of the Polynomial}$$

**Question 1**

State the degree of each polynomial.

**a.** $x^3 - 4x^2 + 5x + 6$

3

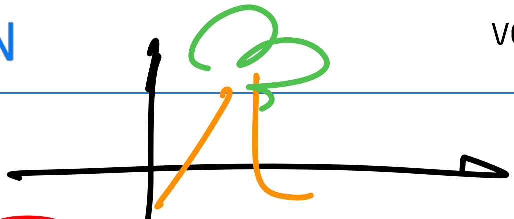
**b.** $3x + 5x^2 - x^7$

7

**c.** A Quadratic.

2

Definition



Roots of Polynomial Functions

$$\text{Roots} = x\text{-intercept}$$

**Discussion:** Can a quadratic have more than 2 roots? Hence, can there be more roots than the degree?

No

cubic $=3$

roots $\leq$ degree

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**Question 2**

x-ints.

Find the roots of the following polynomial:

$$(x-1)^2(x+3)^4 = 0$$

$x-1=0 \quad x+3=0$

$x = 1, -3$

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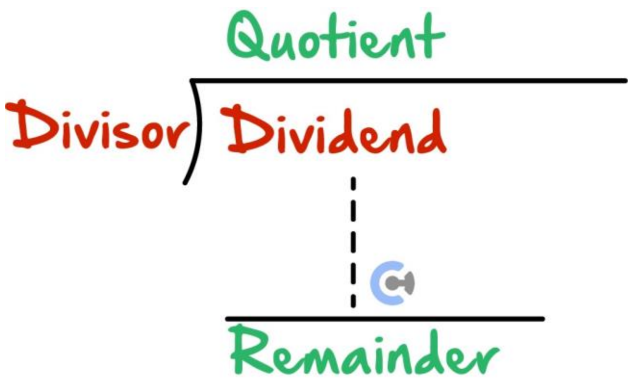
## Sub-Section: Long Division

### Polynomial Long Division

- Division of polynomials:

$$\begin{array}{r} 48 \\ 3\overline{)144} \\ -12\phantom{4} \\ \hline 24 \\ -24 \\ \hline 0 \end{array}$$




$$\frac{\textit{Dividend}}{\textit{Divisor}} = \textit{Quotient} + \frac{\textit{Remainder}}{\textit{Divisor}}$$

**Question 3 Walkthrough.**

Simplify the following using polynomial long division.

$$\frac{3x^2 + 10x + 20}{2x + 4}$$

$$\begin{array}{r} \frac{3}{2}x + 2 \\ 2x+4\overline{)3x^2+10x+20} \\ -\left(3x^2+6x\right)\phantom{+20} \\ \hline 4x+20 \\ -\left(4x+8\right) \\ \hline 12 \end{array}$$

$3x^2 \div 2x = \frac{3}{2}x$

$\frac{3}{2}x + 2 + \frac{12}{2x+4}$

$= \frac{3}{2}x + 2 + \frac{6}{x+2}$

TIP: Always focus on the highest degree term first.

*Your turn!*

**Question 4**

Simplify the following using polynomial long division.

$$\frac{x^2 - 3x + 5}{x - 1} = x - 2 + \frac{3}{x-1}$$

$$\begin{array}{r|l} & x - 2 \\ \hline x-1 & x^2 - 3x + 5 \\ & -)\ x^2 - x \\ \hline & -2x + 5 \\ & -)\ -2x + 2 \\ \hline & 3 \end{array}$$

*Now, a slightly more difficult example!*

**Question 5**

Simplify the following using polynomial long division.

$$\frac{x^3 + x^2 + 2}{x - 3} = x^2 + 4x + 12 + \frac{38}{x-3}$$

$0x$

$$\begin{array}{r|l} & x^2 + 4x + 12 \\ \hline x-3 & x^3 + x^2 + 0x + 2 \\ & -)\ x^3 - 3x^2 \\ \hline & 4x^2 + 0x \\ & -)\ 4x^2 - 12x \\ \hline & 12x + 2 \\ & -)\ 12x - 36 \\ \hline & 38 \end{array}$$

**TIP:** Always remember to fill in any missing powers of $x$ in the numerator or denominator with "placeholders" that have a coefficient of $0$.

**Question 6 Extension.**

Simplify the following using polynomial long division.

$$\frac{x^4 + 4x^3 + 3x^2 - 2x + 3}{x + 3}$$

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## Sub-Section: Remainder Theorem

↳ Find R.

*How can we find the remainder without long division?*

**Exploration: Derivation of the Remainder Theorem**

- Consider $\frac{f(x)}{g(x)}$.

quadratic / linear

$$\frac{f(x)}{g(x)} = q(x) + \frac{R}{g(x)}, \text{ where } R = \text{Remainder}$$

- Let's multiply everything by $g(x)$.

$$f(x) = q(x) \cdot g(x) + R$$

- Remember, we are trying to find the remainder $R$ before we do long division.

  What functions do we already have before long division?

$$f(x) = q(x) \cdot g(x) + R$$

$= 0$

- How can we get $f(x)$ to equal to the remainder $R$?
  - We can substitute a value of $x$ such that, the $g(x)$ (divisor) is equal to 0.

$$f(\alpha) = q(x) \times 0 + R$$

$$f(\alpha) = R$$

**Space for Personal Notes**

quadratic

$x-a$

**Remainder Theorem**

- **Definition**: Finds the remainder of long division without the need of long division.

$= 0$

$$\textit{when } P(x) \textit{ is divided by } (x-\alpha), \textit{the remainder is } P(\alpha)$$

- **Steps:**

1. Find $x$ values which makes the divisor equal to 0.
2. Substitute it into the dividend function.

**Discussion:** How do we find the remainder of $f(x) \div (x-2)$? $=0$

$x-2=0$

$x=2$

**Discussion:** How do we find the remainder of $f(x) \div (2x+1)$? $=0$

$2x+1=0$

$x=-\frac{1}{2}$

**Question 7 Walkthrough.**

Find the remainder of the division $\frac{f(x)}{g(x)}$, where, $f(x) = x^3 + x^2 - 2x + 5$ and $g(x) = x + 1$

$x+1=0$

$x=-1$

$f(-1) = -1+1+2+5$

$=7$

Remainder $=7$

Extra

$$\begin{array}{r|l} & x^2 - 2 \\ \hline x+1 & x^3+x^2-2x+5 \\ - & x^3+x^2 \\ \hline & -2x+5 \\ - & -2x-2 \\ \hline & 7 \end{array}$$

$\frac{f(x)}{g(x)} = x^2-2+\frac{7}{x+1}$

$f(x) = (x^2-2)g(x) + 7$

*Your turn!*

**Active Recall: Remainder Theorem**

1. Find $x$ values which makes the divisor (denom.) equal to 0.
2. Substitute it into the dividend (numerator) function.

**Question 8**

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where, $f(x) = x^3 - 2x^2 + 3x + 1$ and $g(x) = 2x + 4$.

① make $g(x) = 0$

$2x + 4 = 0$

$x = -2$

② sub $x = -2$ $f(x)$

$f(-2) = -8 - 8 - 6 + 1$

$= -21$

Remainder $= -21$

**Question 9 Extension.**

For the polynomial $f(x) = 3x^3 - 2x^2 + (7 - 2a)x + 1$, we get a remainder of 14 when $f(x)$ is divided by $g(x) = x - 1$. Find the value of $a$.

6
2 3 6

## Sub-Section: Factor Theorem

$x=2$ $x-2=0$

**Discussion:** What division could $f(2)$ be the remainder of?

$f(x) \div (x-2)$

Remainder theorem

**Discussion:** Hence, what does it mean when $f(2) = 0$? ← Remainder is 0

$x-2$ is a factor $f(x)$

*This is called the "Factor theorem"*

### Factor Theorem

- For every $x$-intercept, there is a factor:

$$\boldsymbol{if\ P(\alpha) = 0\ then, (x - \alpha)\ is\ a\ factor\ of\ P(x)}$$

**Question 10 Walkthrough.**

$\frac{3x^3+8x^2-20x-16}{x+4}$

Determine if $\underbrace{x + 4}_{x=-4}$ ($=0$) is a factor of $P(x) = 3x^3 + 8x^2 - 20x - 16$. $=0$

$P(-4) = 3\cdot(-64) + 8\cdot 16 - 20\cdot(-4) - 16$
$= -192 + 128 + 80 - 16$
$= -208 + 208$
$= 0$

$\boxed{\text{It is: As subbing } x = -4 \text{ makes } P(x) = 0.}$

$x+4$ is a factor of $P(x)$

*Your turn!*

**Question 11**

Determine if $x+2$ is a factor of $P(x) = 2x^3 - 7x^2 + 7x - 2$.

$=0$

$x=-2$

$P(-2) = 2\cdot(-8) - 7\cdot 4 + 7(-2) - 2$

$= -16 - 28 - 14 - 2$

$= -60$

Remainder $= -60$

$\Rightarrow x+2$ is not a factor

It is not: As subbing $x = -2$ does not make $P(x) = 0$.

**Question 12 Extension.**

Determine if $x - \frac{3}{2}$ is a factor of $P(x) = 6x^3 - x^2 - 20x + 12$.

.

It is, since $P(3/2) = 0$.

## Sub-Section: Factorising Polynomials

### Factorising Polynomials

$P(\ldots) = 0$

Goal: Remainder $= 0$

- The steps are:
  - Find a single root by trial and error.

    (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

18

2 9

1 −1
2 −2
3 −3

**Question 13 Walkthrough.**

Find all the roots of $f(x) = x^3 + 3x^2 - 6x - 8$.

$f(1) = 1 + 3 - 6 - 8 = -10 \neq 0$

$f(-1) = -1 + 3 + 6 - 8 = 0$

Factor: $x = -1$

$x + 1 = 0$

$$\begin{array}{r|l} & x^2 + 2x - 8 \\ \hline x+1 & x^3 + 3x^2 - 6x - 8 \\ & -)\, x^3 + x^2 \\ \hline & 2x^2 - 6x \\ & -)\, 2x^2 + 2x \\ \hline & -8x - 8 \\ & -8x - 8 \\ \hline & 0 \end{array}$$

$f(x) = (x+1)(x^2 + 2x - 8)$

$= (x+1)(x+4)(x-2)$

Roots

$x = -1, -4, 2$

**NOTE:** When the question asks for all roots, you cannot just factorise and end it there!

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*Your turn!*

**Question 14** → factorise

Find all the roots of $f(x) = x^3 + 12x^2 + 17x - 90$.

① trial & error $f(\ldots) = 0$

$f(1) \neq 0$

$f(-1) \neq 0$

$f(2) = 0$

factor: $x-2$

② 
$$\begin{array}{r} x^2+14x+45 \\ x-2 \overline{)\, x^3+12x^2+17x-90} \\ \underline{x^3-2x^2} \\ 14x^2+17x \\ \underline{14x^2-28x} \\ 45x-90 \\ \underline{45x-90} \\ 0 \end{array}$$

$f(x) = (x-2)(x^2+14x+45)$

$= (x-2)(x+5)(x+9)$

$x = 2, -5, -9$

**Question 15**

Find all the roots of $f(x) = -2x^3 - 13x^2 - 5x + 6$.

$f(-1) = 0$

$x+1$

$$\begin{array}{r} -2x^2-11x+6 \\ x+1 \overline{)\, -2x^3-13x^2-5x+6} \\ \vdots \end{array}$$

$f(x) = (x+1)(-2x^2-11x+6)$

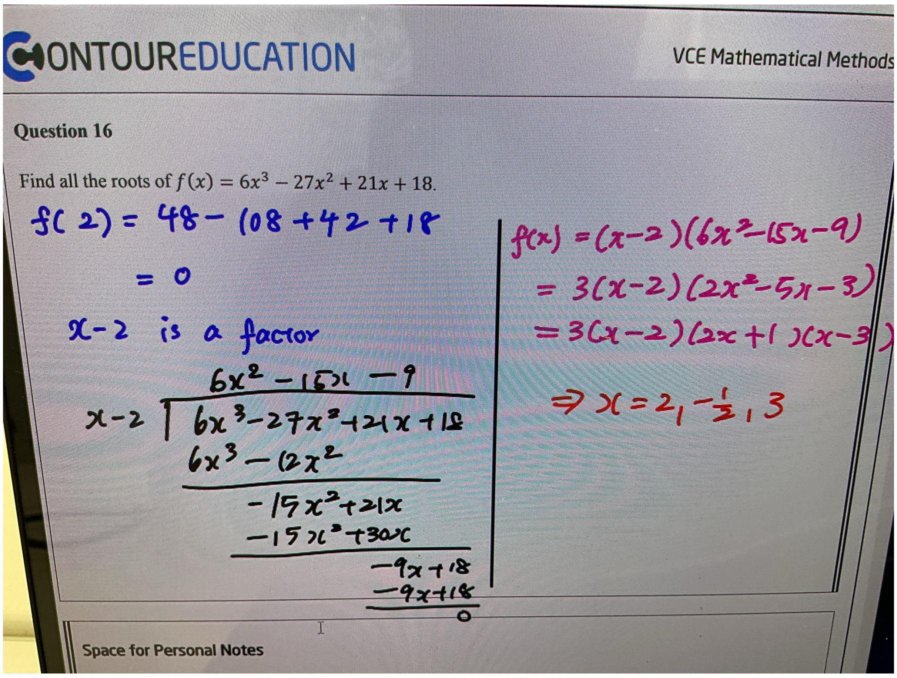
$= (x+1)(2x-1)(-x-6)$

Roots

$x = -1, \frac{1}{2}, -6$

**Question 16**

Find all the roots of $f(x) = 6x^3 - 27x^2 + 21x + 18$.



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Question 16

Find all the roots of $f(x) = 6x^3 - 27x^2 + 21x + 18$.

$f(2) = 48 - 108 + 42 + 18$

$= 0$

$x - 2$ is a factor

$$\begin{array}{r|l} & 6x^2 - 15x - 9 \\ \hline x-2 & 6x^3 - 27x^2 + 21x + 18 \\ & 6x^3 - 12x^2 \\ \hline & -15x^2 + 21x \\ & -15x^2 + 30x \\ \hline & -9x + 18 \\ & -9x + 18 \\ \hline & 0 \end{array}$$

$f(x) = (x-2)(6x^2 - 15x - 9)$

$= 3(x-2)(2x^2 - 5x - 3)$

$= 3(x-2)(2x+1)(x-3)$

$\Rightarrow x = 2, -\frac{1}{2}, 3$

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Space for Personal Notes

## Sub-Section: Rational Root Theorem

**Discussion:** Consider $(2x-1)(3x-1)(6x-1)$. What are the roots and could we have gotten that from trial and error?

$\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

*So, what should we do?*

### Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$\textit{Potential root} = \pm\frac{\textit{Factors of constant term } a_0}{\textit{Factors of leading coefficient } a_n}$$

- If the roots are rational numbers, the roots can only be $\pm\frac{factors\ of\ constant\ term\ a_0}{factors\ of\ leading\ coefficient\ a_n}$.

**Question 17 Walkthrough.**

Find all the roots of $f(x) = 6x^3 + 13x^2 - 14x + 3$.

1, 2, 3, 6    1, 3

8, 1/-8   L-A...

$\frac{1}{1}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{6}$

$\frac{3}{1}$ $\frac{3}{2}$ $\cancel{\frac{3}{3}}$ $\cancel{\frac{3}{6}}$

$f(1) = 6 + 13 - 14 + 3 \neq 0$

$f(-1) = -6 + 13 + 14 + 3 \neq 0$

$f(\frac{1}{2}) = 0$ ✓ $\Rightarrow x - \frac{1}{2} = 0$

or $2x - 1$

→ long divide
→ factorise

**NOTE:** All the roots are part of the suggestion given by the rational root theorem.

**Question 18**

Find all the roots of $f(x) = 2x^3 - x^2 - 22x - 24$.

Discussion: Why is rational root theorem called a rational root theorem?

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**Question 19 Extension.**

Find all the roots of $f(x) = 6x^3 + 19x^2 - 24x - 16$.

find 1 factor

1 2 3 6 1 2 4 8 16

$\frac{1}{1}$ $-\frac{1}{2}$ ✓ $\frac{1}{3}$ $\frac{1}{6}$ $\frac{4}{1}$ $\frac{4}{2}$ $\frac{4}{3}$ $\frac{4}{6}$

$\frac{2}{1}$ $\frac{2}{2}$ $\frac{2}{3}$ $\frac{2}{6}$ $\frac{8}{1}$ $-\frac{8}{2}$ $\frac{8}{3}$ $\frac{8}{6}$

$\frac{16}{1}$ $\frac{16}{2}$ $\frac{16}{3}$ $\frac{16}{6}$

$1$ $-1$ $\frac{1}{2}$ $-\frac{1}{2}$

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## Sub-Section: Sum and Difference of Cubes

### Sum and Difference of Cubes

No 2

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

**Question 20 Walkthrough.**

Factorise the following polynomial as much as possible.

$$x^3 + 125$$

$(x+5)(x^2-5x+25)$

**Question 21**

Factorise the following polynomial as much as possible.

$$8x^3 - 216$$

$(2x)^3 \quad 6^3$

$(2x-6)(4x^2+12x+36)$

$= 2(x-3) \times 4(x^2+3x+9)$

$= 8(x-3)(x^2+3x+9)$

**Question 22 Extension.**

Factorise the following polynomial as much as possible.

$$32x^3 - 256$$

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linear quadratic cubic

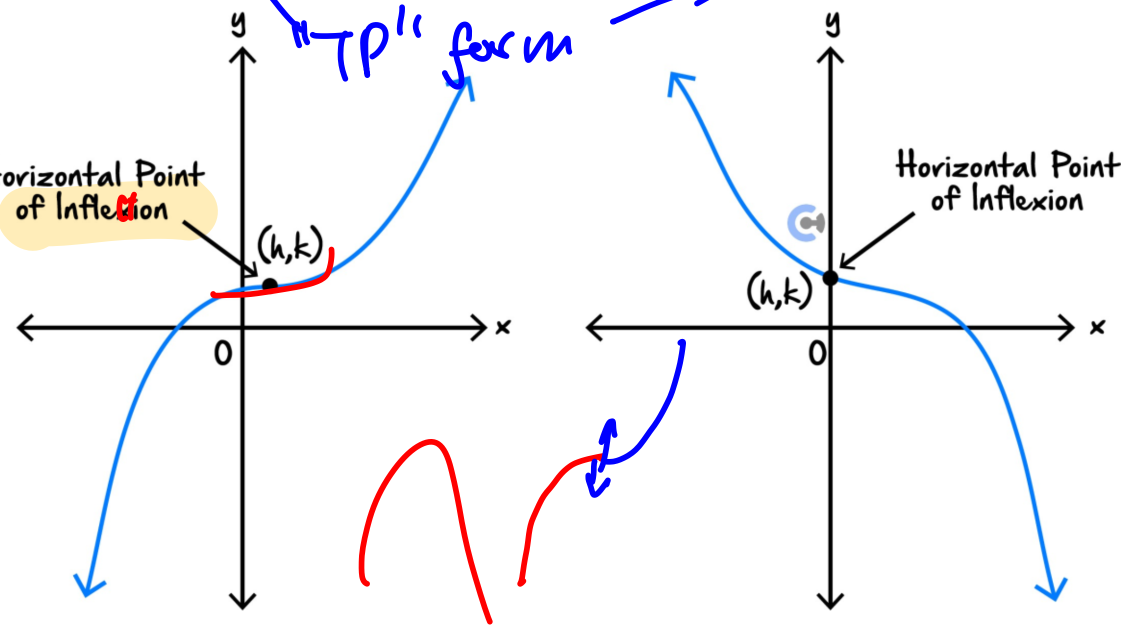
## Section B: Graphs of a Polynomial

### Sub-Section: Graphing Polynomials in the Form of $a(x-h)^n + k$

**Graphs of $a(x-h)^n + k$, where $n$ is an Odd Positive Integer**

- All graphs look like a "cubic".

"TP" form → SPOI form




- The point $(h,k)$ gives us the stationary point of inflection.
- $n$ cannot be 1 for this shape to occur!

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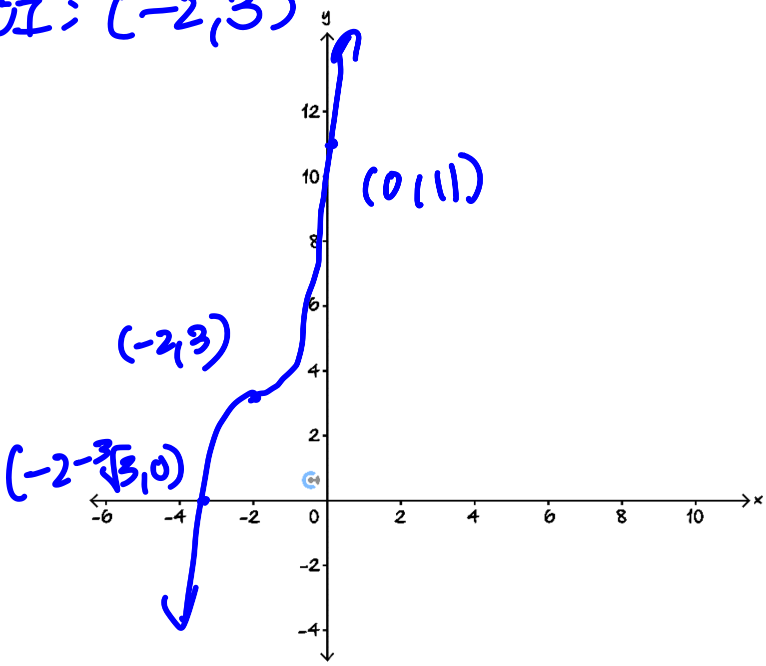
**Question 23 Walkthrough**.

Sketch the graph of $y = (x + 2)^3 + 3$ on the axes below.

SPOI: $(-2,3)$

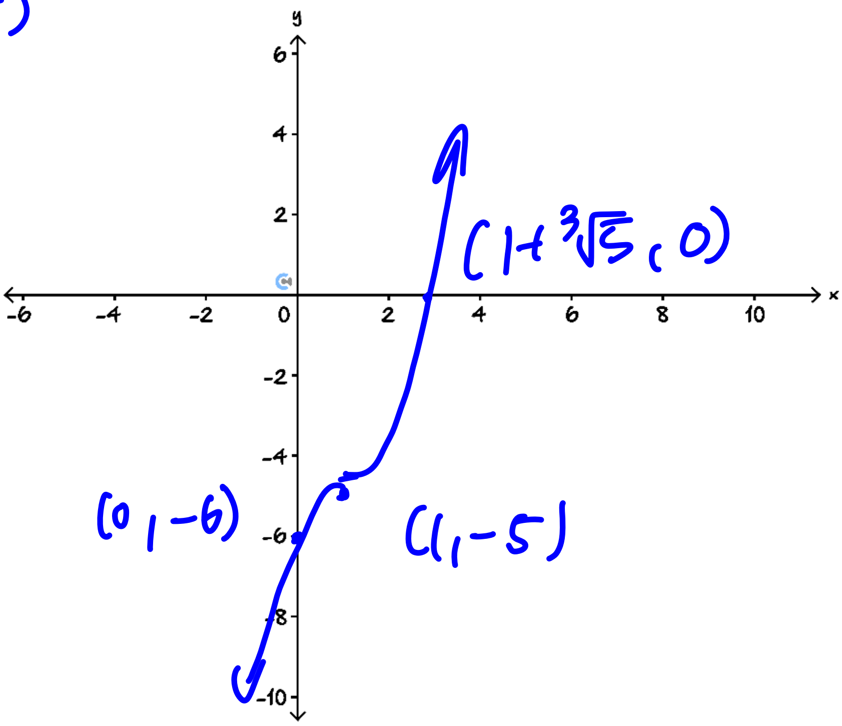



**Question 24**

x-int $(1+\sqrt[3]{5},0)$

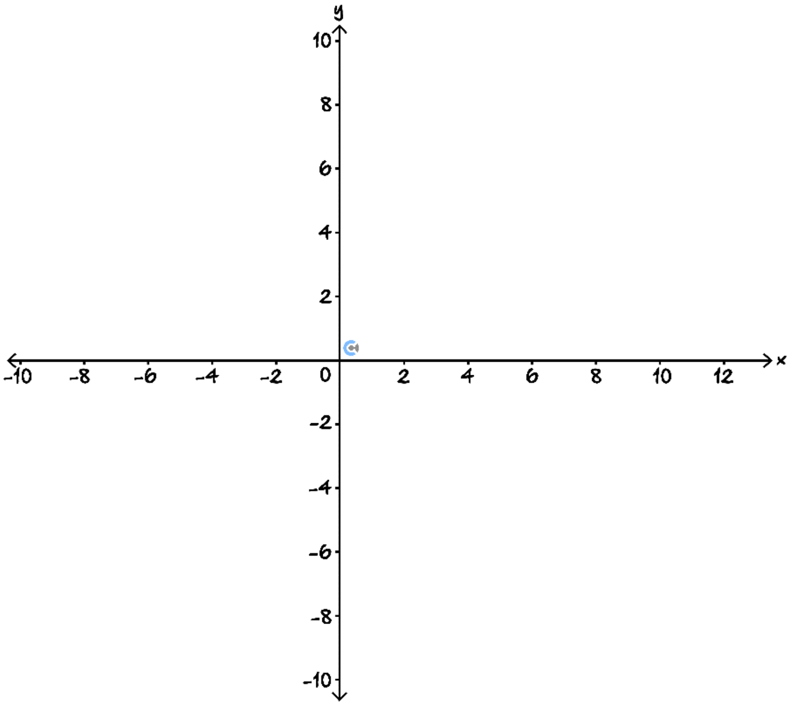
Sketch the graph of $y = (x - 1)^3 - 5$ on the axes below.

SPOI $(1,-5)$



**Question 25**

Sketch the graph of $y = \frac{1}{2}(x-2)^3 + 6$ on the axes below.

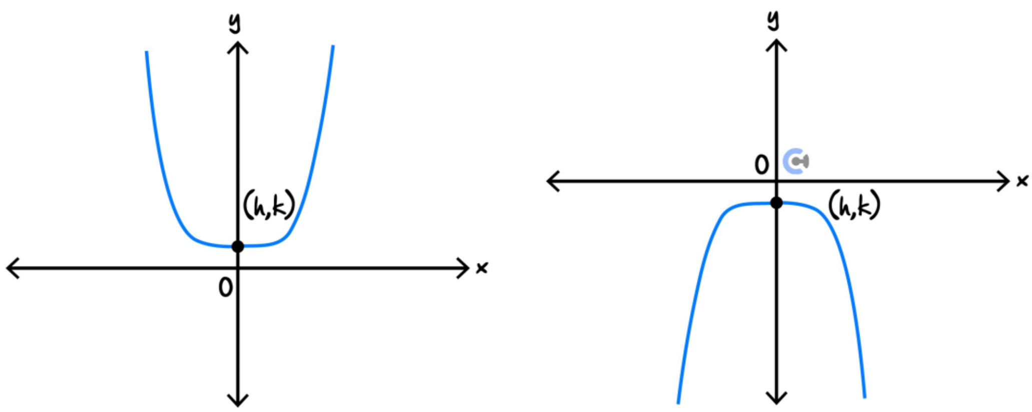



*What about even powers?*

4 6 8

**Graphs of $a(x-h)^n + k$, where $n$ is an Even Positive Integer**

- All graphs look like a "quadratic".



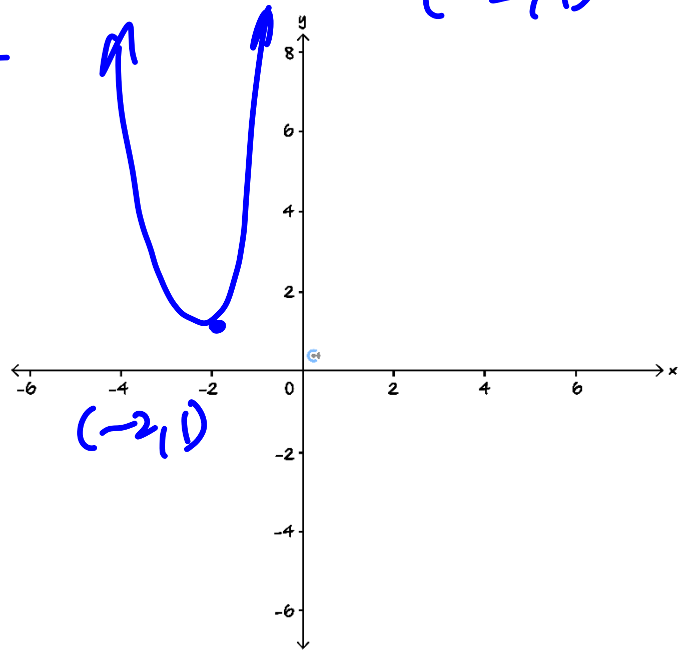

- The point $(h, k)$ gives us the turning point.

**Question 26 Walkthrough.**

Sketch the graph of $y = (x + 2)^4 + 1$ on the axes below.

$(-2, 1)$

$y\text{-int} = 17$




**Question 27**

Sketch the graph of $y = -(x - 1)^4 + 4$ on the axes below.

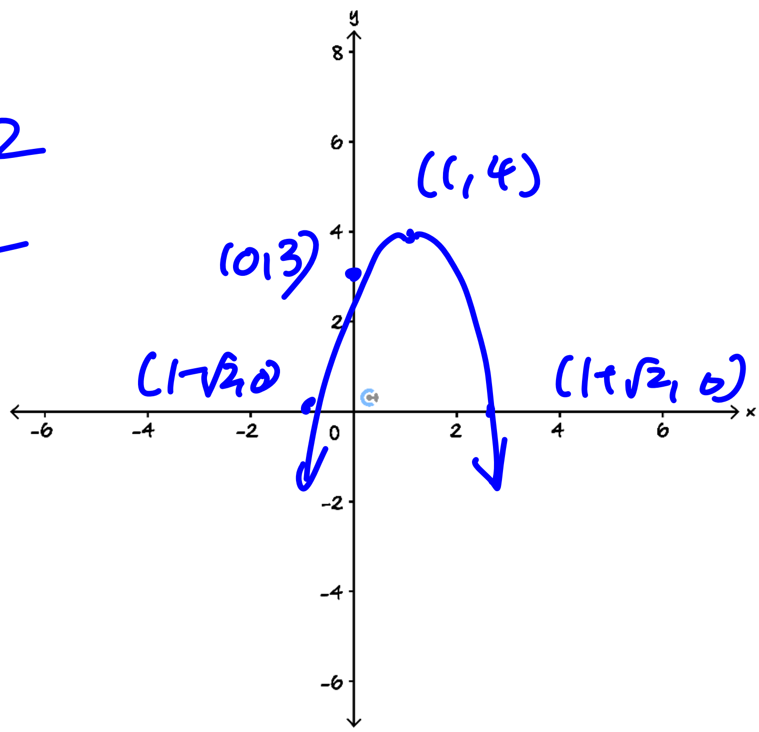
x-ints

$(x-1)^4 = 4$

$(x-1)^2 = \pm 2$

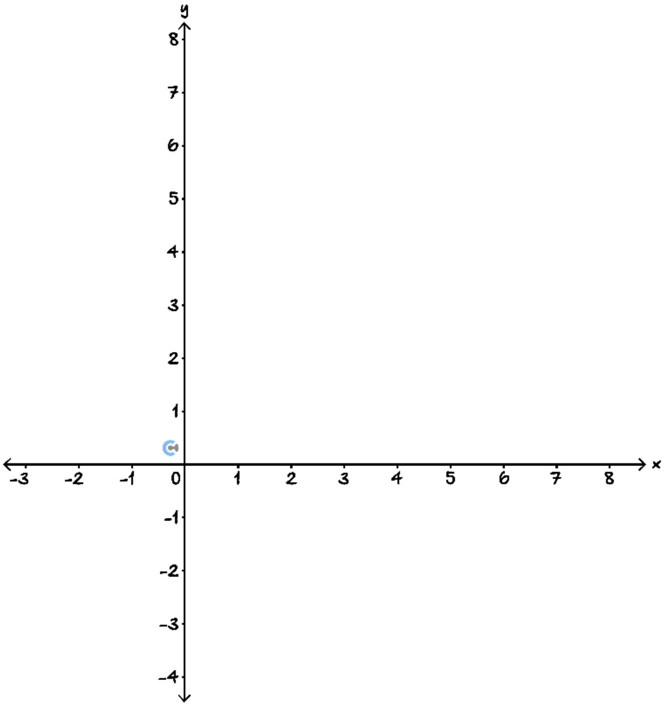
$x - 1 = \pm\sqrt{2}$

$x = 1 \pm \sqrt{2}$



**Question 28 Extension.**

Sketch the graph of $y = \frac{1}{2}(x-2)^4 - 3$ on the axes below.

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## Sub-Section: Graphing Factorised Polynomials

*What about the graph of a factorised polynomial?*

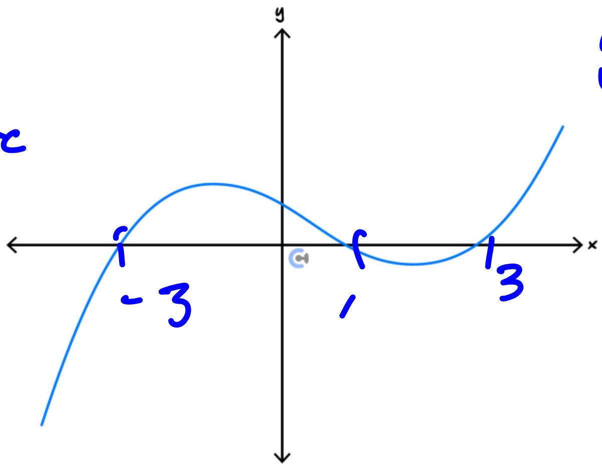
### Exploration: Graphs of Factorised Polynomials

*All* non-repeated *linear factors correspond to* x-ints *of the graph.*

cubic

$y=(x+3)(x-1)(x-3)$

$-3, 1, 3$




- E.g., $f(x) = (x-a)(x-b)(x-c)$ results in $x$-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$.

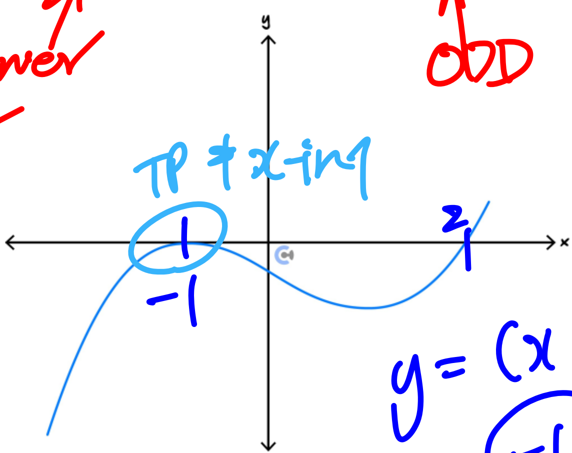
*All* repeated *linear factors* $(x-1)^2$ *correspond to* TP OR SPOI x-ints *of the graph.*

EVEN power $(x-2)^2$

ODD power $(x+1)^3$

SPOI $(-1)$

TP + x-int



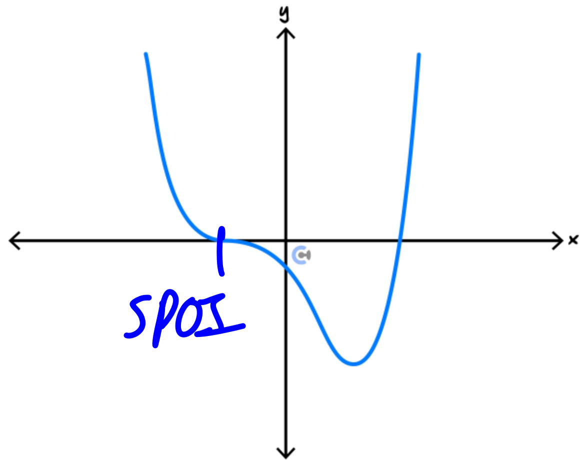

$y=(x+1)^2(x-2)$

$-1$ TP + x-int

just x-int

- E.g., $f(x) = (x-a)^2(x-b)$ will have an $x$-intercept $(a, 0)$ which is also a local minimum/maximum.

*All* odd powers *linear factors correspond to* SPOI *of the graph.*




- E.g., $f(x) = (x-a)^3(x-b)$ has an $x$-intercept $(a, 0)$ which is also a stationary point of inflection.

Definition

<u>Graphs of Factorised Polynomials</u>

- Steps:

1. Plot $x$-intercepts.
2. Determine whether the polynomial is positive or negative.
3. Use the repeated factors to deduce the shape.

   Non-Repeated: Only $x$-intercept.

   Even Repeated: $x$-intercept and a turning point.

   Odd Repeated: $x$-intercept and a stationary point of inflection.

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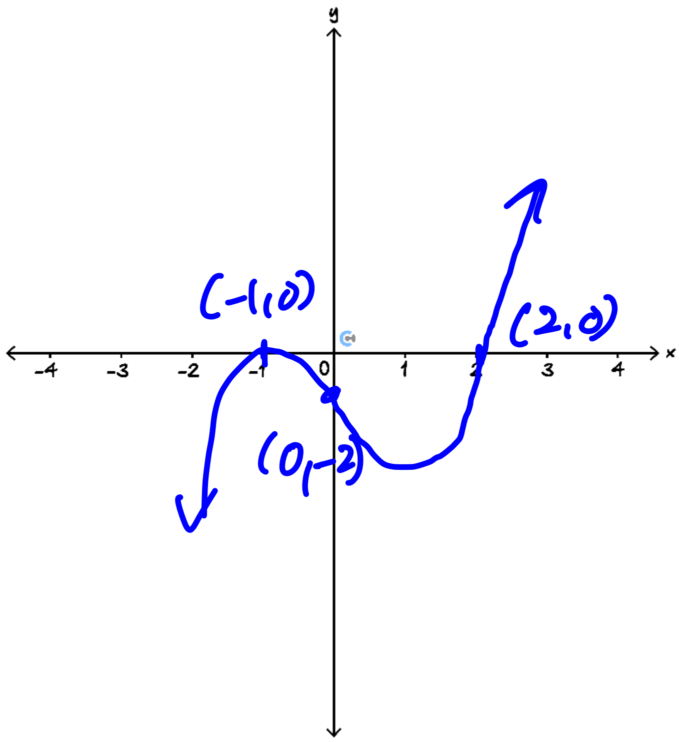
**Question 29 Walkthrough.**

Sketch the graphs of the following functions on the axes provided. Ignore the $y$-axis scale.

**a.** $y = (x+1)^2(x-2)$

$-1$ TP $\quad 2$




**b.** $y = \left(x - \frac{3}{2}\right)^3(4-x) = -\left(x-\frac{3}{2}\right)^2(x-4)$

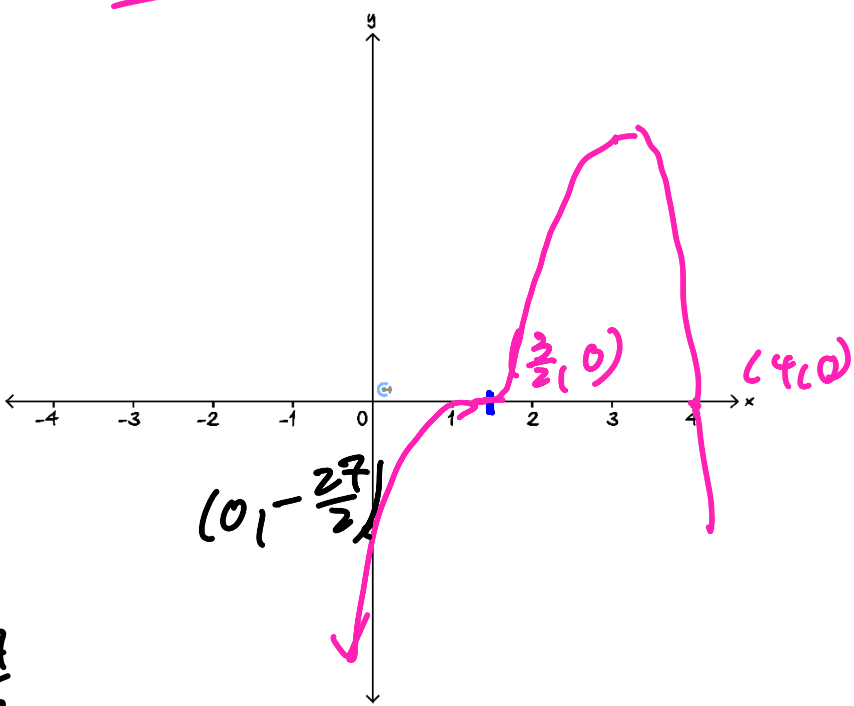
$\frac{3}{2}$

SPOI

$\left(-\frac{3}{2}\right)^3(4)$

$= \frac{-27}{8} \times 4$

$= -\frac{27}{2}$



*Your turn!*

**Question 30**

Sketch the graphs of the following functions on the axes provided. Ignore the $y$-axis scale.

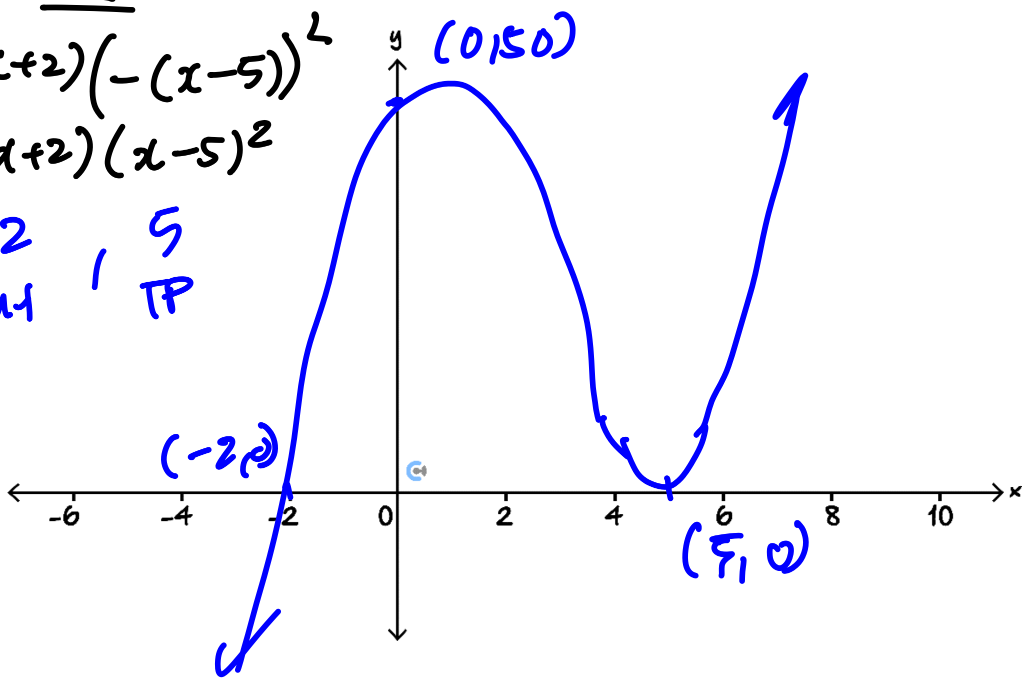
**a.** $y = (2 + x)(5 - x)^2$

$y = (x+2)(-(x-5))^2$

$= (x+2)(x-5)^2$

$-2$ , $5$

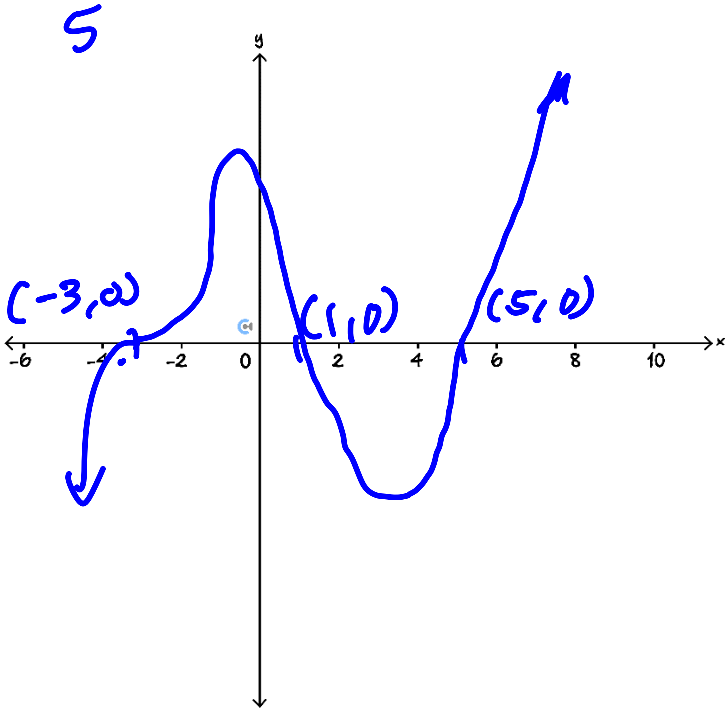
x-int , TP




**b.** $y = (x + 3)^3(x - 1)(x - 5)$

$-3$ $1$ $5$

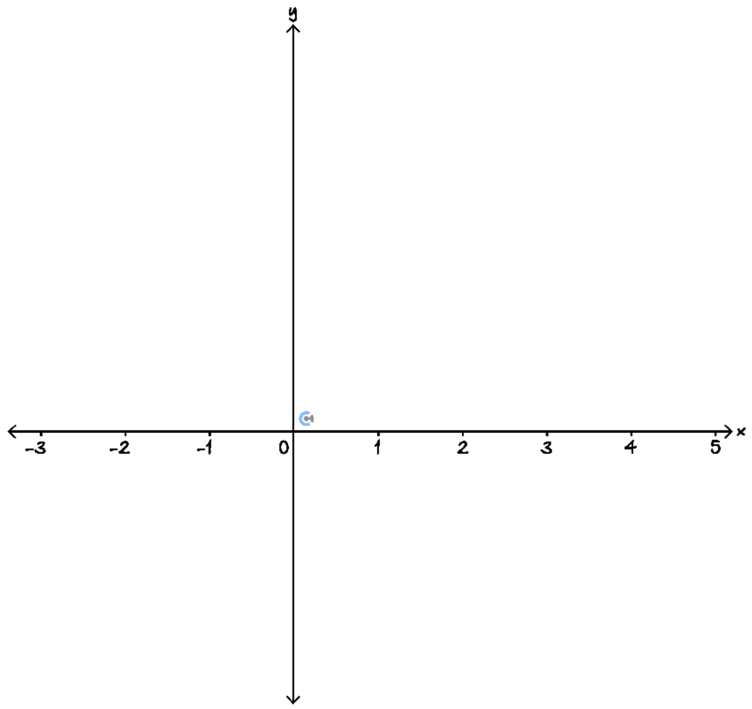
SPOI



**Question 31**

Sketch the graph of the following function on the axes provided. Ignore the $y$-axis scale.

$$y = (x-1)^3(x+1)^2(x-2)$$




Space for Personal Notes

# Contour Check

**Learning Objective: [1.5.1] – Identify the properties of polynomials and solve long division.**

Key Takeaways

- The degree of a polynomial is the polynomial's highest power.
- The roots of a polynomial are its x-ints.
- For polynomial long division:

$$\frac{Dividend}{Divisor} = Quotient + \frac{Remaindor}{Divisor}$$

**Learning Objective: [1.5.2] – Apply remainder and factor theorem to find remainders and factors.**

Key Takeaways

- When $P(x)$ is divided by $(x-\alpha)$ $=0$, the remainder is $P(a)$.
- If $P(\alpha) = 0$ then $(x-\alpha)$ is a factor of $P(x)$.

**Learning Objective:** [1.5.3] - Find factored form of polynomials.

Key Takeaways

- Steps to factor a cubic polynomial are:
  - Find a single root by trial and error.

    (Factor Theorem: Substitute into the function and see if we get = 0.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.
- Rational Root Theorem **narrows down** the possible roots. If the roots are rational numbers, it must be that any.

$$\textbf{Potential root} = \pm\frac{\textbf{Factors of } \text{constant} \quad a_0}{\textbf{Factors of } \text{leading coefficient} \quad a_n}$$

- Sum and difference of cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

**Learning Objective: [1.5.4] - Graph factored and unfactored polynomials.**

**Key Takeaways**

- Graphs of $a(x-h)^n + k$, where $n$ is an Odd Positive Integer that is not equal to $1$:
  - The point $(h, k)$ gives us the stationary point of inflection.
- Graphs of $a(x-h)^n + k$, where $n$ is an Even Positive Integer:
  - The point $(h, k)$ gives us the TP.
  - These graphs look like a quadratic.
- Steps to graphing factorised polynomials:
  1. Plot $x$-intercepts.
  2. Determine whether the polynomial is positive or negative.
  3. Use the repeated factors to deduce the shape:

     Non-Repeated: Only x-int.

     Even Repeated: $x$-intercept and a TP.

     Odd Repeated: $x$-intercept and a SPOI.

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# VCE Mathematical Methods ½

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