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VCE Mathematical Methods ½
Quadratics [1.3]
Workbook

Outline:

Learning Objectives:

- ☐ MM12 [1.3.1] - Find factorised form of quadratics.
- ☐ MM12 [1.3.2] - Find solutions and the number of solutions to quadratic equations.
- ☐ MM12 [1.3.3] - Graph and find rules from the graph of quadratic equations.
- ☐ MM12 [1.3.4] - Solving Quadratic Inequalities and hidden quadratics.

Section A: Basics of Quadratics

Sub-Section: Factorising Quadratics

Let's quickly revise how we factorised quadratics!

Context: Factorising

- Reversing the process of algebraic expansion is known as factorisation.
- When we write a quadratic as the product of two linear terms, we say we factor the quadratic.
  - What does factorising allow us to do?

→ Graph (x-int)

→ Solving

Factorising Quadratics

$$y = (x - a)(x - b)$$

- Steps:

1. ~~Divide by the coefficient of the leading term. (If applicable)~~
2. Consider the factors of the constant term.
3. (If Positive Constant Term): See which pair of factors can add up to the coefficient of the x term.

   (If Negative Constant Term): See which pair of factors can subtract from the coefficient of the x term.
4. Construct the linear factors.

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Question 1 Walkthrough.

Factorise $x^2 + 3x - 10$.

$x^2 + 5x - 2x - 10$

$= x(x+5) - 2(x+5)$

$= (x-2)(x+5)$

$$\begin{array}{ccc} 1 & 5 & 5 \\ 1 & -2 & -2 \\ \hline & & 3 \end{array}$$

Active Recall

- **Steps:**

1. ________________ by the coefficient of the leading term. (If applicable)

2. Consider the factors of the ___________________________.

3. (If Positive Constant Term): See which pair of factors can add up to the coefficient of the x term.

   (If Negative Constant Term): See which pair of factors can subtract from the coefficient of the x term.

4. Construct the ______________________.

Question 2

Factorise the following expressions:

a. $x^2 + 4x + 3$

$(x+3)(x+1)$

b. $x^2 - 7x + 12$

$(x-4)(x-3)$

Question 3 Extension.

$3k \quad k$
$-3k \quad -k$

Express $x^2 - 4kx + 3k^2$ as the product of two linear factors.

$1 \quad -3k$
$1 \quad -k$
$-4k$

$(x-3k)(x-k)$

Question 4 Walkthrough.

$3 \quad -1$
$-3 \quad 1$

Factorise $2x^2 + 5x - 3$.

$2 \quad -1 \quad -1$
$1 \quad 3 \quad 6$
5

$(2x-1)(x+3)$

Question 5

Factorise the following expressions:

a. $-x^2 + 5x - 6$

$= -(x^2 - 5x + 6)$

$= -(x-2)(x-3)$

b. $-6x^2 + x + 1$

$= -(6x^2 - x - 1)$

$= -(2x-1)(3x+1)$

Question 6 Extension.

Write the expression $-6x^2 + kx + 2k^2$ as the product of two linear factors.

Sub-Section: Perfect Squares

Let's quickly revise perfect squares!

Perfect Squares

$(a+b)(a+b)$

$(a+b)^2 =$ $a^2+2ab+b^2$

$(a-b)^2 =$ $a^2-2ab+b^2$

- Perfect squares are special quadratic expressions that are made up of two **identical** linear factors.
- In other words, when a linear factor is squared, it becomes a perfect square.

Question 7

Factorise the following expression using the perfect square formula.

$$x^2 - 14x + 49$$

$(x-7)^2$

Question 8

Factorise the following expressions using the perfect square formula.

a. $x^2 + 4x + 4$

$(x+2)^2$

$(3x)^2$ 2^2

b. $9x^2 - 12x + 4$

$(3x - 2)^2$

Question 9 Extension.

Express $4x^2 - 20kx + 25k^2$ as a perfect square.

TIP: Identify the value of a and b using the form above.

Sub-Section: Difference of Squares

Let's quickly revise the difference between squares!

DOPS difference of perfect square

DOTS two

Difference of Squares

$$a^2 - b^2 = (a+b)(a-b)$$

Question 10 Walkthrough.

Factorise $x^2 - 4$.

$x^2 - 2^2$

$= (x+2)(x-2)$

Question 11

Factorise the following expressions:

a. $x^2 - 25$

$(x+5)(x-5)$

$(2x)^2$

b. $4x^2 - 9$

$(2x+3)(2x-3)$

Question 12 Extension.

Factorise $3x^2 - 25$.

$(\sqrt{3}x)^2$

$(\sqrt{3}x-5)(\sqrt{3}x+5)$

Sub-Section: Completing the Square

↳ TP form

Let's quickly revise completing the square!

Completing the Square

- When we complete the square of a quadratic $x^2 + bx + c$, we write it in the form:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

$$= x^2 + bx + \left(\frac{b}{2}\right)^2$$

- Steps:
  1. We halve the coefficient of x.
  2. Subtract the half of the coefficient of x squared outside the square bracket.

Question 13 Walkthrough.

Complete the square for $x^2 - 4x + 7$.

$$(x-2)^2 - (-2)^2 + 7$$

$$= (x-2)^2 - 4 + 7$$

$$= (x-2)^2 + 3$$

Active Recall: Complete the Square Steps

$$2x^2 + 4x + 7$$

$$= 2\left(x^2 + 2x + \frac{7}{2}\right)$$

- Steps:
  1. We halve the coefficient of x.
  2. Subtract the $\left(\frac{b}{2}\right)^2$ outside the square bracket.

Question 14

Complete the square for each quadratic.

a. $x^2 - 6x + 1$

$(x-3)^2 - (-3)^2 + 1$

$= (x-3)^2 - 9 + 1$

$= (x-3)^2 - 8$

b. $2x^2 + 20x + 3$

$= 2\left(x^2 + 10x + \frac{3}{2}\right)$

$= 2\left((x+5)^2 - (5)^2 + \frac{3}{2}\right)$

$= 2\left((x+5)^2 - 25 + \frac{3}{2}\right)$

$= 2\left((x+5)^2 + \frac{-50}{2} + \frac{3}{2}\right)$

$= 2\left((x+5)^2 - \frac{47}{2}\right)$

$= 2(x+5)^2 - 47$

Question 15 Extension.

Complete the square for $2x^2 - 4kx + 2k^2 + 3$.

Key Takeaways

- ☑ A perfect square is in the form of $x^2 + 2ax + a^2 = (x+a)^2$.
- ☑ The difference in squares is in the form of $a^2 - b^2 = (a-b)(a+b)$.
- ☑ Complete the square form of $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$.

Section B: Quadratic Equations

Sub-Section: Solving by Factorisation

Solving by Factorisation

$x-a=0$ OR $x-b=0$

$$(x-a)(x-b)=0$$

$$x = a \text{ and } b$$

- **Steps:**
  1. Factorise the quadratic.
  2. Equate each factor to 0 and solve for x.

Question 16

Solve each of the following quadratic equations for x:

a. $x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 1, 4$

b. $2x^2 - 5x - 7 = 0$

$(2x-7)(x+1) = 0$

$x = \frac{7}{2}, -1$

$2 \quad -7$

$1 \quad 1$

-7

2

-5

Sub-Section: Quadratic Formula

Discussion: What do we do if the quadratic is not easy to factorise?

The Quadratic Formula

$$\text{for } ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Where does this come from?

Extension: Derivation of Quadratic Formula

- The quadratic formula can be algebraically derived from attempting to complete the square for the general form of a quadratic equation $ax^2 + bx + c = 0$.
  - Since $a \neq 0$, we can divide the whole equation by a to make the coefficient of x^2 equal to 1:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

  - Now, complete the square!

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = 0$$

  - Rearrange the constant term to the other side!

$$\left(x + \frac{b}{2a}\right)^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

  - Expand the RHS!

$$\left(x + \frac{b}{2a}\right)^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

  - Now, solve for x and see what happens!

NOTE: That was the ultimate satisfaction.

Question 17

Solve each of the following quadratic equations for x.

a. $x^2 - 3x - 7 = 0$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-7)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{9 + 28}}{2}$$

$$= \frac{3 \pm \sqrt{37}}{2}$$

b. $2x^2 + 3x - 1$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{9 + 8}}{4}$$

$$= \frac{-3 \pm \sqrt{17}}{4}$$

Sub-Section: Discriminant

↳ No. of solution

Active Recall: Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discussion: What happens if the inside of the root is negative/zero/positive?

$\pm\sqrt{0} \Rightarrow$ one solution

$\pm\sqrt{-ve} \Rightarrow$ No solutions

$\pm\sqrt{+ve} \Rightarrow$ two solution

"normal"

The Discriminant

- **Definition:**
  - The discriminant, often denoted by **Δ** (Delta), is the part **inside** the square root of the quadratic formula.

$$\textit{discriminant} = \Delta = b^2 - 4ac$$

if $\Delta > 0$, there are 2 solutions.

if $\Delta = 0$, there is 1 solution.

if $\Delta < 0$, there are No solutions.

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Question 18

Determine how many unique roots exist in each of the following quadratic equations:

different solutions

$$x^2 - 4x + 20 = 0$$

$\Delta = b^2 - 4ac$

$= (-4)^2 - 4(1)(20)$

$= 16 - 80$

$= -64 < 0$

No solution

Question 19 Extension.

Find the value(s) of m that makes $3x^2 + 4x = 2m$ have no real solutions.

Key Takeaways

- We can solve quadratic equations by first factorising.
- Alternatively, we can use the quadratic formula given by $x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$.
- The discriminant is given by $b^2 - 4ac$ which dictates the number of solutions.

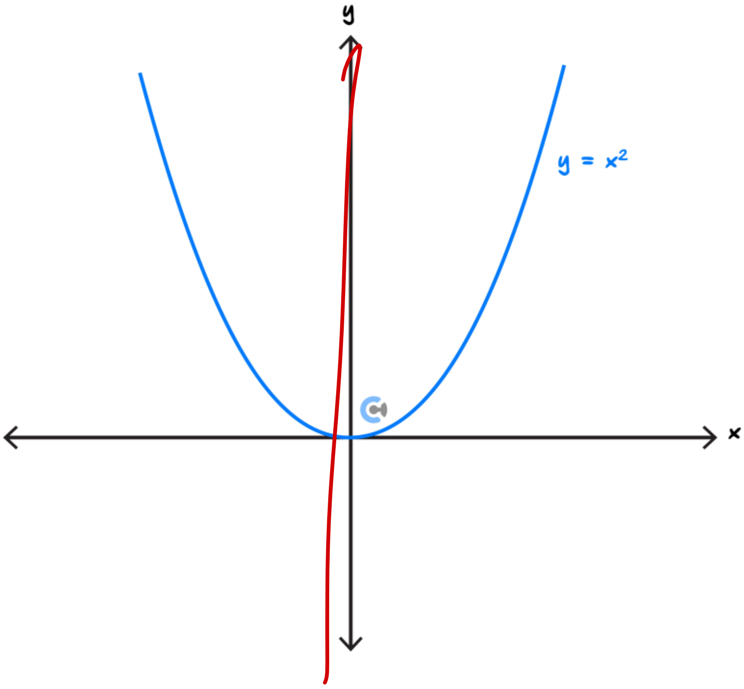
Section C: Graphs of Quadratic Equations

Sub-Section: Parabola and Symmetry

Parabola

- **Definition:**

The shape of the graph of a quadratic is known as a parabola.



Discussion: What is the parabola symmetrical to?

Axis of Symmetry ⇒ vertical line which includes the TP

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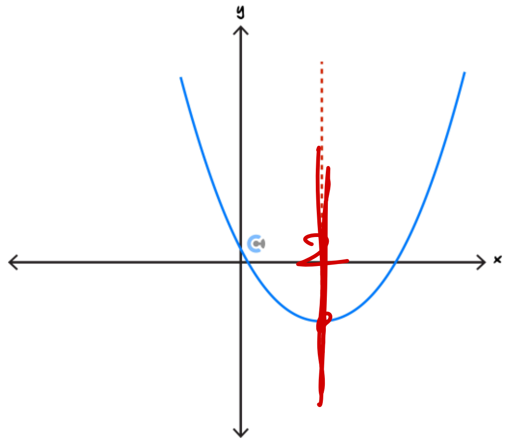
Definition

Axis of Symmetry



$$\text{axis of symmetry}: x = -\frac{b}{2a}$$

$x_{TP} = -\frac{b}{2a}$

Question 20

Find the axes of symmetry of each of the following quadratic hence, the coordinate of turning point.

$$y = 2x^2 - 3x + 5$$

$x = \frac{-(-3)}{2(2)} = \frac{3}{4}$

$x = \frac{3}{4}$

$y = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 5$

$= 2\left(\frac{9}{16}\right) - \frac{9}{4} + 5$

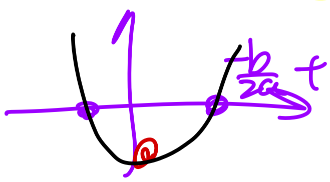
$= \frac{9}{8} - \frac{9}{4} + 5$

$= -\frac{9}{8} + 5 = \frac{31}{8}$

$\left(\frac{3}{4}, \frac{31}{8}\right)$

NOTE: When a question asks for coordinates, you must mention both the x- and y-value of the point.

Discussion: Given that $ax^2 + bx + c$ has an x-intercept of $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, what do you notice about the line of symmetry at $x = -\frac{b}{2a}$?



$-\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$

TP is halfway between any x-ints

Sub-Section: Graphing Quadratics

Turning Point Form

complete the square

- The turning point form of a quadratic is given by:

$$y = a(x - h)^2 + k$$

$$\textit{turning point} = \underline{(h, k)}$$

- The turning point form is obtained by **completing the square.**

$y = 2(x-3)^2 - 5$

TP: $(3, -5)$

Discussion: Can every quadratic be put into turning point form? (Does every quadratic have a turning point?)

Yes → Complete the square

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Question 21 Walkthrough.

Sketch the graph of the following quadratic equation, labelling the coordinates of all axes intercepts and the turning point.

$$y = (x+1)^2 - 1$$

TP $(-1,-1)$

x-int

$y=0$

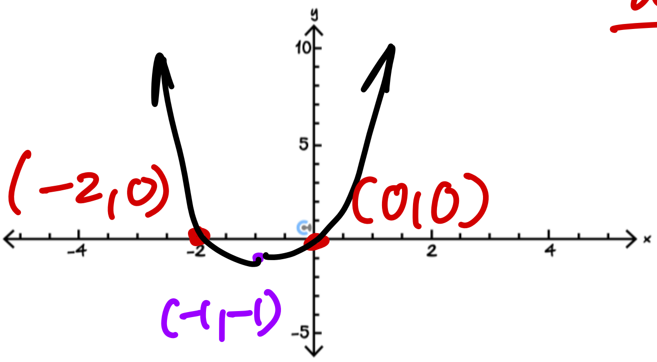
$0=(x+1)^2-1$

$1=(x+1)^2$

$x+1=\pm 1$

$x=-1\pm 1$

$x=-2, 0$



y-int

$y=(1)^2-1=0$

Question 22

Sketch the graph of the following quadratic equation, labelling the coordinates of all axes intercepts and the turning point.

$\left(-\frac{3}{2}, -4\right)$

$$y = -\left(x+\frac{3}{2}\right)^2 - 4$$

y-int

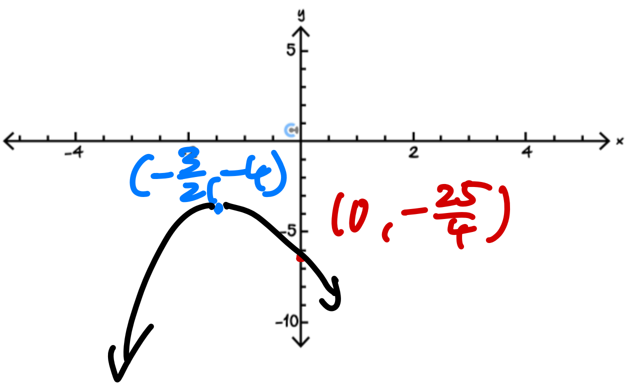
$x=0$

$y=-\left(\frac{3}{2}\right)^2-4$

$y=-\frac{9}{4}-4$

$y=\frac{-9-16}{4}$

$y=-\frac{25}{4}$

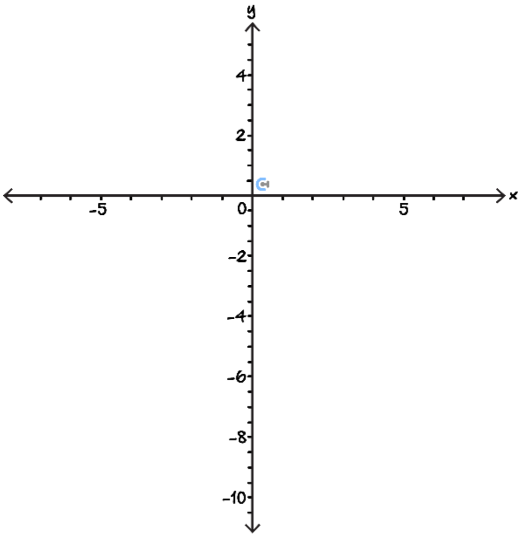


Discussion: What would the discriminant of $-\left(x+\frac{3}{2}\right)^2 - 4 = 0$ equation be? (Graph from part b. above.)

Question 23 Extension.

Sketch the graph of the following quadratic equation, labelling the coordinates of all axes intercepts and the turning point. Begin by putting the equation into turning point form.

$$y = -x^2 + 6x - 5$$



Intercept Form

TP: halfway between x-ints

- The x-intercept form of a quadratic is given by:

$$y = a(x-b)(x-c)$$

$$x\text{-}intercepts: (b,0) \text{ and } (c,0)$$

- The axis of symmetry is located exactly in the middle of the two x-intercepts.

Question 24 Walkthrough.

Sketch the graph of the following quadratic equation, labelling the coordinates of all axes intercepts, as well as the turning point.

$$y = x^2 - 3x + 2$$

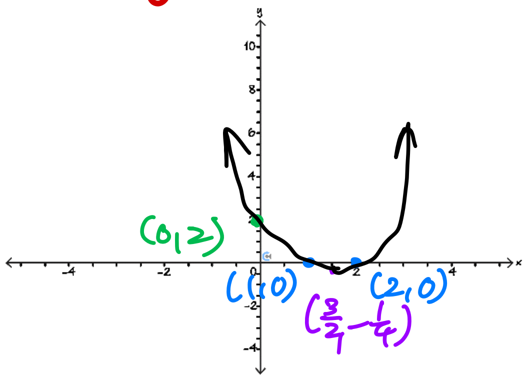
$y = (x-2)(x-1)$

y-int
$x = 0$
$\Rightarrow y = 2$

x-int
$y = 0$
$x = 1, 2$

$x_{TP} = \frac{1+2}{2} = \frac{3}{2}$

$y_{TP} = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 2$
$= \frac{9}{4} - \frac{9}{2} + 2$
$= -\frac{9}{4} + 2$
$= -\frac{1}{4}$



Question 25

Sketch the graph of the following quadratic equation, labelling the coordinates of all axes intercepts, as well as the turning point.

$$y = -x^2 + 4x + 5$$

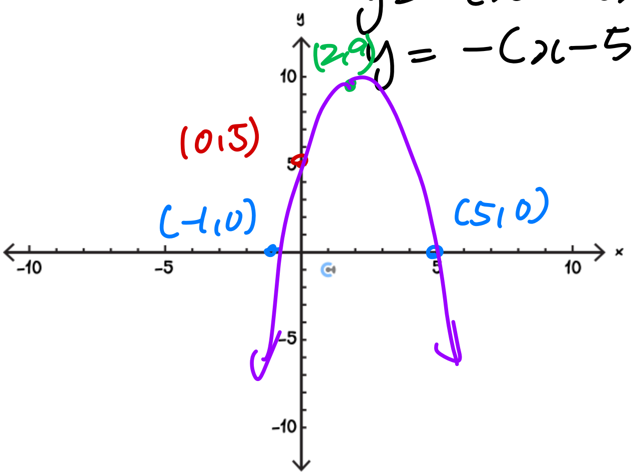
y-int
$y = 0 + 0 + 5 = 5$

$y = -(x^2 - 4x - 5)$
$y = -(x-5)(x+1)$

x-int
$x = -1, 5$

$x_{TP} = \frac{5-1}{2} = 2$

$y_{TP} = -2^2 + 4(2) + 5$
$= -4 + 8 + 5$
$= 9$

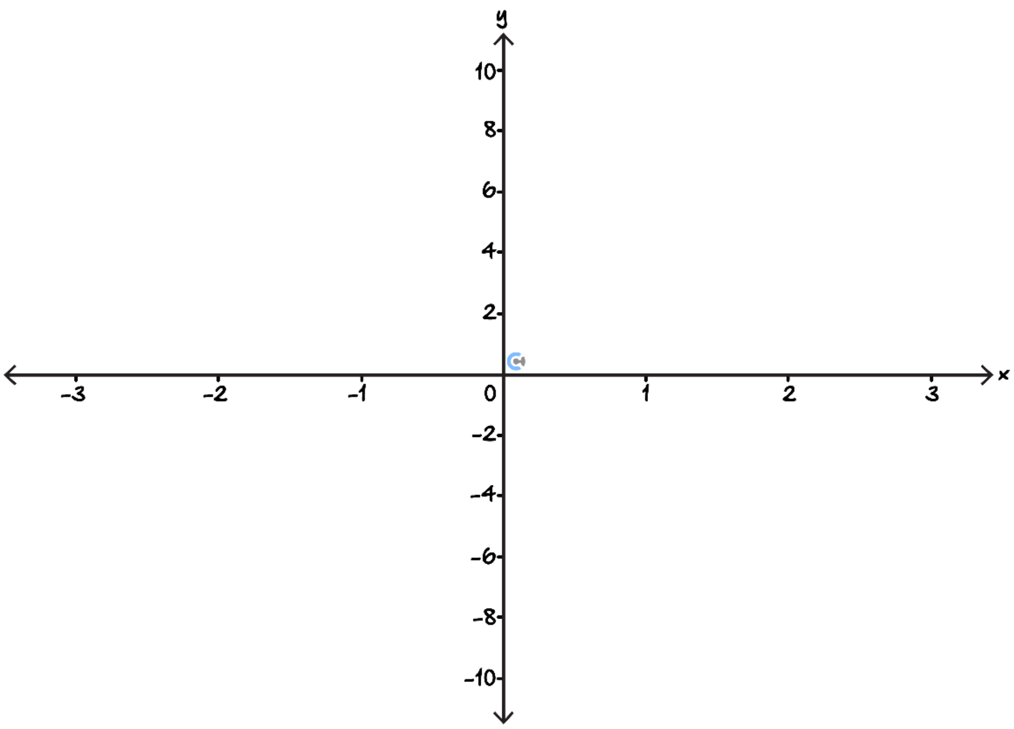


NOTE: When a is negative, the x-intercepts stay the same, but the **shape** of the parabola becomes a **negative** parabola instead.

Question 26 Extension.

Sketch the graph of the following quadratic equation, labelling the coordinates of all axes intercepts, as well as the turning point.

$$y = 4x^2 - 5x - 6$$



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Sub-Section: Finding a Rule of a Quadratic From a Graph

Let's try to do it the other way around!

Finding the Equation of a Quadratic

- **Form 1: Turning Point Form**

$$y = a(x-h)^2 + k$$ ← TP

  - Recommended when a turning point is easy to identify.

- **Form 2: x-intercept Form**

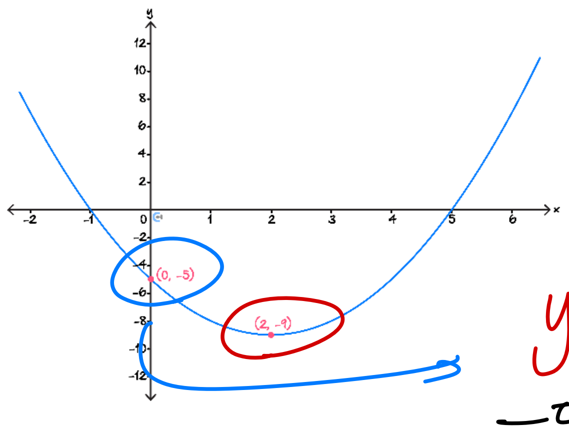
$$y = a(x-b)(x-c)$$ ← x-int

  - Recommended when both x-intercepts are easy to identify.

Question 27 Walkthrough.

Find the equations of the quadratics graphed below. Show your working.

a. Find the equation in turning point form.



$y = a(x-2)^2 - 9$

$-5 = a(-2)^2 - 9$

$-5 = 4a - 9$

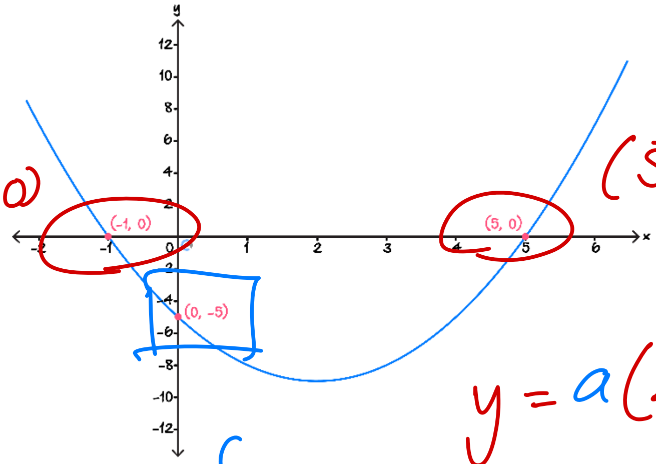
$4a = 4$

$a = 1$

$y = (x-2)^2 - 9$

b. Find the equation in intercept form.

$(-1, 0)$ $(5, 0)$



$y = a(x+1)(x-5)$

$-5 = a(0+1)(0-5)$

$-5 = -5a$

$a = 1$

$y = (x+1)(x-5)$

Question 28

Find the equations of the quadratics graphed below. Show your working.

a.

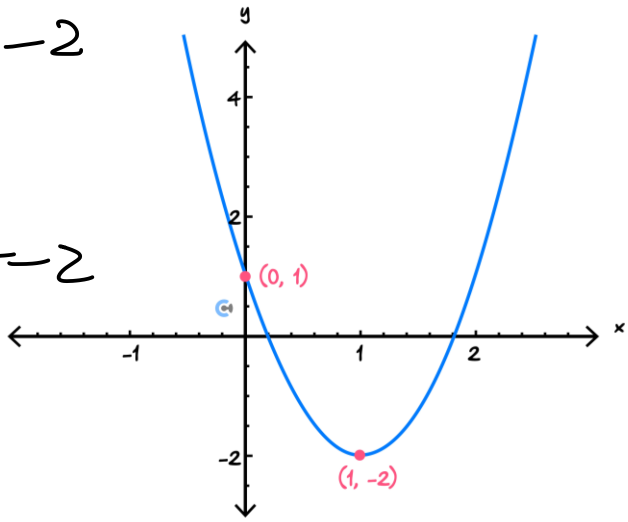
$y = a(x-1)^2 - 2$

$(0, 1)$

$1 = a(0-1)^2 - 2$

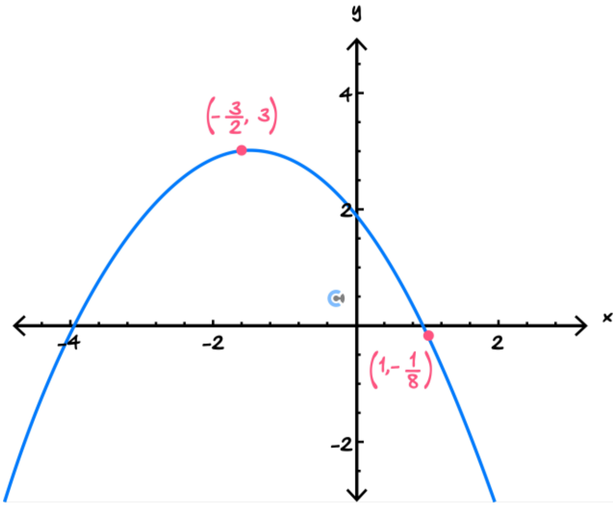
$1 = a - 2$

$a = 3$



$y = 3(x-1)^2 - 2$

b.



$y = a(x+\frac{3}{2})^2+3$

$(1, -\frac{1}{8})$

$-\frac{1}{8} = a(1+\frac{3}{2})^2+3$

$-\frac{1}{8} = a(\frac{5}{2})^2+3$

$-\frac{25}{8} = \frac{25}{4}a$

$a = -\frac{25}{8} \div \frac{25}{4}$

$= -\frac{25}{8} \times \frac{4}{25}$

$= -\frac{4}{8}$

$= -\frac{1}{2}$

NOTE: Never forget the a coefficient!

$y = -\frac{1}{2}(x+\frac{3}{2})^2+3$

Key Takeaways

- Every quadratic can be put into the turning point given by $y = a(x-h)^2 + k$.
- Not all quadratic can be put into the x-intercept form given by $y = a(x-b)(x-c)$.
- We can use x-intercept form or turning point form to find the rule.

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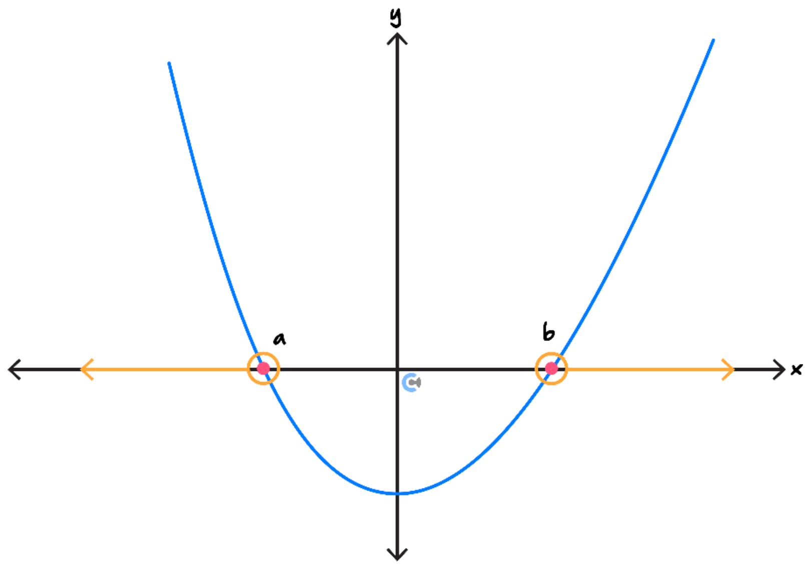
Section D: Advanced Algebra of Quadratics

Sub-Section: Quadratic Inequalities

How can we tackle quadratic inequalities?

$5 - x > 2$

Quadratic Inequalities



- For quadratic inequalities, we always Sketch the function.
- **Steps:** → Just find x-int
  1. Sketch the function.
  2. See where the y-value is within the inequality.
  3. Find the corresponding x-values.

Discussion: Why do we look at y-value < 1 if the function < 1?

Question 29 Walkthrough.

Solve $x^2 - 1 > 0$ for x.

Hint: This is the same question as asking, "For what values of x is the graph of $y = x^2 - 1$ greater than 0?"

$y = x^2 - 1$

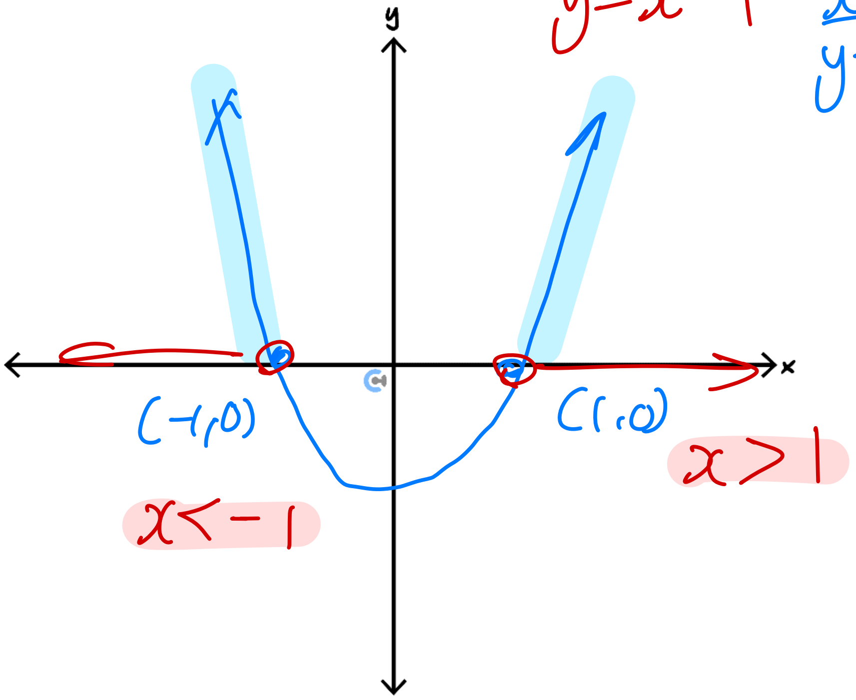
$\underline{x\text{-int?}}$

$y = 0$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$



$$\boxed{x < -1 \text{ or } x > 1}$$

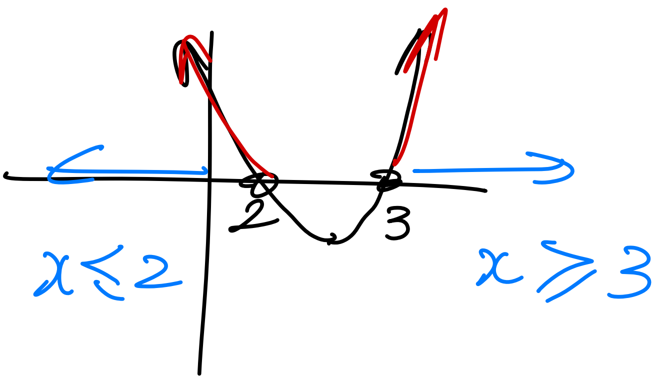
Active Recall: Quadratic Inequalities

1. Sketch the function.
2. See where the y value is within the inequality.
3. Find the corresponding x values.

Question 30

Solve each of the following for x:

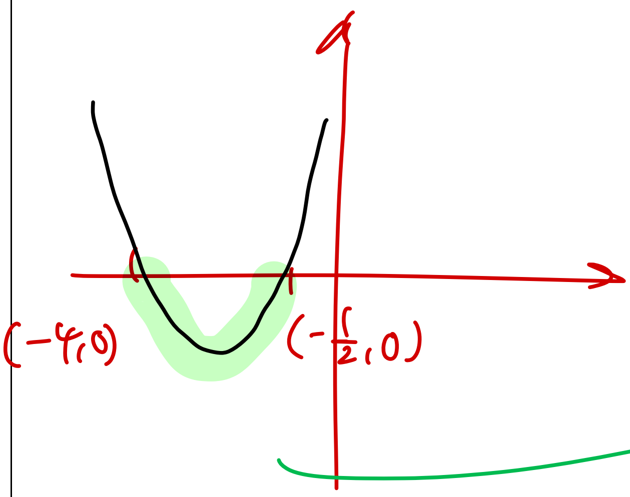
a. $(x-2)(x-3) \geq 0$



$\Rightarrow x \leq 2$ or $x \geq 3$

b. $2x^2 + 9x + 4 < 0$

$(2x+1)(x+4)$ $\quad x = -\frac{1}{2}, -4$



$\boxed{-4 < x < -\frac{1}{2}}$

Question 31 Extension.

Solve $kx^2 + 4x - 4 > 0$ for x, where $k \in R \setminus \{0\}$. Give two answers depending on whether $k > 0$ or $k < 0$.

Sub-Section: Hidden Quadratics

Let's take a look at hidden quadratics!

Hidden Quadratics

- Instead of:

$$af(x)^2 + bf(x) + c = 0$$

- We can let $f(x) = X$ to have:

$$aX^2 + bX + c = 0$$

Question 32 Walkthrough.

Solve $x^4 - 13x^2 + 36 = 0$ for x.

$(x^2)^2$

$A = x^2$

$A^2 - 13A + 36 = 0$

$(A-9)(A-4) = 0$

$A = 9, 4$

$x^2 = 9 \quad x^2 = 4$

$x = \pm 3, \pm 2$

Question 33

a. Solve $(x-2)^2 - 7(x-2) + 12 = 0$ for x.

$A = x - 2$

$A^2 - 7A + 12 = 0$

$(A-3)(A-4) = 0$

$A = 3, 4$

$x - 2 = 3 \quad x - 2 = 4$

$x = 5, 6$

$(\sqrt{x})^2$

b. Solve $x + 2\sqrt{x} - 15 = 0$ for x.

$A = \sqrt{x}$

$A^2 - 2A - 15 = 0$

$(A-5)(A+3) = 0$

$A = -3, 5$

$\sqrt{x} = -3$ ✗ $\quad \sqrt{x} = 5$

$\Rightarrow$ Reject $\quad x = 25$

$\sqrt{9} = 3$

$x^2 = 9 \Rightarrow x = \pm 3$

Question 34 Extension.

Solve $x^4 - 4x^2 - k = 0$ for x, where k is a real number.

Key Takeaways

- For quadratic inequalities, we always sketch.
- For hidden quadratics, look for the pattern of something and something squared.

Contour Check

Learning Objective: [1.1.1] – Find factorised form of quadratics

Key Takeaways

- Perfect square is in the form of $(x+a)^2 = x^2 + 2ax + a^2$.
- Differences of squares are in the form of $a^2 - b^2$ $= (a-b)(a+b)$.
- Complete the square form of $x^2 + bx + c =$ $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$.

Learning Objective: [1.1.2] – Find solutions and number of solutions to quadratic equations

Key Takeaways

- We can solve for quadratic equations by first factorised.
- Alternatively, we can use the quadratic formula given by $x =$ $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$.
- The discriminant is given by $b^2 - 4ac$ which dictates the number of solutions.

Learning Objective: [1.1.3] – Graph and find rules from the graph of quadratic equations

Key Takeaways

- Every quadratic can be put into the turning point given by $y =$ $a(x-h)^2 + k$.
- Not all quadratic can be put into the x-intercept form given by $y =$ $a(x-b)(x-c)$.
- We can use x-intercept form or turning point form to find the rule.

Learning Objective: [1.1.4] – Solving Quadratic Inequalities and hidden quadratics

Key Takeaways

- For quadratic inequalities, we always Sketch.
- For hidden quadratics, look for the pattern of something and something Squared.

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