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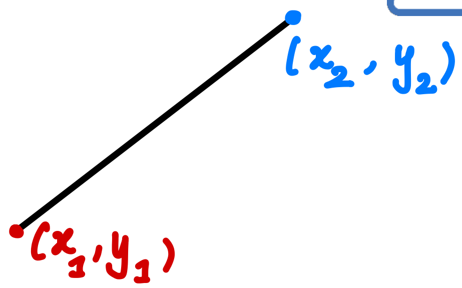
VCE Mathematical Methods ½
Linear & Coordinate Geometry [1.1]
Workbook

Outline:

Section A: Linear Functions and Graphs

Linear Equations

- **Definition:** Equations where the highest power of a variable is 1.
  - **Gradient-intercept form:**



$$y = mx + c$$

$$\text{where } m = \text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{and } c = \text{y-intercept}$$

- No singular solution for a linear equation in two variables.
  - All pairs of coordinates (x, y) that satisfy the equation lie on a **line.** (Hence, *linear* equations).

Space for Personal Notes

Let's have a look at sketching some of these equations.

Question 1 Walkthrough.

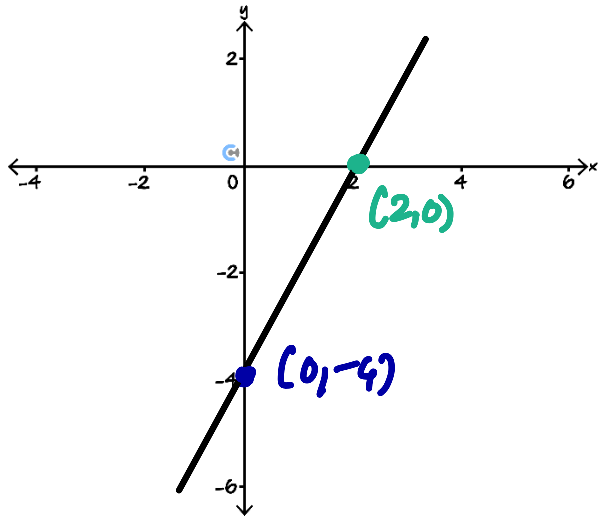
Sketch the graphs of each of the following linear relations, labelling all axes intercepts with their coordinates.

$y = 2x - 4$

Y-int: $x=0$

$y=-4$

$(0,-4)$



X-int: $y=0$

$2x-4=0$

$x=2$

$(2,0)$

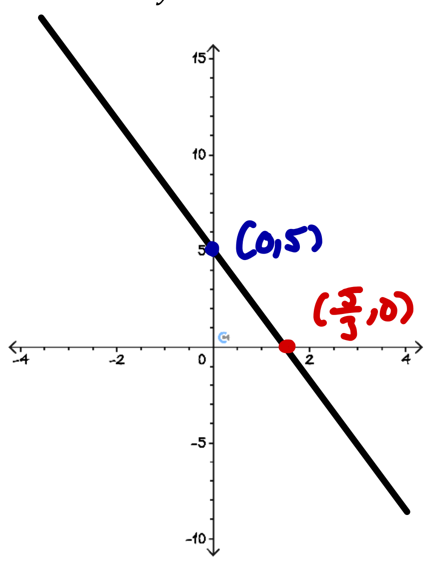
Question 2

Sketch the graphs of each of the following linear relations, labelling all axes intercepts with their coordinates.

$y = -3x + 5$

Y-int: $x=0$

$(0,5)$



X-int: $y=0$

$-3x+5=0$

$3x=5$

$x=\frac{5}{3}$

$(\frac{5}{3},0)$

Sub-Section: Inequality

Definition

Inequalities Rule

$$x > \frac{b}{a}, \text{where } a < 0$$

- Multiplying both sides by a negative number flip the inequality sign.

Question 3

Solve each of the following for x:

$$9 - 6x \leq 2$$

Option 1: Rearrange

+6x +6x

$9 \leq 2 + 6x$

−2 −2

$7 \leq 6x$

÷6 ÷6

$\frac{7}{6} \leq x$

Inequalities Rule:

−9 −9

$-6x \leq -7$

÷−6 ÷−6

$x \geq \frac{-7}{-6}$

$\Downarrow$

$x \geq \frac{7}{6}$

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<u>Exploration</u>: **Graphs of Linear Inequalities**

- Instead of just representing a line on the Cartesian plane, the graph of a linear inequality represents an entire shaded region (x, y) which satisfies the inequality.
- **Step 1:** Put the equation into the form of $y = mx + c$
- **Step 2:** Sketch the linear **equation,** ignoring the inequality sign.
  - If the inequality is inclusive ($\leq$ *or* $\geq$), we draw a [solid] / [dotted] line.
  - If the inequality is exclusive ($<$ or $>$), we draw a [solid] / [dotted] line
- **Step 3:** Shade the region either **above** or **below** the line
  - If $y >$ or $y \geq$: above
  - If $y <$ or $y \leq$: below

① Find line (make y subject)

② Find intercepts (Make x=0 & y=0)

③ Sketch Above/Below Inclusive/Exclusive

Question 4 Walkthrough.

Sketch the linear inequality $y - 1 - x \geq 0$ on the axes below.

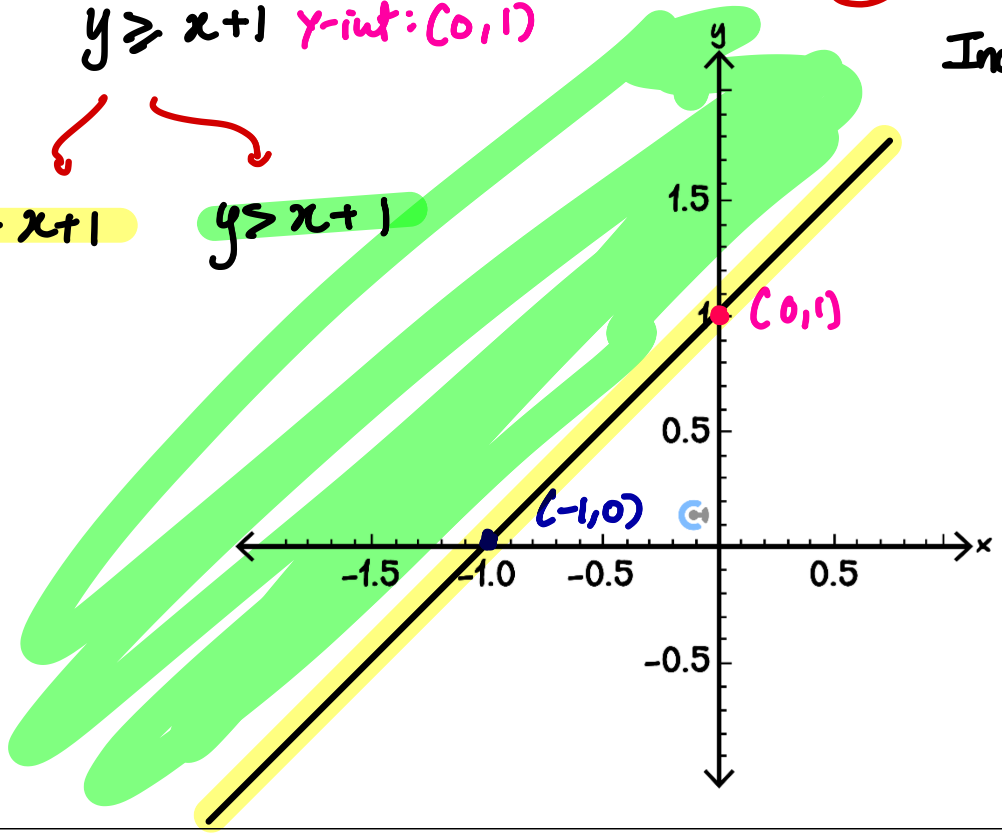
$y = mx + c$

① $y \geq x + 1$

x-int: $(-1, 0)$

y-int: $(0, 1)$

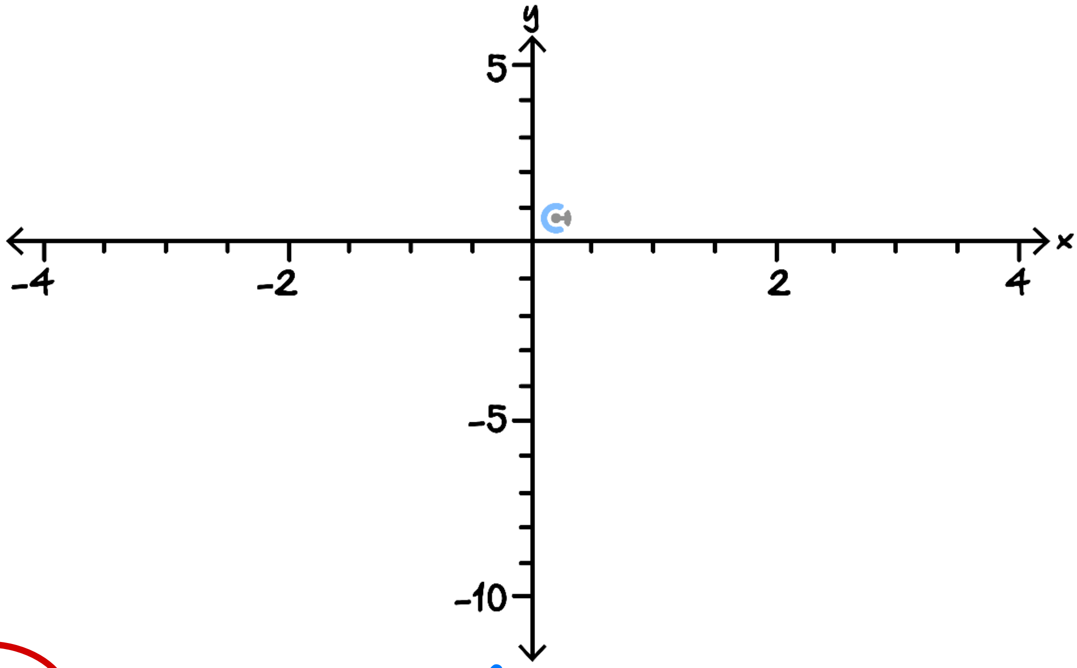
$y = x + 1$ $\quad$ $y > x + 1$



Question 5

Sketch the graphs of each of the following linear inequalities, labelling all axes intercepts with their coordinates, and shading the appropriate regions.

a. $4x + 2y = -6$



45s

b. $5x - 2y - 8 \leq 0$

$2y \geq 5x - 8$

$y \geq \frac{5}{2}x - 4$

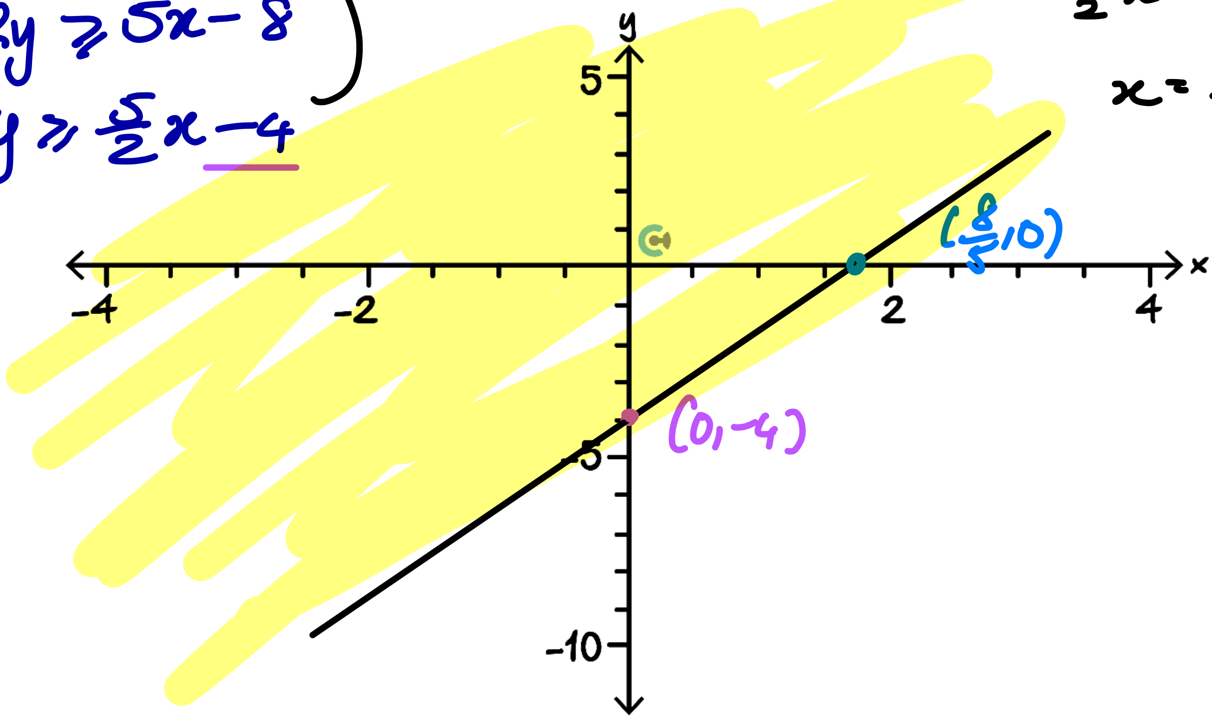
X-int: $(\frac{8}{5}, 0)$

Y-int: $(0, -4)$

$\frac{5}{2}x - 4 = 0$

$\frac{5}{2}x = 4$

$x = \frac{8}{5}$



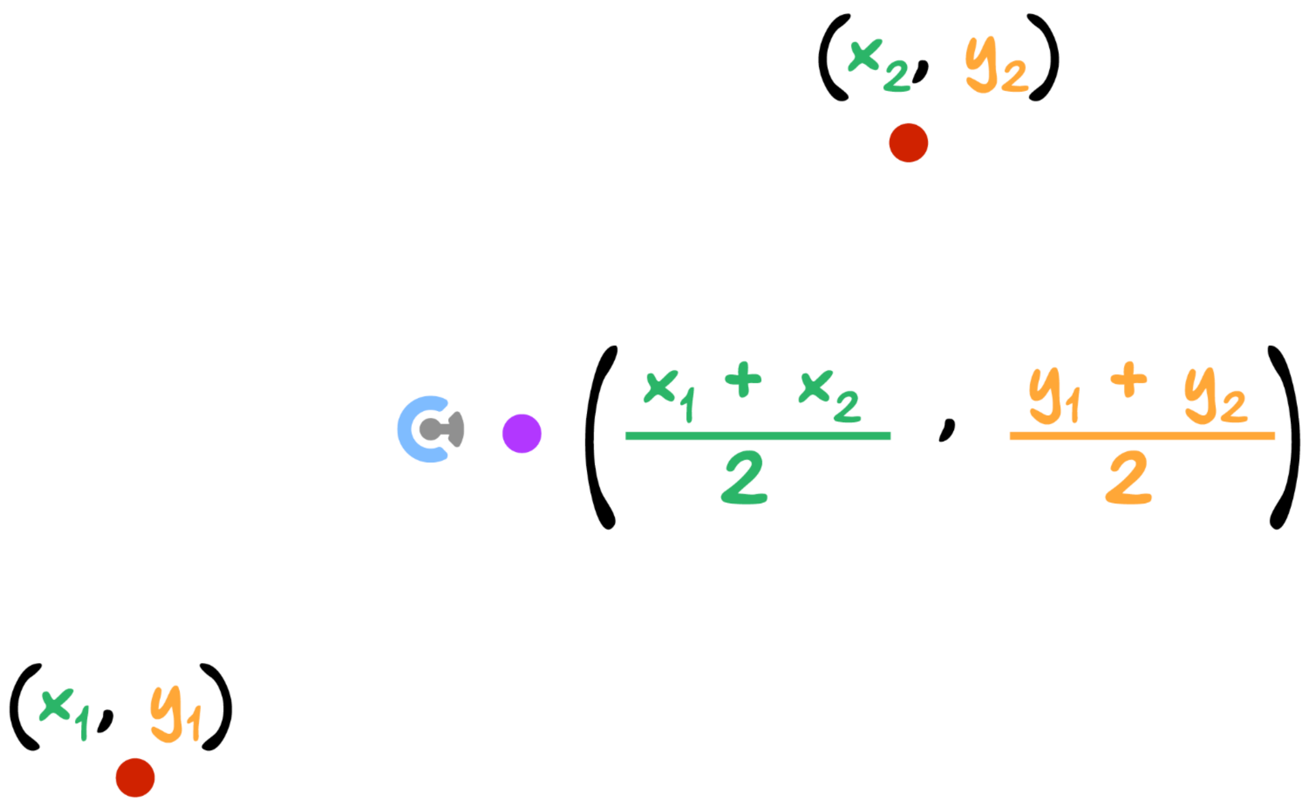
Section B: Midpoint and Distances

Sub-Section: Midpoint

Discussion: How might we find a midpoint between two points?

Taking the average!

Midpoint



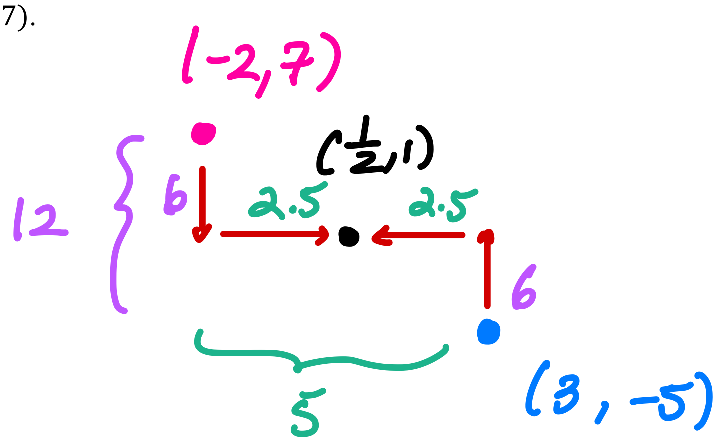
- **Definition:** The midpoint, M, of two points A and B is the point halfway between A and B.

$$M(x_m, y_m) = \left(\frac{x_A+x_B}{2}, \frac{y_A+y_B}{2}\right)$$

- The midpoint can be found by taking the average of the x-coordinate and y-coordinate of the two points.

Question 6 10s

Find the midpoint between $(3, -5)$ and $(-2, 7)$.



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Sub-Section: Distance Between Two Points

Distance Between Two Points

- **Definition:** The distance between two points (x_1, x_2) and (y_1, y_2) can be found using Pythagoras' theorem:

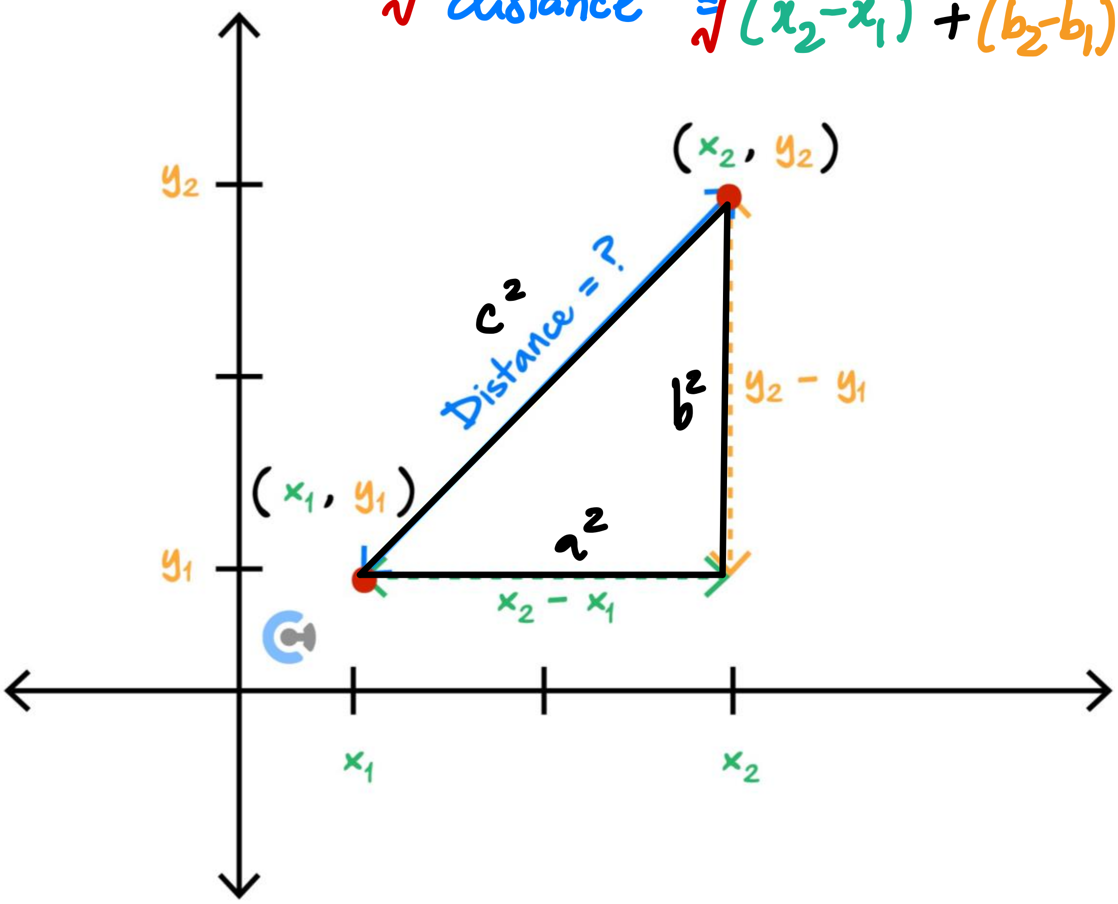
$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

How does this formula work?

$c^2 = a^2 + b^2$

Exploration

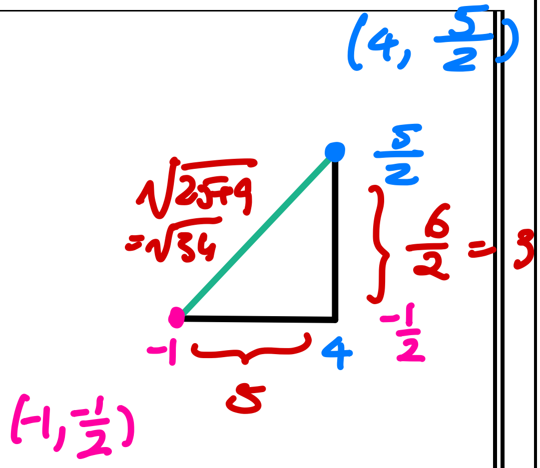
$\sqrt{\text{distance}^2} = \sqrt{(x_2 - x_1)^2 + (b_2 - b_1)^2}$



- Try to construct a Pythagoras' theorem with the three sides above ☺

Question 7

Find the distance between $\left(4, \frac{5}{2}\right)$ and $\left(-1, -\frac{1}{2}\right)$.



$$\text{distance} = \sqrt{\left(4-(-1)\right)^2 + \left(\frac{5}{2} - \left(-\frac{1}{2}\right)\right)^2}$$

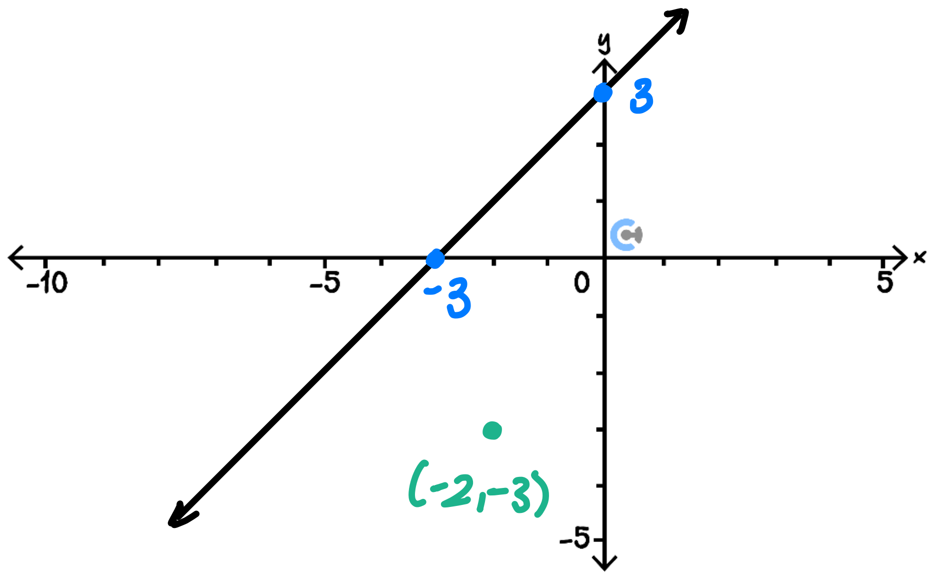
$$= \sqrt{(5)^2 + (3)^2}$$

$$= \sqrt{34} \text{ units}$$

~~**Question 8 Extension.**~~ CAS

a. Find a point(s) on the line $y = x + 3$ which has a distance of 4 from the point $(-2, -3)$.

b. Give a reason as to why there is more than 1 more point found in **part a.**



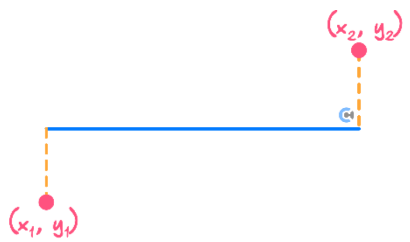
TIP: Don't hesitate to define a point by letting its y-value be the function (linear in the above question!)

Sub-Section: Vertical Distance VS Horizontal Distance

Discussion: How can we find a horizontal distance between two points?

Change in x.

Horizontal Distance



$$\text{Horizontal Distance} = x_2 - x_1 \text{ where } x_2 > x_1$$

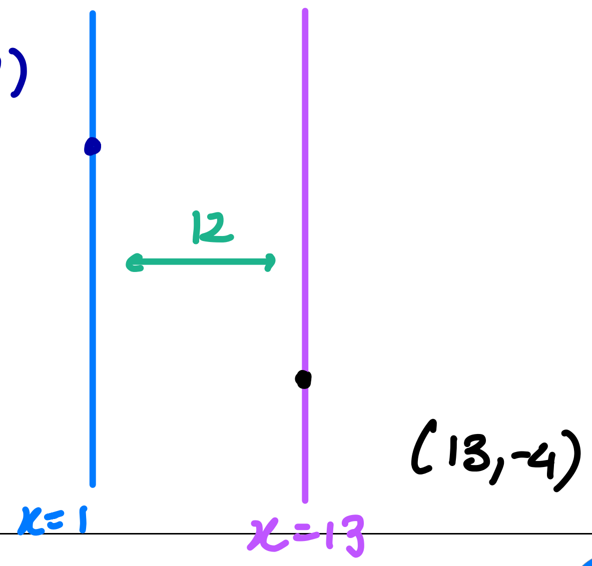
- Find the difference between their x-values.

Question 9

Find the horizontal distance between the two points $(1,9)$ and $(13,-4)$.

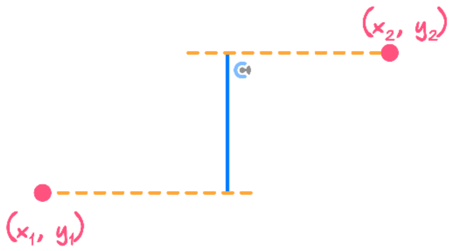
$\cancel{-12?}$

distance $\neq$ $-$ve



What about vertical distance then?

Vertical Distance

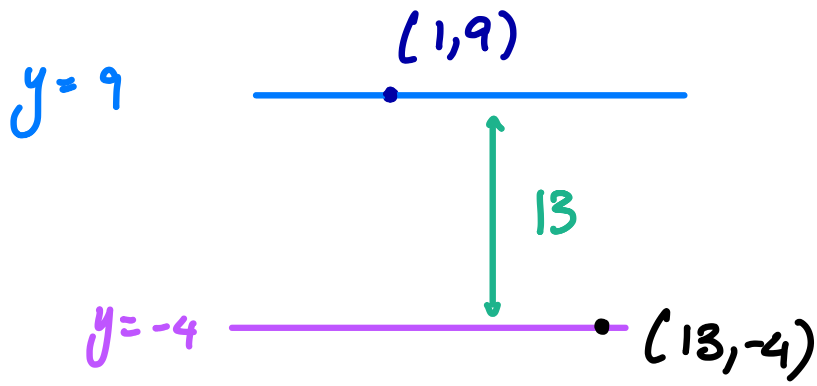


$$\text{Vertical Distance} = y_2 - y_1 \text{ where } y_2 > y_1$$

- Find the difference between their y values.

Question 10

Find the vertical distance between the two points $(1,9)$ and $(13,-4)$.



Key Takeaways

- A midpoint is simply an average point.
- The distance between two points is derived from Pythagoras' theorem.
- Horizontal distance is simply the difference in their x-values
- Vertical distance is simply the difference in their y-values.

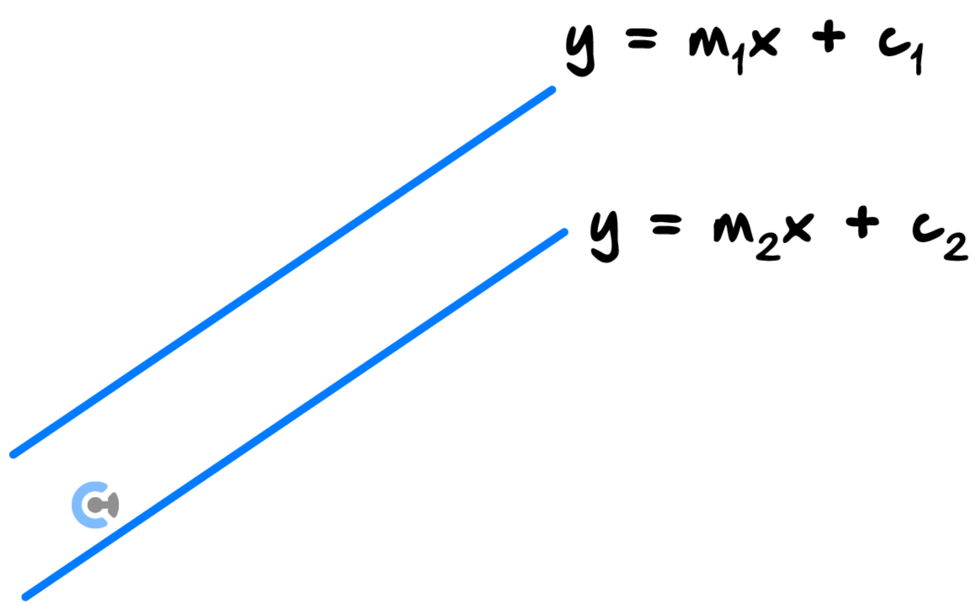
Section C: Line Geometry

Sub-Section: Parallel and Perpendicular Lines

Discussion: What do we need for the two lines to be parallel?

Same m (gradient)

Parallel Lines



- Parallel lines have the ___Same___ gradient.

$$m_1 = m_2$$

Space for Personal Notes

Question 11

Find a line which is parallel to $y = 2x - 1$ passing through the point $(-1,3)$.

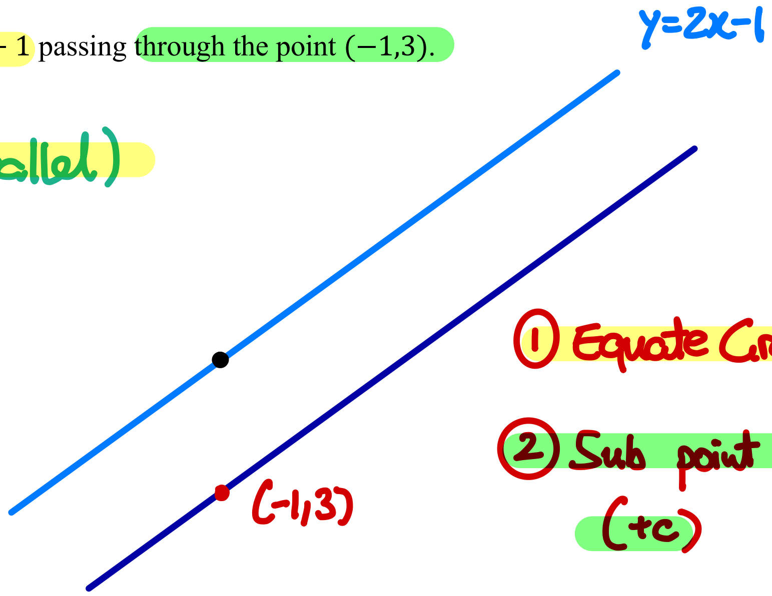
① $m = 2$ (parallel)

② Sub $(-1,3)$

$\therefore y = 2x + c$

$3 = 2(-1) + c$

$\therefore c = 5$ → $\therefore y = 2x + 5$



① Equate Gradient

② Sub point to solve (+c)

TIP: Try to ignore the constant term of the line we must be parallel to. Simply focus on its gradient.

They intersect at 90°

Discussion: What about perpendicular lines?

→ negative reciprocal of other gradient.

$m_1 \times m_2 = -1$

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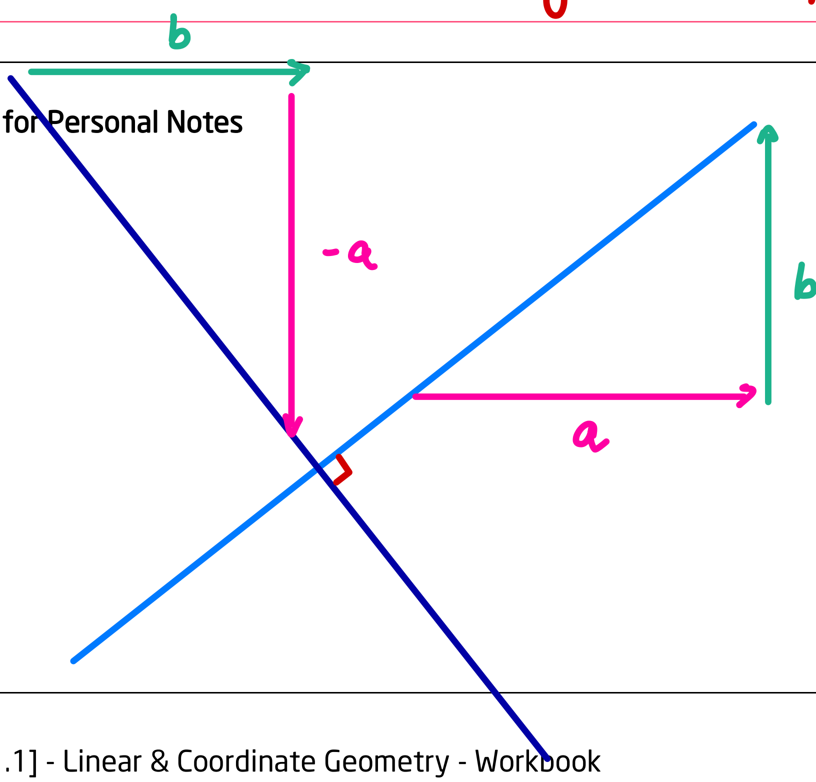


$m = \dfrac{b}{a}$

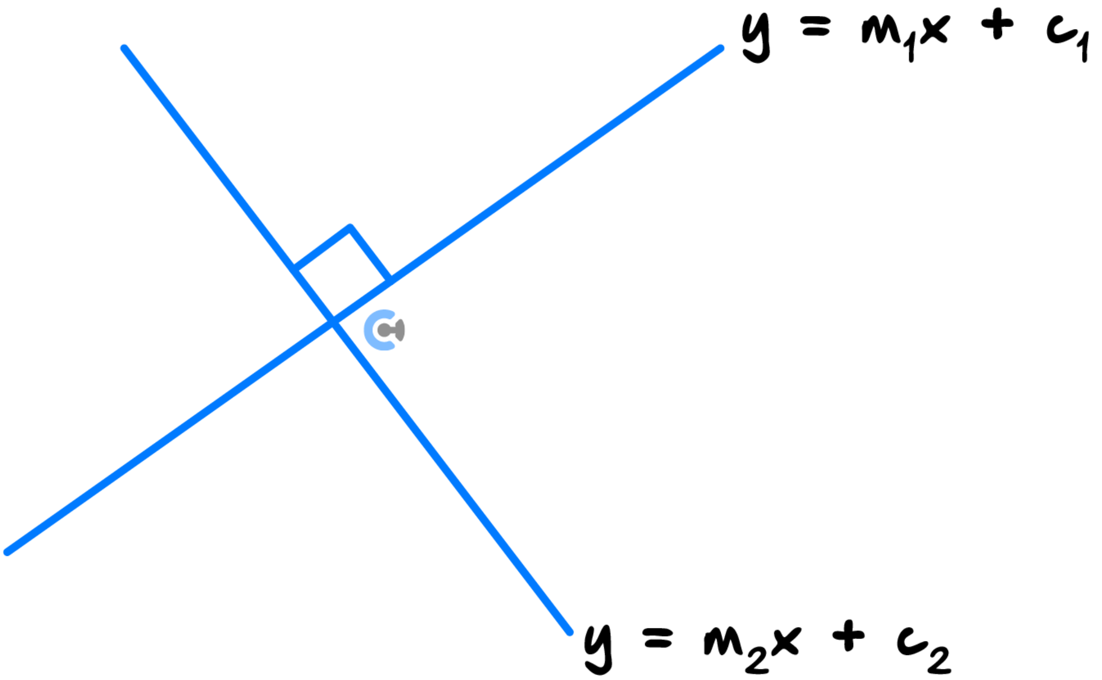
$m = \dfrac{-a}{b}$

$m \cdot m = \dfrac{b}{a} \cdot \dfrac{-a}{b}$

$= -1$

Definition

Perpendicular Lines



➤ A line which is perpendicular to another line has a gradient which is the negative reciprocal of the gradient of the first line.

Question 12

Find a line which is perpendicular to $y = -3x - 1$ passing through the point $(5, -1)$.

$m = \frac{-1}{-3} = \frac{1}{3}$

$\therefore y = \frac{1}{3}x + c$

Sub $(5, -1)$:

$-1 = \frac{1}{3}(5) + c$

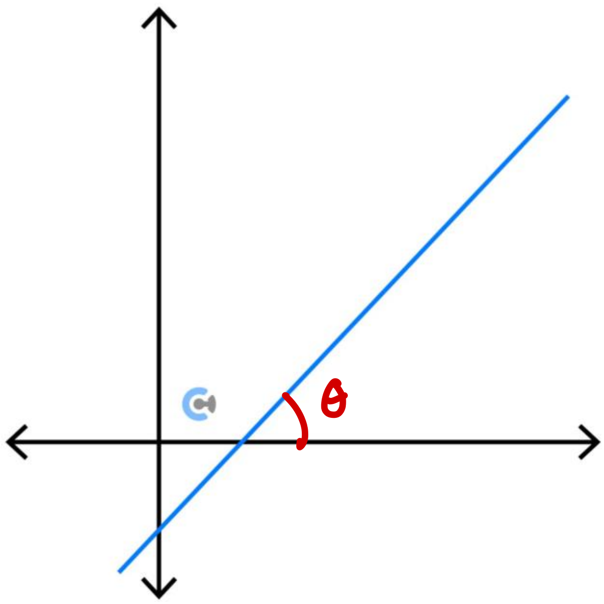
$\therefore c = \frac{-8}{3}$

$\therefore y = \frac{1}{3}x - \frac{8}{3}$

<u>Sub-Section</u>: Angle Between a Line and the x-axis

How do we find the angle between a line and the x-axis?

<u>Angle between a Line and the x-axis</u>



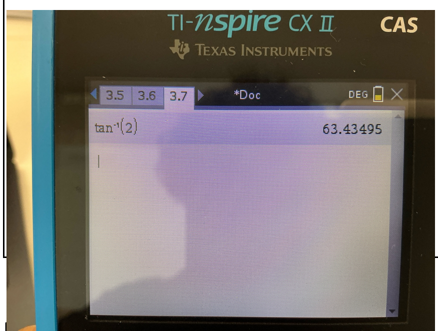
- The angle between a line and the ___positive___ **direction of the x-axis** (anticlockwise) is given by

$$\tan(\theta) = m$$

Question 13 Tech-Active. (CAS)

Find the angle made between the line $y = 2x - 6$ and the x-axis measured in the anticlockwise direction. Give your answer in degrees correct to two decimal places.

$\tan(\theta) = 2$



$\theta = \tan^{-1}(2)$

$= 63.43^{\circ}$

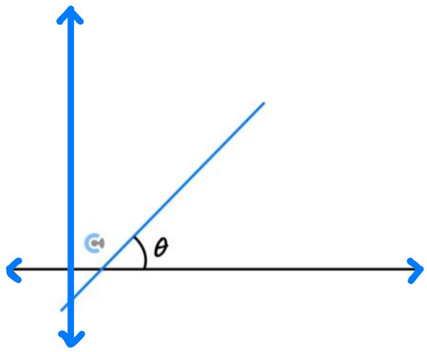
NOTE: Angles from the x-axis measured anticlockwise = positive angles.

- Don't worry about it too much, it's just convention! (More on this in circular functions.)

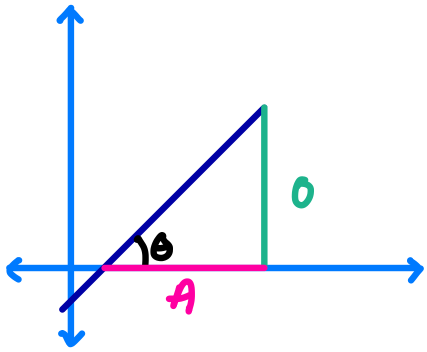
How does this formula work?

Exploration: Angle between a line and x-axis.

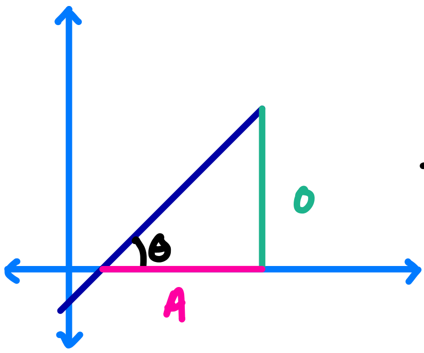
- **Consider a line in the visual below.**



- Construct a right-angle triangle with the angle θ.



- Consider the opposite and adjacent sides of the right-angle triangle. What can we call them?



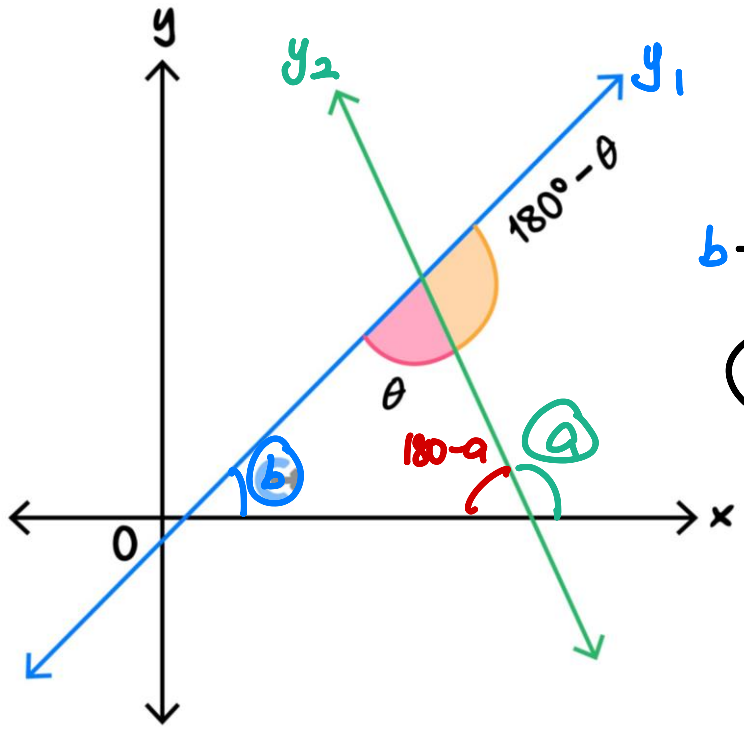
$$\tan\theta = \frac{O}{A} = \frac{\text{Rise}}{\text{Run}} = m$$

- Hence what does $\tan(\theta)$ equal to given that $\tan = \text{opposite}/\text{adjacent}$?

Sub-Section: Angle Between the Two Lines

Slightly more complicated now!
How about an angle between two lines?

Acute Angle Between Two Lines



$b + \theta + 180 - a = 180$

$\theta = a - b$

$$\theta = \left|\underbrace{\tan^{-1}(m_1)}_{b} - \underbrace{\tan^{-1}(m_2)}_{a}\right|$$

magnitude
abs value
" Makes inside +ve

- Alternatively:

$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$|3| = 3$

$|-2| = 2$

For your understanding, note that this formula is derived from the tan compound angle formula covered in SM12.

NOTE: $|x|$ just takes the positive value of x.

Question 14 Tech-Active. (CAS)

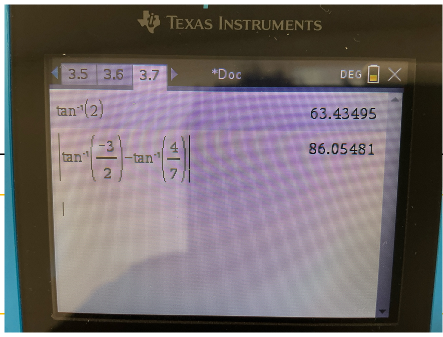
Find the acute angle between the lines $3x + 2y = 2$ and $y = \frac{4}{7}x + 1$. Give your answer in degrees correct to two decimal places.

$2y = -3x + 2$

$y = \frac{-3}{2}x + 1, \quad y = \frac{4}{7}x + 1$

$\theta = \left|\tan^{-1}\left(\frac{-3}{2}\right) - \tan^{-1}\left(\frac{4}{7}\right)\right|$

$\approx 86.05°$



TIP: Make sure your CAS is in degrees.

Let's see if it's consistent with parallel lines!

<u>Exploration</u>: **Understanding parallel lines using the angle between two lines formula**

- When two lines are parallel, what must be the angle θ between them?

$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

- Let's substitute the value of θ and see what we get! $\theta = 0°$

$$\tan(0°) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$0 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$m_1 - m_2 = 0$

$m_1 = m_2$

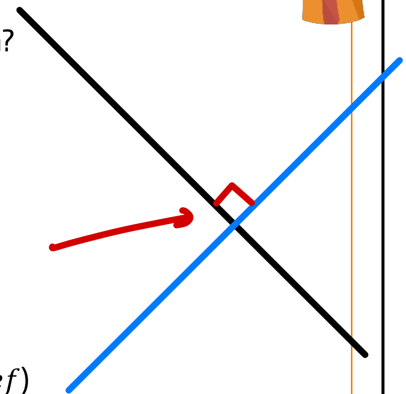
$\therefore$ They are parallel!

- This looks rather familiar, doesn't it?

And now perpendicular lines!

Exploration: Understanding perpendicular lines using the angle between two lines formula

- When two lines are perpendicular, what must be the angle θ between them?



$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$\theta = 90°$

- Let's substitute the value of θ and see what we get! (Note: $\tan(90) = Undef$)

$$\tan(90°) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

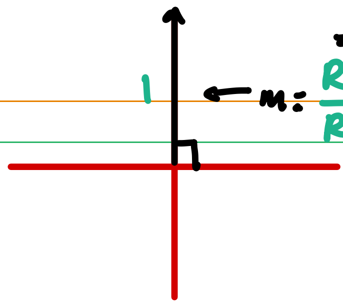
$$\text{Undefined} = \left|\frac{m_1 - m_2}{\underbrace{1 + m_1 m_2}_{0}}\right|$$

$\therefore 1 + m_1 m_2 = 0$

$m_1 m_2 = -1$

They are perpendicular!

- This looks rather familiar, doesn't it?



$m = \tan(\theta)$

$= \tan(90°)$

$m = \frac{\text{Rise}}{\text{Run}} = \frac{1}{0}$

$=$ undefined

Key Takeaways

- Parallel lines have the same gradient.
- Perpendicular lines have a negative reciprocal gradient.
- The angle between a line and x-axis is given by $\tan^{-1}(m)$.
- Tangent of the angle between two lines is given by $\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.
- The parallel lines and perpendicular lines formula is consistent with the angle between the two lines formula.

Kahoot Break!

(10min)

Section D: Simultaneous Equations

Sub-Section: Finding Simultaneous Equations for Two Variables

Simultaneous Linear Equations

1. Elimination Method:

   - Add or subtract one equation from the other in order to eliminate one of the variables. Then have an equation in one variable that can be solved easily.

2. Substitution Method

   - Make one of the variables the subject (generally x or y) and Sub that value into the other equation.

Question 15 Walkthrough.

Solve the following simultaneous linear equations using either elimination or substitution.

(2) $2x - y = 8$ and $2y + 5x = 11$ (1)

$5x + 2y = 11$ ...(1)

$+ (4x - 2y = 16)$... $2\times$(2)

$9x = 27$

$x = 3$

$5(3) + 2y = 11$ (Sub in (1))

$2y = -4$

$y = -2$

Elimination:

↳ similar form, one of the variables are not the subject

Substitution:

↳ When a variable is the subject (easy to sub)

Question 16

Solve the following equations for x and y.

a. $2x - 5y = 4$ and $2x + y = 16$

$$2x + y = 16$$
$$-(2x - 5y = 4)$$
$$6y = 12$$
$$y = 2$$

$$2x - 5(2) = 4$$
$$2x = 14$$
$$\therefore x = 7$$

b. $-3x + 2y = 2$ and $2 - 2y = x$

$$-3x + 2y = 2$$
$$+(2 - 2y = x)$$
$$-3x + 2 = x + 2$$
$$4x = 0$$
$$x = 0$$

$$2y = 2$$
$$\therefore y = 1$$

~~**Question 17 Extension.**~~

Solve the following:

$$-3x + 2y = 10 \text{ and } \underline{-10 + y = \tfrac{3}{2}x}$$

$-\frac{3}{2}x + y = 10$

$-3x + 2y = \underline{\underline{10}}$ $\quad$ $-3x + 2y = \underline{\underline{20}}$

$\therefore$ No solutions //

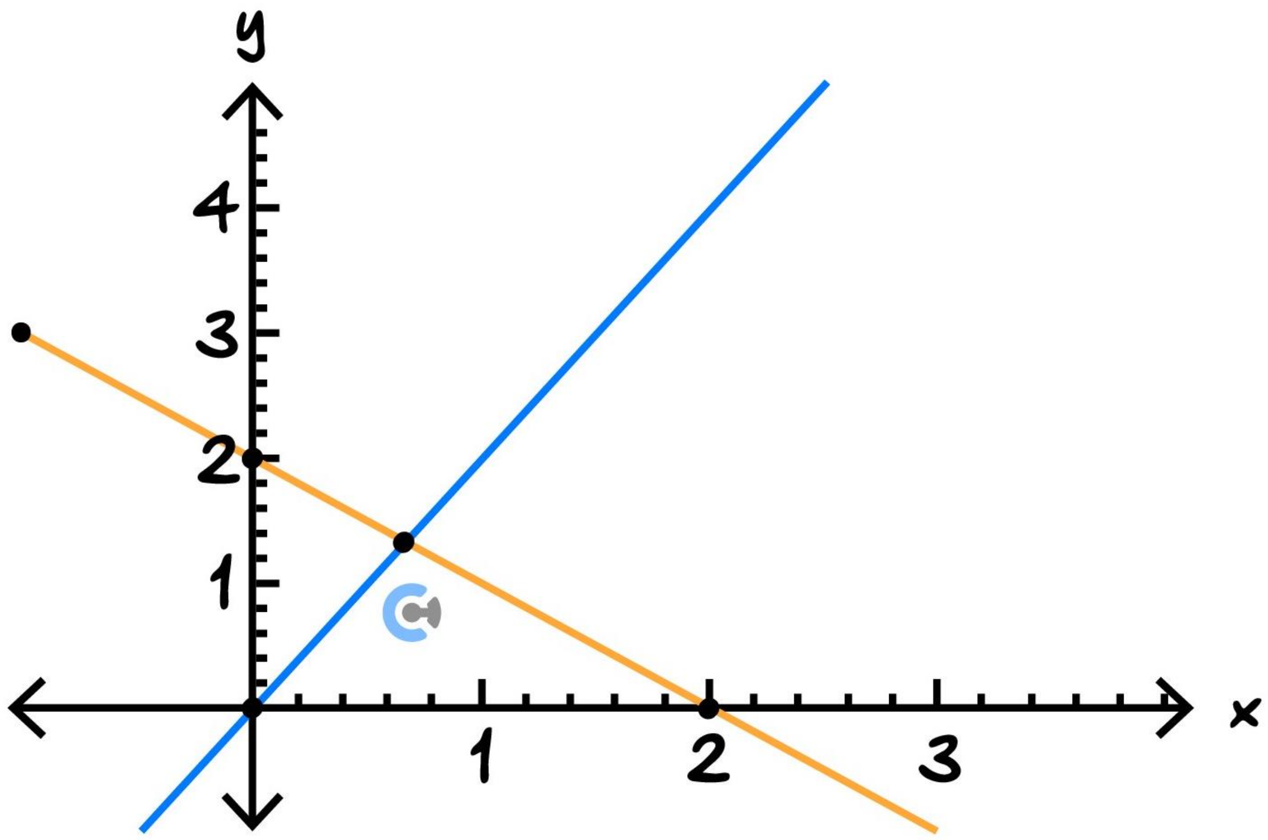
Sub-Section: Number of Solutions For Two Variables

What does the geometry look like for each number of solutions?

Exploration: Geometry of the number of solutions between linear graphs

- Unique Solution 1 intersection

$$m_1 \neq m_2$$

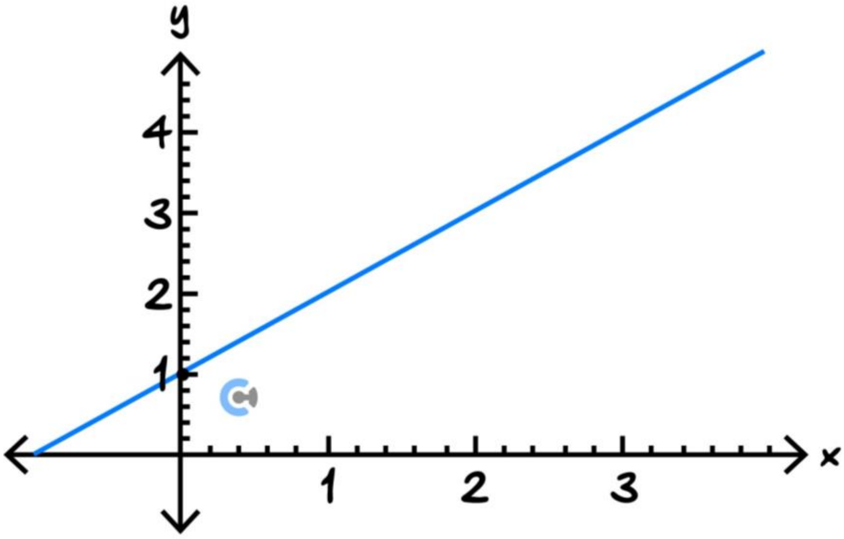


They just need to have different gradient

∞ intersections

- **Infinite Solutions**

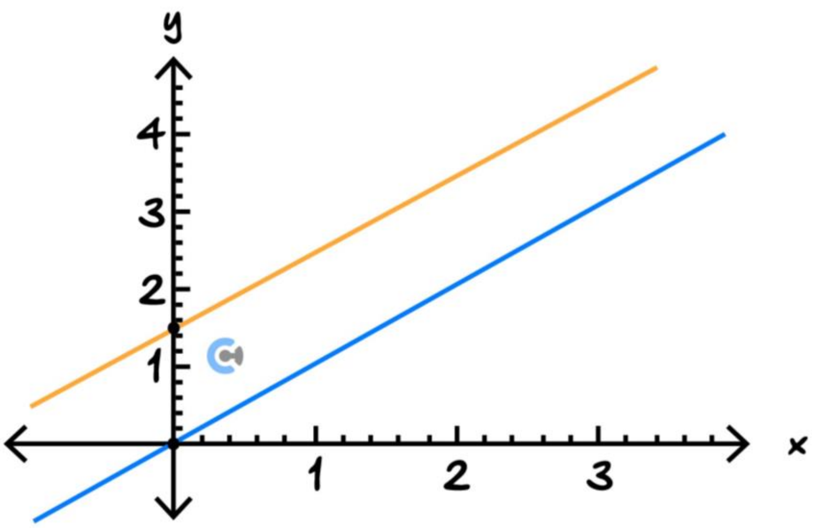
$$m_1 = m_2 \text{ AND } c_1 = c_2$$



- They just need to have the same gradient and the same y-intercept.
- In other words, they have to be the same line.

- **No Solutions**

$$m_1 = m_2 \text{ AND } c_1 \neq c_2$$



- They need to have the same gradient but different $+c$.
- They have to be two different parallel lines.

General Solutions of Simultaneous Linear Equations

- Two linear equations are either:
  - The same line is expressed in a different form. In this case, they have ____∞____ solutions.
  - Unique lines which are **parallel**. In this case, they have ____0____ solutions.
  - Unique lines which are not parallel. In this case, they have ____1____ solution.

Question 18 Walkthrough.

Consider the following pair of simultaneous equations in terms of $k \in \mathbb{R}\backslash\{0\}$:

$$y = kx + 5$$

$$y = \frac{2x}{k-1} - 5k$$

a. Find the value(s) of k for which there are no solutions to the simultaneous equations.

"parallel" + "different c"

$m_1 = m_2 \cap c_1 \neq c_2$

$k = \frac{2}{k-1} \cap 5 \neq -5k$

$k \neq -1$

$k(k-1) = 2$

$k^2 - k - 2 = 0$

$(k-2)(k+1) = 0$

$\underbrace{\therefore k = 2, -1 \cap k \neq -1}$

$\therefore k = 2$

b. Find the value(s) of k for which there is a unique solution to the simultaneous equations.

$m_1 \neq m_2$

$m_1 = m_2$

$k = 2, k = -1$

$\therefore k \neq 2$ and $k \neq -1$

$\therefore k \in \mathbb{R} \setminus \{2, -1, 0\}$

except

c. Find the value of k for which there are infinite solutions to the simultaneous equations.

$m_1 = m_2$ & $c_1 = c_2$

$c_1 \neq c_2$

$k \neq -1$

$\underbrace{k = 2, k = -1 \text{ \& } k = -1}$

$\therefore k = -1$

TIP: It's a good idea to substitute your answer back into the equations to see if the criteria are met for each part.

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Question 19 2 min

Consider the following pair of simultaneous equations in terms of $k \in \mathbb{R}\backslash\{0\}$:

$$y = \frac{1x}{1-2k} - 2k$$

$$y = -kx - 2$$

a. Find the value(s) of k for which there is a unique solution to the simultaneous equations.

$m_1 \neq m_2$

$\frac{1}{1-2k} \neq -k$

$2k^2 - k \neq 1$

$2k^2 - k - 1 \neq 0$

$(2k+1)(k-1) \neq 0$

$\therefore k \neq 1$ and $k \neq \frac{-1}{2}$

$\therefore k \in \mathbb{R} \backslash \{-\frac{1}{2}, 1, 0\}$

b. Find the value of k for which there are infinite solutions to the simultaneous equations.

$m_1 \neq m_2$

$k \neq 1$ and $\frac{-1}{2}$

$m_1 = m_2$ & $c_1 = c_2$

$\therefore k = 1, \frac{-1}{2}$ & $-2k = -2$

$k = 1, \frac{-1}{2}$ & $k = 1$ $\Rightarrow$ $\therefore k = 1$

c. Find the value of k for which there are no solutions to the simultaneous equations.

$m_1 = m_2$ & $c_1 \neq c_2$

$k = 1, \frac{-1}{2}$ & $k \neq 1$

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$\therefore k = \frac{-1}{2}$

~~**Question 20 Extension.**~~

Consider the following pair of simultaneous equations in terms of $a \in \mathbb{R}\backslash\{0\}$:

Box Method

$$ax - 2y = -5$$

$$-3x + (a-1)y = 5$$

a. Find the value(s) of a for which there are no solutions to the simultaneous equations.

$a = -2$

b. Find the value(s) of a for which there is a unique solution to the simultaneous equations.

$a \neq -2$ or 3

$a \in \mathbb{R} \backslash \{-2, 3, 0\}$

c. Find the value(s) of a for which there are infinite solutions to the simultaneous equations.

$a = 3$

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Learning Objective: [1.1.1] – Solve and Graph Linear Equations and Inequalities

Key Takeaways

- Linear equations are in the form of $y =$ mx+c where m is the gradient and c is the y-intercept.
- The inequality sign flips when you multiply by a negative.

Learning Objective: [1.1.2] – Find Midpoint, Distance (Horizontal & Vertical) Between Two Points Or Functions

Key Takeaways

- Midpoint is simply the average of 2 points.
- Distance formula is derived from Pythagoras Theorem.
- Horizontal distance is the distance between x values.
- Vertical distance is the distance between y values.

Learning Objective: [1.1.3] – Find Parallel and Perpendicular Lines

Key Takeaways

- Parallel lines have the same gradient.
- Perpendicular lines have negative reciprocal gradient.

Learning Objective: [1.1.4] - Find the Angle Between a Line and x axis or Two Lines

Key Takeaways

- To find the angle between a line and the x axis we can use equation $m = \underline{\tan\theta}$.
- To find the angle between two lines we can use $\theta = \left|\tan^{-1}(m_1) - \tan^{-1}(m_2)\right|$ or $\tan(\theta) = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$.

Learning Objective: [1.1.5] - Find The Unknown Value for Systems of Linear Equations

Key Takeaways

- Two linear equations have unique solution if they have <u>diffrent</u> gradients.
- Two linear equations have infinitely many solutions when they have <u>same</u> gradient and <u>same</u> constant.
- Two linear equations have no solution when they have <u>same</u> gradient and <u>different</u> constant.

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VCE Mathematical Methods ½

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