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VCE Mathematical Methods ½
Polynomials [1.5]
Workbook

Outline:

Learning Objectives:

- ☐ **MM12 [1.5.1]** - Identify the properties of polynomials and solve long division.
- ☐ **MM12 [1.5.2]** - Apply remainder and factor theorem to find remainders and factors.
- ☐ **MM12 [1.5.3]** - Find factored form of polynomials.
- ☐ **MM12 [1.5.4]** - Graph factored and unfactored polynomials.

Section A: Algebra of Polynomial Functions

Sub-Section: Terminologies of Polynomials

Degree of Polynomial Functions

$$\textit{Degree} = \textit{Highest Power of the Polynomial}$$

Question 1

State the degree of each polynomial.

a. $x^3 - 4x^2 + 5x + 6$

b. $3x + 5x^2 - x^7$

c. A Quadratic.

Roots of Polynomial Functions

$$\text{Roots} = x\text{-intercept}$$

Discussion: Can a quadratic have more than 2 roots? Hence, can there be more roots than the degree?

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Question 2

Find the roots of the following polynomial:

$$(x-1)^2(x+3)^4$$

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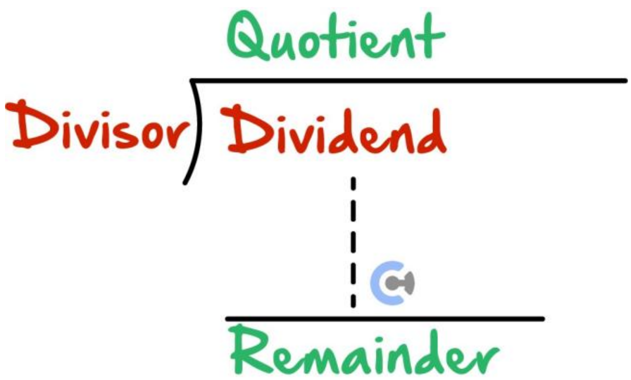
Sub-Section: Long Division

Polynomial Long Division

- Division of polynomials:



$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

Question 3 Walkthrough.

Simplify the following using polynomial long division.

$$\frac{3x^2 + 10x + 20}{2x + 4}$$

TIP: Always focus on the highest degree term first.

Your turn!

Question 4

Simplify the following using polynomial long division.

$$\frac{x^2 - 3x + 5}{x - 1}$$

Now, a slightly more difficult example!

Question 5

Simplify the following using polynomial long division.

$$\frac{x^3 + x^2 + 2}{x - 3}$$

TIP: Always remember to fill in any missing powers of x in the numerator or denominator with “placeholders” that have a coefficient of 0.

Question 6 Extension.

Simplify the following using polynomial long division.

$$\frac{x^4 + 4x^3 + 3x^2 - 2x + 3}{x + 3}$$

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Sub-Section: Remainder Theorem

How can we find the remainder without long division?

Exploration: Derivation of the Remainder Theorem

- Consider $\frac{f(x)}{g(x)}$.

$$\frac{f(x)}{g(x)} = q(x) + \frac{R}{g(x)}, \text{where } R = \text{Remainder}$$

- Let's multiply everything by $g(x)$.

$$f(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Remember, we are trying to find the remainder R before we do long division.

  What functions do we already have before long division?

$$f(x) = q(x) \cdot g(x) + R$$

- How can we get $f(x)$ to equal to the remainder R?
  - We can substitute a value of x such that, the ____________________ is equal to 0.

$$f(\alpha) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$f(\alpha) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

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Remainder Theorem

- **Definition**: Finds the remainder of long division without the need of long division.

$$\textit{when } P(x) \textit{ is divided by } (x-\alpha), \textit{ the remainder is } P(\alpha)$$

- **Steps:**

1. Find x values which makes the divisor equal to 0.
2. Substitute it into the dividend function.

Discussion: How do we find the remainder of $f(x) \div (x-2)$?

Discussion: How do we find the remainder of $f(x) \div (2x+1)$?

Question 7 Walkthrough.

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where, $f(x) = x^3 + x^2 - 2x + 5$ and $g(x) = x + 1$.

Your turn!

Active Recall: Remainder Theorem

1. Find x values which makes the___________________ equal to 0.
2. Substitute it into the ________________ function.

Question 8

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where, $f(x) = x^3 - 2x^2 + 3x + 1$ and $g(x) = 2x + 4$.

Question 9 Extension.

For the polynomial $f(x) = 3x^3 - 2x^2 + (7 - 2a)x + 1$, we get a remainder of 14 when $f(x)$ is divided by $g(x) = x - 1$. Find the value of a.

Sub-Section: Factor Theorem

Discussion: What division could $f(2)$ be the remainder of?

Discussion: Hence, what does it mean when $f(2) = 0$?

This is called the "Factor theorem"

Factor Theorem

- For every x-intercept, there is a factor:

$$if\ P(\alpha) = 0\ then, (x - \alpha)\ is\ a\ factor\ of\ P(x)$$

Question 10 Walkthrough.

Determine if $x + 4$ is a factor of $P(x) = 3x^3 + 8x^2 - 20x - 16$.

Your turn!

Question 11

Determine if $x + 2$ is a factor of $P(x) = 2x^3 - 7x^2 + 7x - 2$.

Question 12 Extension.

Determine if $x - \frac{3}{2}$ is a factor of $P(x) = 6x^3 - x^2 - 20x + 12$.

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Sub-Section: Factorising Polynomials

Factorising Polynomials

- The steps are:
  - Find a single root by trial and error.

    (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

Question 13 Walkthrough.

Find all the roots of $f(x) = x^3 + 3x^2 - 6x - 8$.

NOTE: When the question asks for all roots, you cannot just factorise and end it there!

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Your turn!

Question 14

Find all the roots of $f(x) = x^3 + 12x^2 + 17x - 90$.

Question 15

Find all the roots of $f(x) = -2x^3 - 13x^2 - 5x + 6$.

Question 16

Find all the roots of $f(x) = 6x^3 - 27x^2 + 21x + 18$.

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Sub-Section: Rational Root Theorem

Discussion: Consider $(2x-1)(3x-1)(6x-1)$. What are the roots and could we have gotten that from trial and error?

So, what should we do?

Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$\textit{Potential root} = \pm \frac{\textit{Factors of constant term } a_0}{\textit{Factors of leading coefficient } a_n}$$

- If the roots are rational numbers, the roots can only be $\pm\frac{factors\ of\ constant\ term\ a_0}{factors\ of\ leading\ coefficient\ a_n}$.

Question 17 Walkthrough.

Find all the roots of $f(x) = 6x^3 + 13x^2 - 14x + 3$.

NOTE: All the roots are part of the suggestion given by the rational root theorem.

Question 18

Find all the roots of $f(x) = 2x^3 - x^2 - 22x - 24$.

Discussion: Why is rational root theorem called a rational root theorem?

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Question 19 Extension.

Find all the roots of $f(x) = 6x^3 + 19x^2 - 24x - 16$.

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Sub-Section: Sum and Difference of Cubes

Sum and Difference of Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Question 20 Walkthrough.

Factorise the following polynomial as much as possible.

$$x^3 + 125$$

Question 21

Factorise the following polynomial as much as possible.

$$8x^3 - 216$$

Question 22 Extension.

Factorise the following polynomial as much as possible.

$$32x^3 - 256$$

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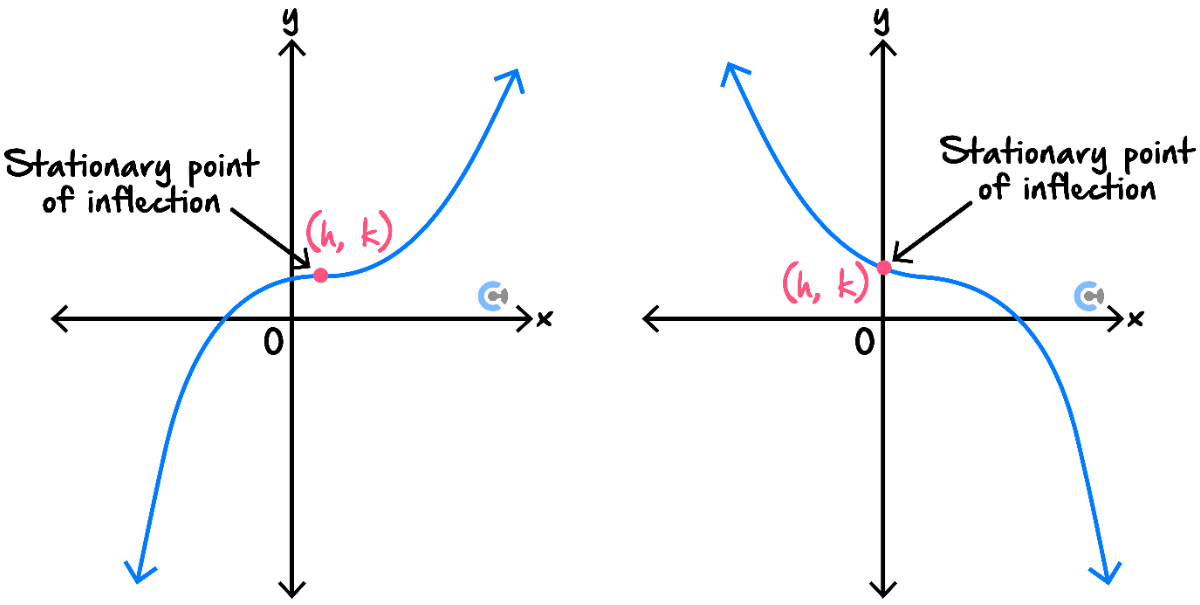
Section B: Graphs of a Polynomial

Sub-Section: Graphing Polynomials in the Form of $a(x-h)^n + k$

Graphs of $a(x-h)^n + k$, where n is an Odd Positive Integer

- All graphs look like a "cubic".



- The point (h, k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

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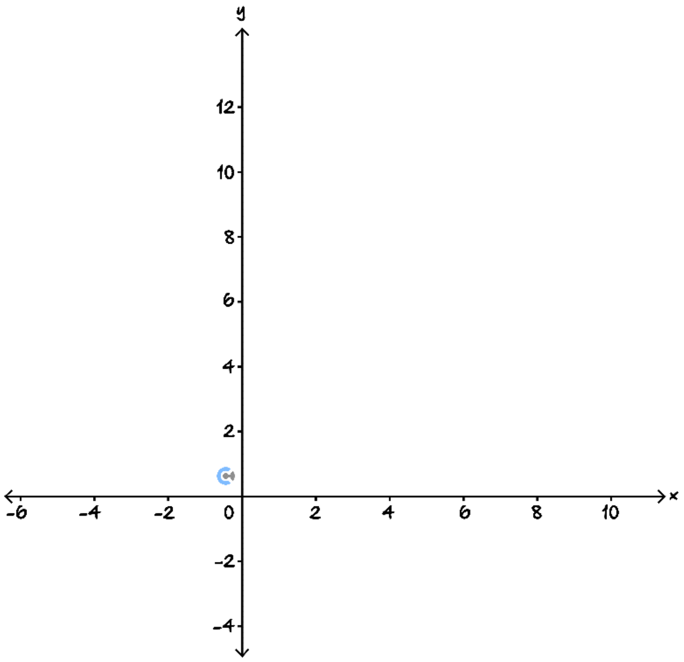
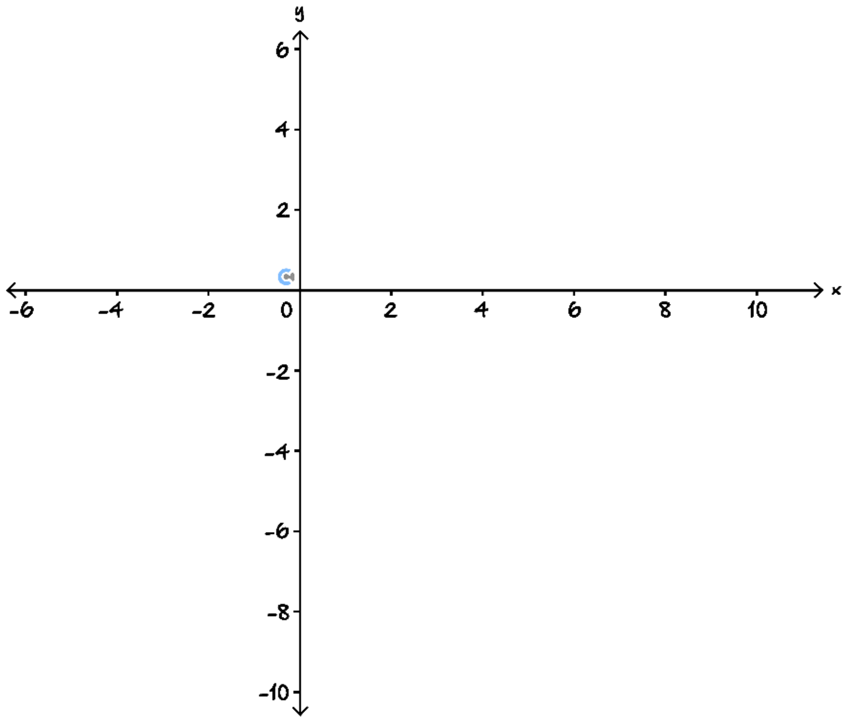
Question 23 Walkthrough.

Sketch the graph of $y = (x + 2)^3 + 3$ on the axes below.



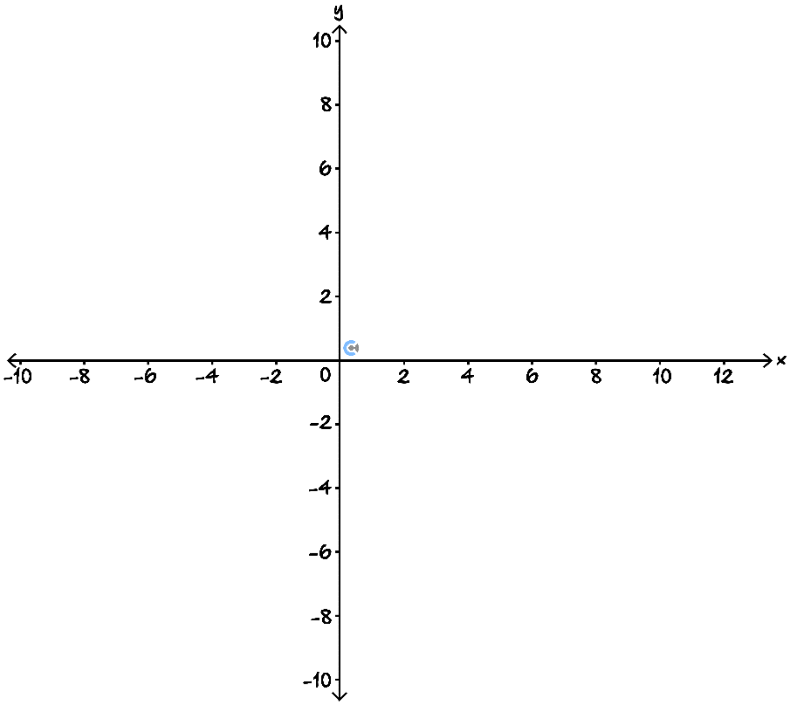
Question 24

Sketch the graph of $y = (x - 1)^3 - 5$ on the axes below.



Question 25

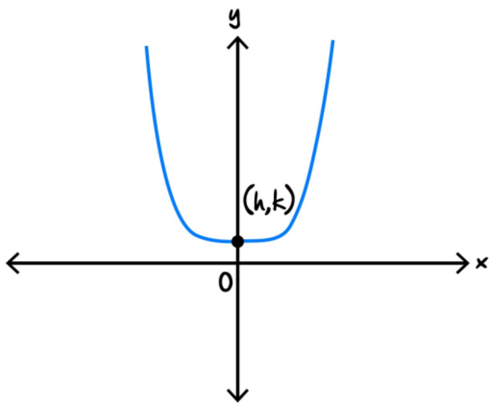
Sketch the graph of $y = \frac{1}{2}(x-2)^3 + 6$ on the axes below.

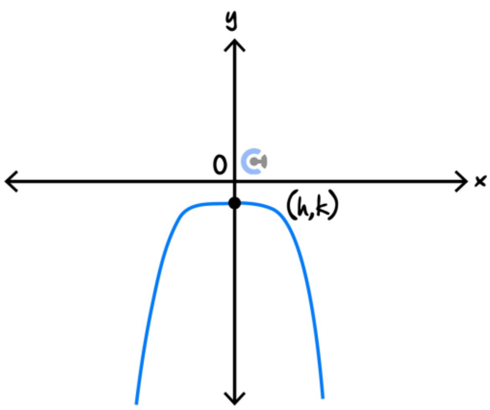


What about even powers?

Graphs of $a(x-h)^n + k$, where n is an Even Positive Integer

- All graphs look like a "quadratic".

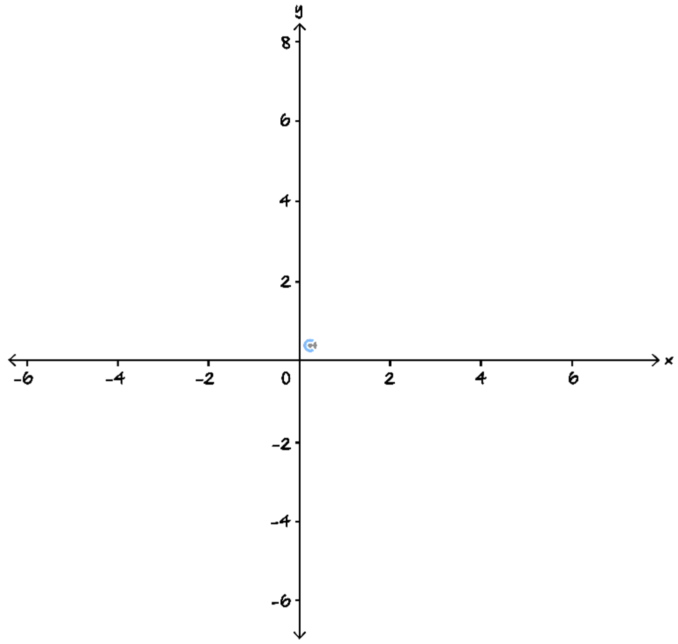




- The point (h, k) gives us the turning point.

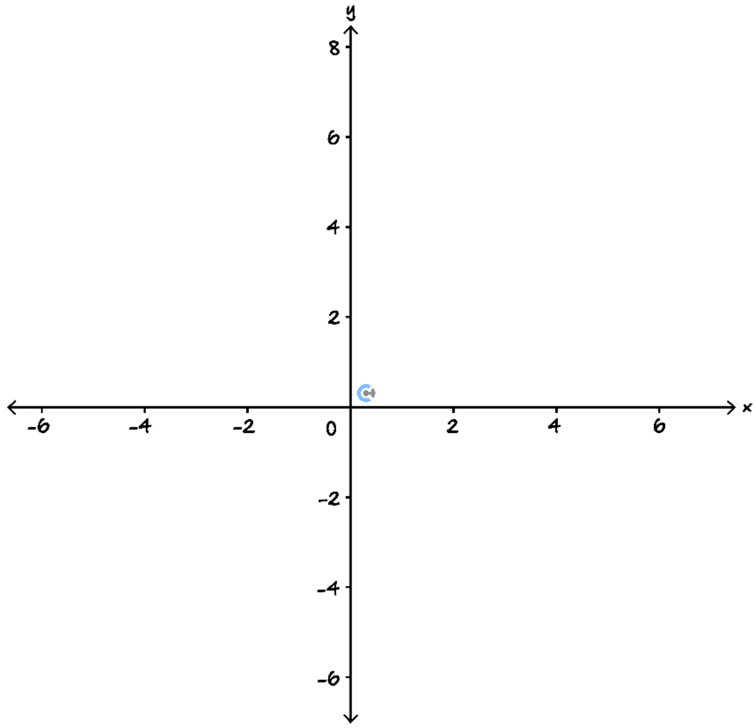
Question 26 Walkthrough.

Sketch the graph of $y = (x+2)^4 + 1$ on the axes below.



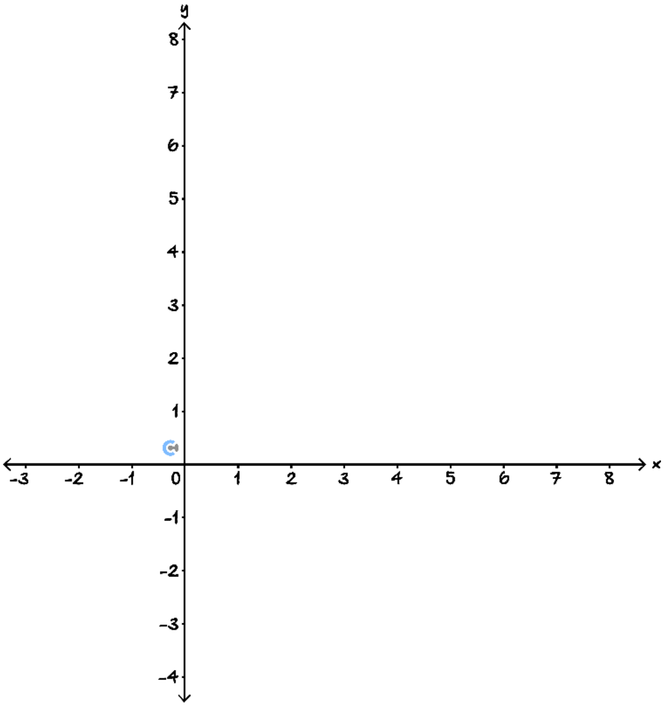
Question 27

Sketch the graph of $y = -(x-1)^4 + 4$ on the axes below.



Question 28 Extension.

Sketch the graph of $y = \frac{1}{2}(x-2)^4 - 3$ on the axes below.



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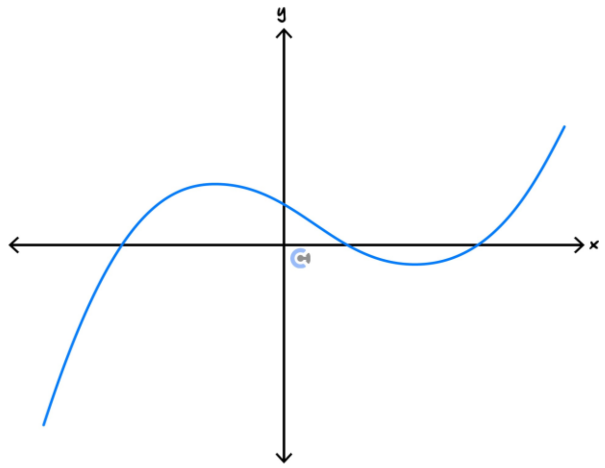
Sub-Section: Graphing Factorised Polynomials

What about the graph of a factorised polynomial?

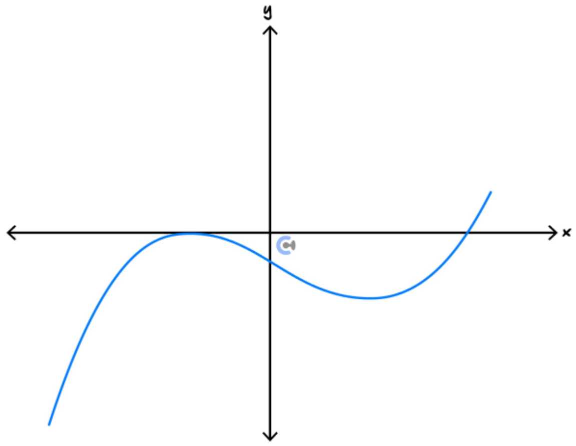
Exploration: Graphs of Factorised Polynomials

All ______________ linear factors correspond to ______________________ of the graph.



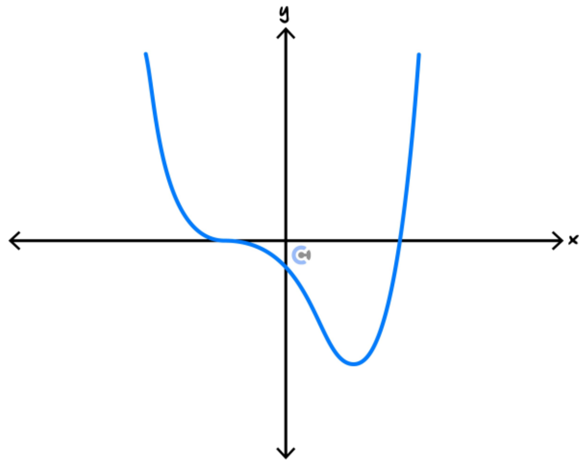
- E.g., $f(x) = (x-a)(x-b)(x-c)$ results in x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$.

All ______________ linear factors correspond to __________________________________ of the graph.



- E.g., $f(x) = (x-a)^2(x-b)$ will have an x-intercept $(a, 0)$ which is also a local minimum/maximum.

All ______________ linear factors correspond to ___________________________________ of the graph.



- E.g., $f(x) = (x-a)^3(x-b)$ has an x-intercept $(a, 0)$ which is also a stationary point of inflection.

Definition

<u>Graphs of Factorised Polynomials</u>

- **Steps:**

1. Plot x-intercepts.
2. Determine whether the polynomial is positive or negative.
3. Use the repeated factors to deduce the shape.

   Non-Repeated: Only x-intercept.

   Even Repeated: x-intercept and a turning point.

   Odd Repeated: x-intercept and a stationary point of inflection.

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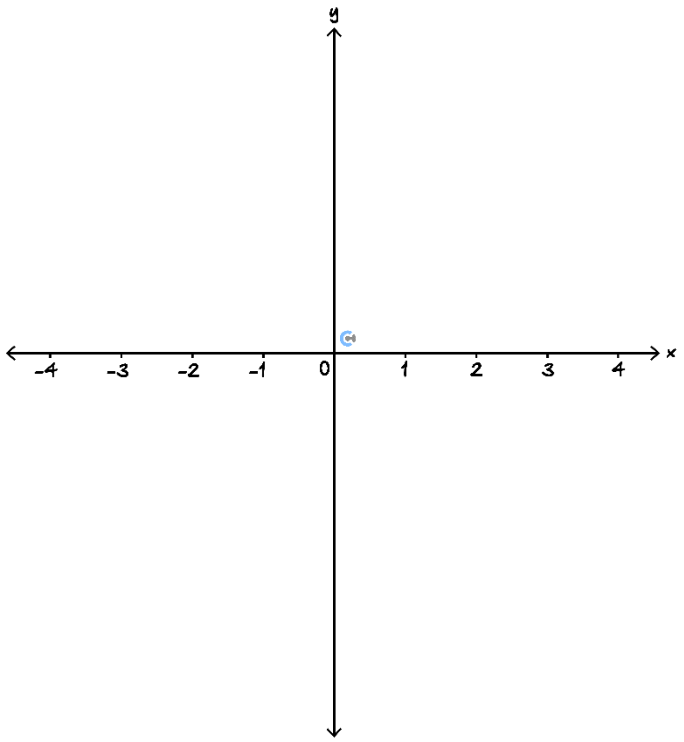
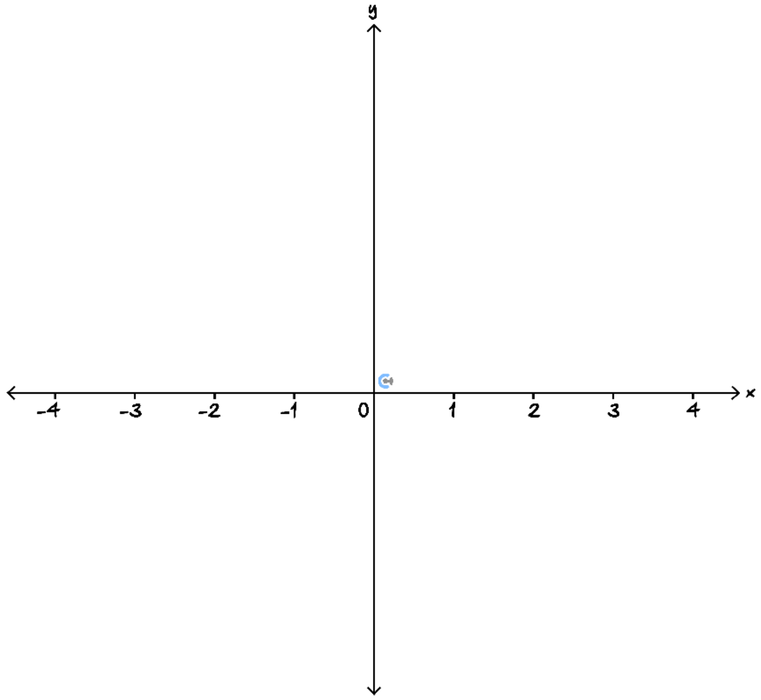
Question 29 Walkthrough.

Sketch the graphs of the following functions on the axes provided. Ignore the y-axis scale.

a. $y = (x+1)^2(x-2)$



b. $y = \left(x - \frac{3}{2}\right)^3 (4 - x)$

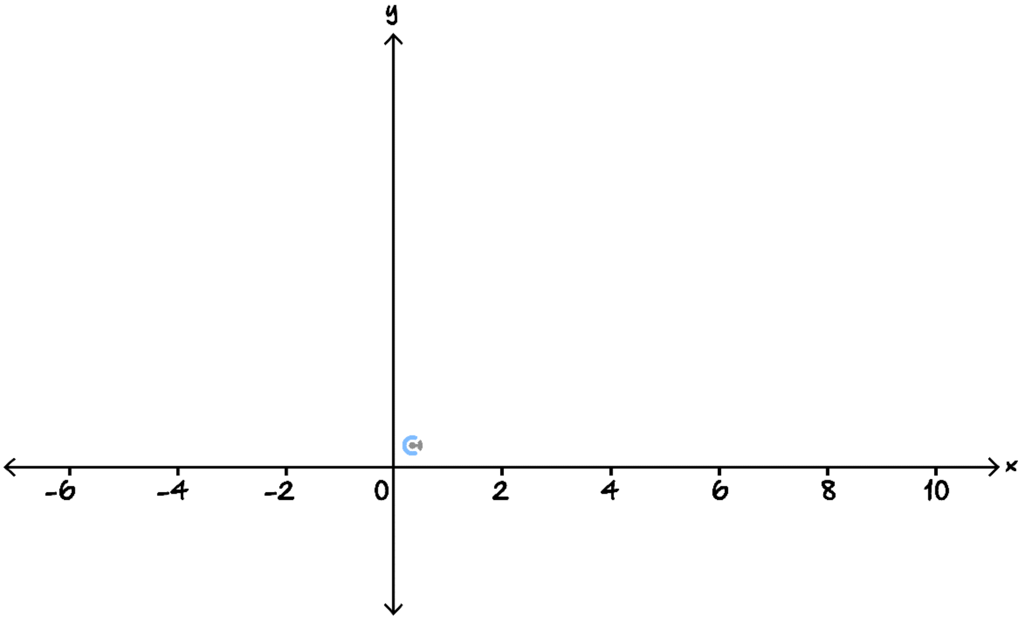


Your turn!

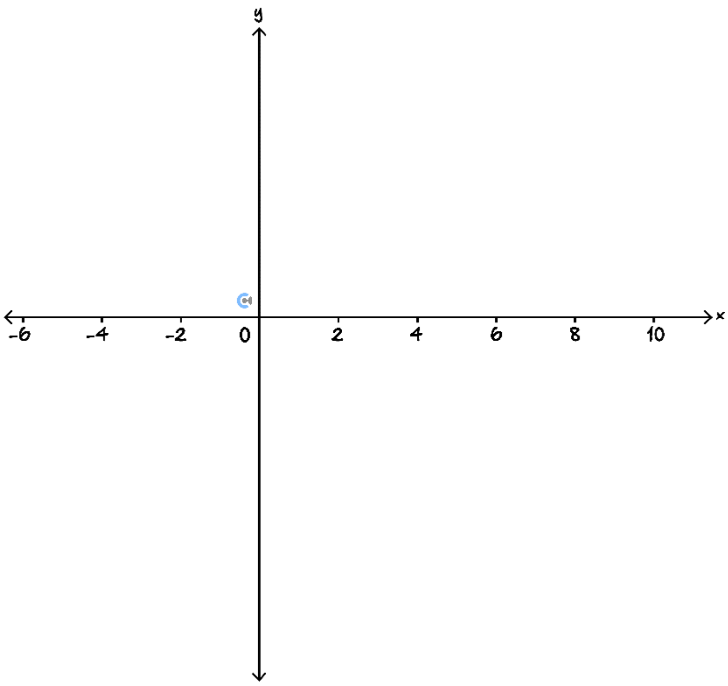
Question 30

Sketch the graphs of the following functions on the axes provided. Ignore the y-axis scale.

a. $y = (2 + x)(5 - x)^2$



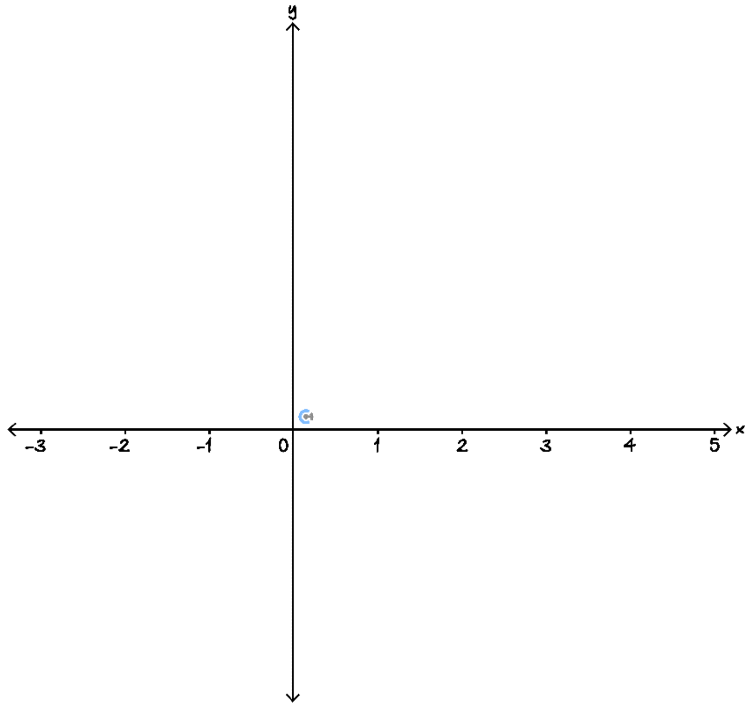
b. $y = (x + 3)^3(x - 1)(x - 5)$



Question 31

Sketch the graph of the following function on the axes provided. Ignore the y-axis scale.

$$y = (x-1)^3(x+1)^2(x-2)$$



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Contour Check

Learning Objective: [1.5.1] - Identify the properties of polynomials and solve long division.

Key Takeaways

- [ ] The degree of a polynomial is the polynomial's ________________________ power.
- [ ] The roots of a polynomial are its ________________________.
- [ ] For polynomial long division:

$$\frac{\textbf{Dividend}}{\textbf{Divisor}} = \textbf{Quotient} + \underline{\hspace{5cm}}$$

Learning Objective: [1.5.2] - Apply remainder and factor theorem to find remainders and factors.

Key Takeaways

- [ ] When $P(x)$ is divided by $(x - \alpha)$, the remainder is ________.
- [ ] If $P(\alpha) = 0$ then $(x - \alpha)$ is a ________________ of $P(x)$.

Learning Objective: [1.5.3] - Find factored form of polynomials.

Key Takeaways

- Steps to factor a cubic polynomial are:
  - Find a single root by trial and error.

    (Factor Theorem: Substitute into the function and see if we get ________________.)
  - Use __________________ to find the quadratic factor.
  - Factorise the quadratic.
- Rational Root Theorem **narrows down** the possible roots. If the roots are rational numbers, it must be that any.

$$\textbf{Potential root} = \pm\frac{\textbf{Factors of } \_\_\_\_\_\_\_\_\_\_ \; a_0}{\textbf{Factors of } \_\_\_\_\_\_\_\_\_\_ \; a_n}$$

- Sum and difference of cubes:

$$a^3 + b^3 = (\_\_\_\_\_\_\_\_\_\_)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (\_\_\_\_\_\_\_\_\_\_)(a^2 + ab + b^2)$$

Learning Objective: [1.5.4] - Graph factored and unfactored polynomials.

Key Takeaways

- Graphs of $a(x-h)^n + k$, where n is an Odd Positive Integer that is not equal to 1:
  - The point (h, k) gives us the stationary point of ______________.
- Graphs of $a(x-h)^n + k$, where n is an Even Positive Integer:
  - The point (h, k) gives us the ____________________.
  - These graphs look like a __________________.
- Steps to graphing factorised polynomials:
  1. Plot x-intercepts.
  2. Determine whether the polynomial is positive or negative.
  3. Use the repeated factors to deduce the shape:

     Non-Repeated: Only ___________________.

     Even Repeated: x-intercept and a ________________________.

     Odd Repeated: x-intercept and a _____________________________________.

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