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VCE Mathematical Methods ½

AOS 1 Revision [1.0]

Contour Check Part 1 Solutions

Contour Check

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Section A: [1.1] - Linear & Coordinate Geometry (Checkpoints)

Sub-Section [1.1.1]: Solve and Graph Linear Equations and Inequalities

Question 1

Solve the following linear equations and inequalities for x:

a. $3x + 8 = 20$

$$\boxed{3x = 12 \implies x = 4}$$

b. $2x + 6 = 3(x - 2)$

$$\boxed{2x + 6 = 3x - 6 \implies x = 12}$$

c. $5x + 2 < 4x + 10$

$$\boxed{x < 8}$$

Question 2

Solve the following linear equations and inequalities for x:

a. $3x + 2 = 12x + 3$

$$\boxed{9x = -1 \implies x = -\frac{1}{9}}$$

b. $\frac{2x+3}{3} > 3(x-5)$

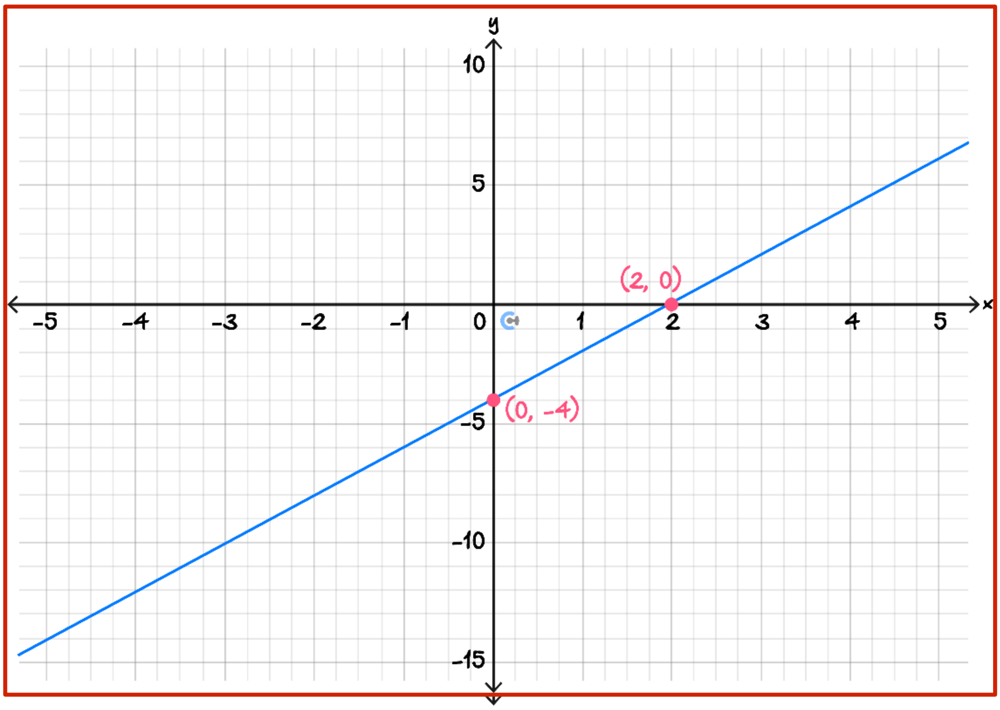
$$2x + 3 > 9x - 45 \implies 48 > 7x \implies x < \frac{48}{7}$$

c. $\frac{5x+3}{4} \leq 10x + 8$

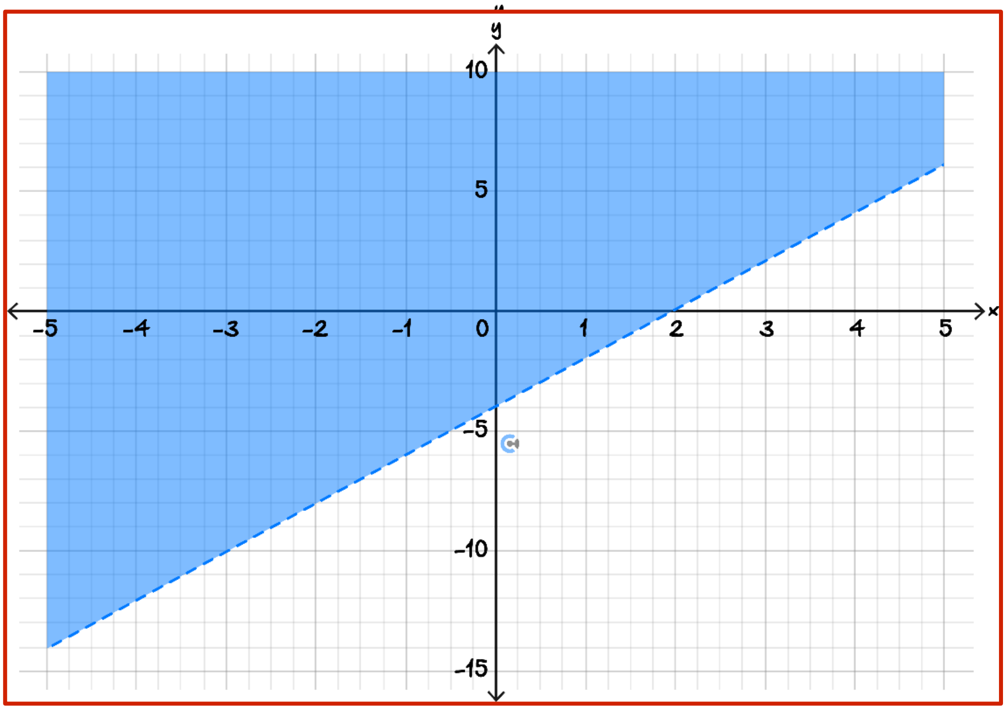
$$5x + 3 \geq 40x + 32 \implies -29 \geq 35x \implies x \geq -\frac{29}{35}$$

Question 3

a. Sketch the line governed by the equation $2y - 4x = -8$ on the axis below. Label all axes intercepts.



b. Shade the region governed by the equation $2y - 4x > -8$ on the axis below.



Question 4

Solve the inequality $\frac{1}{4}(5x - 3) \geq 2x + 8$ for x.

$x \leq -\frac{35}{3}$.

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Sub-Section [1.1.2]: Find The Midpoint and Distance Between Two Points or Functions

Question 5

a. Find the midpoint of $(1, -3)$ and $(6, -10)$.

$$\left(\frac{1+6}{2}, \frac{-3-10}{2}\right) = \left(\frac{7}{2}, -\frac{13}{2}\right)$$

b. The points (a, b) and $(3, 4)$ have a midpoint $(2, 3)$. Find the values of a and b.

$$\frac{a+3}{2} = 2 \implies a = 1 \text{ and } \frac{b+4}{2} = 3 \implies b = 2.$$

Question 6

a. Find the distance between points $(2, 5)$ and $(5, 2)$.

$$d = \sqrt{9+9} = 3\sqrt{2}.$$

b. The curve $y = (x-1)^2 + k$ and the line $y = 3$ has a minimum vertical distance of 4. Find the value of k.

The parabola has a minimum at $(1, k)$. Therefore $4 = k - 3 \implies k = 7$.

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Question 7

The distance between the point $(2, 2)$ and a point P on the line $y = 2x + 2$ is 4 units. Find all possible coordinates for P.

$$d = \sqrt{(x-2)^2 + (2x+2-2)^2} = 4$$
$$\implies x^2 - 4x + 4 + 4x^2 = 16$$
$$5x^2 - 4x - 12 = 0$$
$$x = \frac{4 \pm \sqrt{256}}{10}$$
$$= -\frac{6}{5}, 2$$

Substitute these x-values into the line $y = 2x + 2$ to get possible coordinates for P as

$$\left(-\frac{6}{5}, -\frac{2}{5}\right) \quad \text{and} \quad (2, 6).$$

Question 8

The distance between the point $(1, 2)$ and a point P on the line $y = 3x - 1$ is 4 units. Find all possible coordinates for P.

$d^2 = (x-1)^2 + (3x-3)^2 = 16 \implies x = \dfrac{5 \pm 2\sqrt{10}}{5}$. Therefore points are

$$\left(\frac{5-2\sqrt{10}}{5}, -\frac{2}{5}(3\sqrt{10}-5)\right) \quad \text{and} \quad \left(\frac{5+2\sqrt{10}}{5}, \frac{2}{5}\left(3\sqrt{10}+5\right)\right)$$

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Sub-Section [1.1.3]: Find Parallel and Perpendicular Lines

Question 9

State whether the following lines are parallel or perpendicular:

a. $y = 3x + 1$ and $y = 3x + 3$

$m_1 = m_2 \implies$ parallel.

b. $y = 2x + 3$ and $y = -\frac{1}{2}x + 2$

$m_1 \times m_2 = -1 \implies$ perpendicular.

Question 10

Find the equation of the line that is parallel to the line $y = 2x + 1$ and passes through the point $(5, 2)$.

Since the line is parallel to $y = 2x + 1$ it must have the form $y = 2x + c$.
Sub in the point $(5, 2) \implies 2 = 10 + c \implies c = -8$
The line is $y = 2x - 8$.

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Question 11

Find the equation of the line that is perpendicular to $y = 3x + 6$ and passes through the point $(6, 3)$.

The line has gradient $-\frac{1}{3}$ and passes through $(6, 3)$ therefore,

$$y - 3 = -\frac{1}{3}(x - 6)$$
$$y = -\frac{1}{3}x + 5.$$

Question 12

Find the equation of the line that is perpendicular to $y = \sqrt{3}x + 1$ and passes through the point $(2, 4)$.

The line has gradient $-\frac{1}{\sqrt{3}}$ and passes through $(2, 4)$ therefore,

$$y - 4 = -\frac{1}{\sqrt{3}}(x - 2)$$
$$y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}} + 4.$$

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Sub-Section [1.1.4]: Finding The Angle Between a Line and the x-axis or Between Two Lines

Question 13

Find the angle that $y = -x + 1$ makes with the positive direction of the x-axis.

$\theta = \tan^{-1}(-1) = 135°$.

Question 14

A line that makes an angle of $30°$ with the positive x-axis passes through the point $(1, 1)$. Find the equation of the line.

We know that the gradient $m = \tan(30) = \frac{1}{\sqrt{3}}$. Now since we know the gradient and a point we can find the equation of the line.

$$y - 1 = \frac{1}{\sqrt{3}}(x - 1)$$

$$y = \frac{1}{\sqrt{3}}x + 1 - \frac{1}{\sqrt{3}}$$

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Question 15

It is known that the lines $y = mx + 3$ and $y = 4x - 2$ make an angle of $45°$ when they intersect. Find all possible values of m.

The formula for the angle θ between two lines with gradients m_1 and m_2 is given by

$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

here we have $m_1 = m$ and $m_2 = 4$ and $\theta = 45° \implies \tan(\theta) = 1$. Therefore,

$$\left|\frac{m-4}{1+4m}\right| = 1$$

$$\frac{m-4}{1+4m} = \pm 1$$

solve the two cases separately.

$$m - 4 = 1 + 4m \implies m = -2$$

$$m - 4 = -1 - 4m \implies m = -\frac{5}{3}.$$

Conclude that $m = -\frac{5}{3}$ or $m = \frac{3}{5}$.

Question 16

Find the acute angle made between the lines $y = \sqrt{3}x + 1$ and $y = \frac{x}{\sqrt{3}} - 1$. Give your answer in degrees correct to two decimal places.

$$\theta = \left|\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right| = 60 - 30 = 30°$$

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Sub-Section [1.1.5]: Find The Unknown Value for a System of Linear Equations

Question 17

Consider the simultaneous linear equations:

$$y = kx + 6$$

$$y = 2x + 5$$

Where $x, y \in R$ and k is a real constant.

a. Find the value(s) of k for which the system of equations has no solution.

Lines must be parallel for no solution $\implies k = 2$. Then we have $y = 2x + 6$ and $y = 2x + 5$ which never intersect. Hence no solution if $k = 2$.

b. Find the value(s) of k for which the system of equations has infinitely many solutions.

No value of k will result in the system having infinitely many solutions. This is because there can only be infinitely many solutions if the lines are identical. No value of k will make the lines identical.

c. Find the value(s) of k for which the system of equations has a unique solution.

$k \neq 2$.

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Question 18

Consider the simultaneous linear equations:

$$-3kx + y = k$$

$$-3x + ky = -1$$

Where $x, y \in R$ and k is a real constant.

a. Find the value(s) of k for which the system of equations has no real solution.

The gradients must be equal for a possibility of there being no real solutions.

$$3k = \frac{3}{k} \implies 3k^2 = 3 \implies k = \pm 1.$$

If $k = 1$ the lines are $-3x + y = 1$ and $-3x + y = -1$. Therefore no solution if $k = 1$.
If $k = -1$ the lines are $3x + y = -1$ and $-3x - y = -1 \implies 3x + y = 1$. Therefore no solution if $k = -1$.

b. Find the value(s) of k for which the system of equations has infinitely many solutions.

From above. The system will never have infinitely many solutions.

c. Find the value(s) of k for which the system of equations has a unique solution.

$k \in \mathbb{R} \setminus \{-1, 1\}$.

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Question 19

Consider the simultaneous linear equations:

$$kx + y = 2$$

$$2x + (k-2)y = 4$$

Where $x, y \in R$ and k is a real constant.

a. Find the value(s) of k for which the system of equations has no real solution.

The gradients must be equal for a possibility of there being no real solutions.

$$k = \frac{2}{k-2} \implies k = 1 \pm \sqrt{3}.$$

The y-intercepts are the same if $2 = \dfrac{4}{k-2} \implies k = 4.$
Therefore no real solution if $k = 1 \pm \sqrt{3}$

b. Find the value(s) of k for which the system of equations has infinitely many solutions.

From above. The system will never have infinitely many solutions.

c. Find the value(s) of k for which the system of equations has a unique solution.

$k \in \mathbb{R} \setminus \{1 - \sqrt{3}, 1 + \sqrt{3}\}.$

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Question 20

Consider the simultaneous linear equations:

$$(k-2)x+3y=5$$

$$4x+(k+1)y=k+7$$

Where $x, y \in R$ and k is a real constant.

Find the value(s) of k for which the system has no real solution.

Solve $\frac{k-2}{3}=\frac{4}{k+1} \implies k=\frac{1\pm\sqrt{57}}{2}$. We check and see that both these values of k result in no solution.

$$k=\frac{1\pm\sqrt{57}}{2}.$$

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Section B: [1.2] - Linear & Coordinate Geometry Exam Skills (Checkpoints)

Sub-Section [1.2.1]: Applying Midpoint to Find Reflected Points

Question 21

Find the reflection of the point $(4, 6)$ about the line $y = 4$.

$\boxed{(4, 2)}$

Question 22

The point $(3, 2)$ is reflected in the line $y = b$, and to become the point $(3, -6)$. Find the value of b.

$\boxed{b = -2}$

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Question 23

Consider the function $f(x) = x^2 + 1$.

a. The point $A(1, 1)$ on the graph of $y = f(x)$ is reflected about the line $y = 0$. Find the coordinates of the reflected points' position.

$\boxed{(1, -1)}$

b. The entire graph of $y = f(x)$ is reflected about the line $y = 0$. Find the equation of this new graph.

$\boxed{y = -x^2 - 1}$

Question 24

The function $y = (x - 1)^2 + 3$ is reflected about the line $x = 3$ and then reflected about the line $y = 2$. Find the equation of the graph after these reflections.

First reflection: $y = (x - 5)^2 + 3$
After second reflection: $y = -(x - 5)^2 + 1$

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Sub-Section [1.2.2]: Find Vertical and Horizontal Distance Between Functions

Question 25

Find the vertical distance between $f(x) = 3x + 1$ and $g(x) = x + 3$ when $x = 2$.

$$\boxed{\text{vertical distance} = f(2) - g(2) = 7 - 5 = 2}$$

Question 26

Find the horizontal distance between the function $f(x) = x + 1$ and $g(x) = x^2 - 1$ when $y = 8$.

$$\boxed{f(x) = 8 \implies x = 7 \text{ and } g(x) = 8 \implies x = 3. \text{ Therefore horizontal distance} = 4}$$

Space for Personal Notes

Question 27

Consider the functions $y = x + 3$ and $y = x^2 + 1$.

a. Solve the inequality $x + 3 > x^2 + 1$.

$$\boxed{-1 < x < 2}$$

b. Hence, determine the vertical distance between the two functions when $x = 1$.

$$\boxed{4 - 2 = 2}$$

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Sub-Section [1.2.3]: Finding Distance Between a Point and a Function

Question 28

Find the distance between the point $(1, 2)$ and the function $y = x^2$, when $x = 3$.

Distance between $(1, 2)$ and $(3, 9)$
$d = \sqrt{53}$

Question 29

The distance between the point $A(4, 1)$ and the point $B(-3, m)$ is 7, find the possible value(s) of m.

We must solve $\sqrt{(4+3)^2 + (1-m)^2} = 7$. Therefore,

$$\begin{aligned} 49 + m^2 - 2m + 1 &= 49 \\ (m-1)^2 &= 0 \\ m &= 1. \end{aligned}$$

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Question 30

Find the point(s) on the line $y = 3x + 3$ which have a distance of 5 from the point $(1, 1)$.

$\boxed{(-2, -3) \text{ or } (1, 6).}$

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Section C: [1.3] - Quadratics (Checkpoints)

Sub-Section [1.3.1]: Rewriting Quadratics in Different Forms

Question 31

Find the factorised forms of these quadratics:

a. $x^2 - 4$

We apply difference of squares (recall $a^2 - b^2 = (a-b)(a+b)$) to get,

$$x^2 - 4 = (x-2)(x+2)$$

b. $x^2 - 3x$

We can simply factorise an x out from our expression, hence,

$$x^2 - 3x = x(x-3)$$

c. $5x^2 + 10x$

We factorise a factor of $5x$ out of our expression to get,

$$5x^2 + 10x = 5x(x+2)$$

Space for Personal Notes

Question 32

a. Express $x^2 + 4x + 3$ in intercept form, $(a(x - b)(x - c))$.

$$x^2 + 4x + 3 = x^2 + 3x + x + 3 = x(x + 3) + (x + 3) = (x + 3)(x + 1)$$

b. Express $x^2 - 2x + 3$ in turning point form, $(a(x - h)^2 + k)$.

Recall that $x^2 - 2x + 1 = (x - 1)^2$.
Thus $x^2 - 2x + 3 = x^2 - 2x + 1 + 2 = (x - 1)^2 + 2$.

c. Factorise $x^2 + 6x + 9$.

We observe that $x^2 + 6x + 9 = x^2 + 3x + 3x + 9 = x(x + 3) + 3(x + 3) = (x + 3)^2$.

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Question 33

a. Factorise $3x^2 - 12x - 15$.

Since 3 is a factor of $3, -12, -15$ we see that,

$$3x^2 - 12x - 15 = 3(x^2 - 4x - 5)$$

Thus we need to factorise $x^2 - 4x - 5$.
Since $x^2 - 4x - 5 = x^2 - 5x + x - 5 = x(x-5) + (x-5) = (x+1)(x-5)$, we see that,

$$3x^2 - 12x - 15 = 3(x-5)(x+1).$$

b. Express $2x^2 - 12x + 9$ in turning point form.

Observe that $2(x-3)^2 = 2x^2 - 12x + 18$.
Hence $2x^2 - 12x + 9 = 2x^2 - 12x + 18 - 9 = 2(x-3)^2 - 9$.

c. Express $2(x-1)(x+3)$ in turning point form.

We first expand $2(x-1)(x+3)$ out to get,

$$2(x-1)(x+3) = 2(x^2 + 2x - 3) = 2x^2 + 4x - 6$$

Since $2(x+1)^2 = 2x^2 + 4x + 2$ we see that,

$$2(x-1)(x+3) = 2x^2 + 4x + 2 - 8 = 2(x+1)^2 - 8$$

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Question 34

Factorise $6x^2 - \sqrt{5}x - 5$.

We apply the quadratic formula to find the roots of the equation $6x^2 - \sqrt{5}x - 5 = 0$. Thus

$$x = \frac{\sqrt{5} \pm \sqrt{5+120}}{12} = \frac{\sqrt{5} \pm 5\sqrt{5}}{12} = -\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{2}$$

Hence our expression can be factorised as,

$$6x^2 - \sqrt{5}x - 5 = 6\left(x - \frac{\sqrt{5}}{2}\right)\left(x + \frac{\sqrt{5}}{3}\right) = (2x - \sqrt{5})(3x + \sqrt{5})$$

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Sub-Section [1.3.2]: Find Solutions and Number of Solutions to Quadratic Equations

Question 35

Find all real solutions to the following equations:

a. $x^2 = -5x$

We factorise our expression and apply the null factor law.

$$\begin{aligned} x^2 &= -5x \\ \implies x^2 + 5x &= x(x+5) = 0 \\ \implies x &= 0, -5 \end{aligned}$$

b. $4x^2 - 16 = 0$

We factorise our expression and apply the null factor law.

$$\begin{aligned} 4x^2 - 16 &= 4(x+2)(x-2) = 0 \\ \implies x + 2 &= 0 \text{ or } x - 2 = 0 \\ \implies x &= -2, 2 \end{aligned}$$

c. $2x^2 - 18x = 0$

We factorise our expression and apply the null factor law.

$$\begin{aligned} 2x^2 - 18x &= 2x(x-9) \\ \implies x &= 0, 3 \end{aligned}$$

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Question 36

a. Find all real solutions to the equation $x^2 - 10x + 25 = 0$.

$$\begin{aligned} & x^2 - 10x + 25 = 0 \\ \implies & (x-5)^2 = 0 \\ \implies & x = 5 \end{aligned}$$

b. How many solutions does the equation $x^2 + 2x - 15$ have?

Recall that for a quadratic equation $ax^2 + bx + c$, the discriminant, $\Delta = b^2 - 4ac$ determines the number of real solutions, with

- $\Delta > 0$ implying two real solutions.
- $\Delta = 0$ implying one real solution.
- $\Delta < 0$ implying no real solutions.

Since our discriminant is equal to $4 - 4(-15) = 64 > 0$ our equation has two real solutions.

c. Find all real solutions to the equation $3(x+1)^2 = 12$.

$$\begin{aligned} & 3(x+1)^2 = 12 \\ \implies & (x+1)^2 = 4 \\ \implies & x + 1 = \pm 2 \\ \implies & x = -3, 1 \end{aligned}$$

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Question 37

a. Find all real solutions to the equation $x^2 - 6x = 4$.

We rearrange our to be in the form $ax^2 + bx + c$, getting,

$$x^2 - 6x - 4 = 0$$

From here we apply the quadratic formula $\left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$ to our equation to get,

$$x = \frac{6 \pm \sqrt{36 - 4(-4)(1)}}{2} = \frac{6 \pm 2\sqrt{13}}{2} = 3 \pm \sqrt{13}$$

b. For what values of a does the equation $ax^2 - 6x = 18$ have no real solutions?

We rearrange our to be in the form $ax^2 + bx + c$, getting,

$$ax^2 + 6x - 18 = 0$$

Since we have no solutions our discriminant is $\Delta = 36 + 72a$ is less than 0. Hence $a < -\frac{1}{2}$.

c. Find all real solutions to the equation $5x^2 + 20x = 15$.

We observe that $5(x+2)^2 = 5x^2 + 20x + 20$, hence,

$$\begin{aligned} 5x^2 + 20x = 15 &\implies 5x^2 + 20x + 20 = 35 \\ &\implies 5(x+2)^2 = 35 \\ &\implies (x+2)^2 = 7 \\ &\implies x + 2 = \pm\sqrt{7} \\ &\implies x = -2 \pm \sqrt{7} \end{aligned}$$

Question 38

For what values of b does the equation $2x(b - x) = 5$ have no real solutions?

We rearrange our equation to be in the form $ax^2 + bx + c = 0$, getting,

$$2x^2 - 2bx + 5 = 0$$

Since we desire no real solutions, we require that

$$\Delta = 4b^2 - 4(2)(5) = 4(b^2 - 10) < 0.$$

From the graph of $x^2 - 10$, we see that this occurs if $b \in (-\sqrt{10}, \sqrt{10})$.

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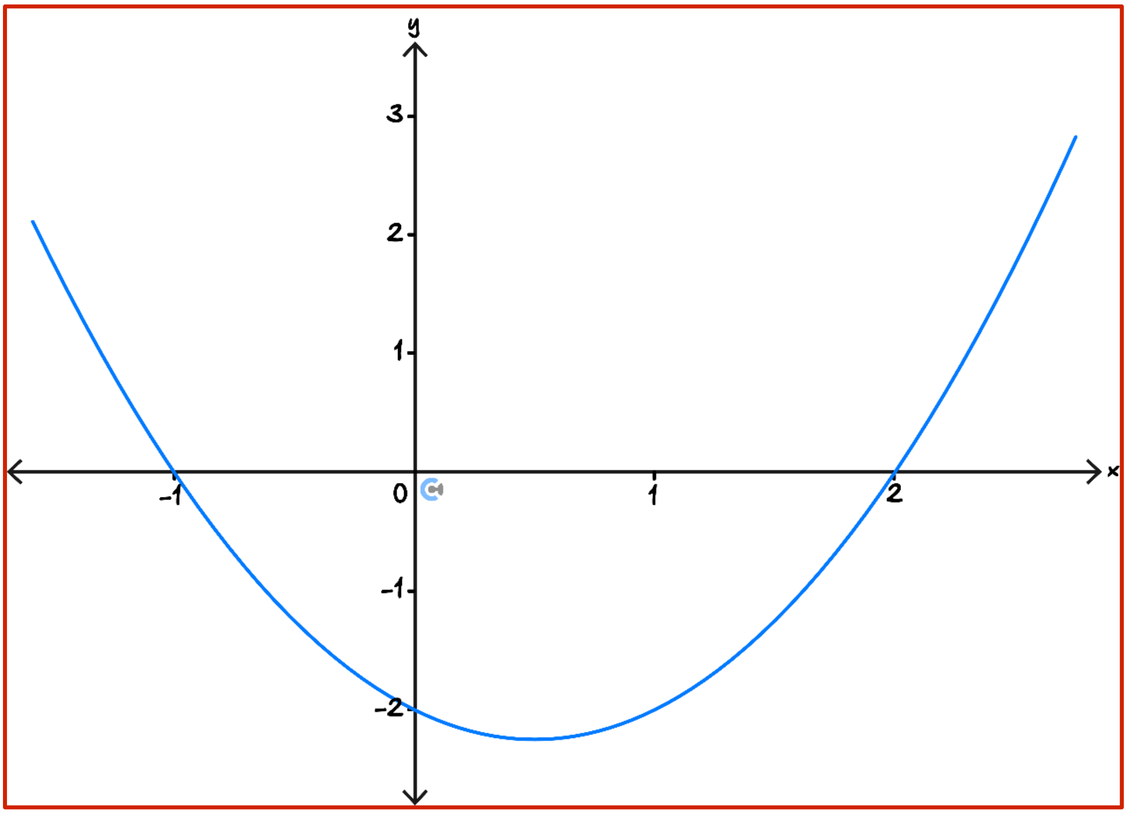
Sub-Section [1.3.3]: Graph and Find Rules From the Graph of Quadratic Equations

Question 39

Sketch the graph of $y = (x + 1)(x - 2)$ on the axis below.



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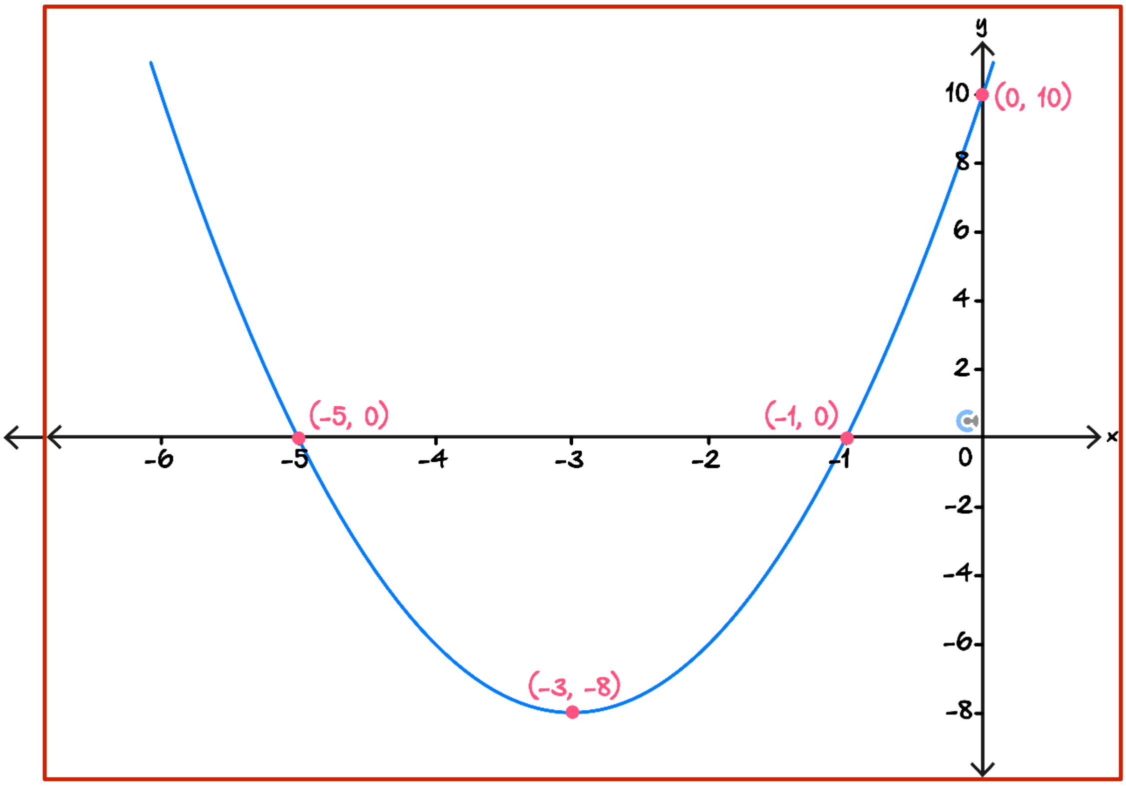
Question 40

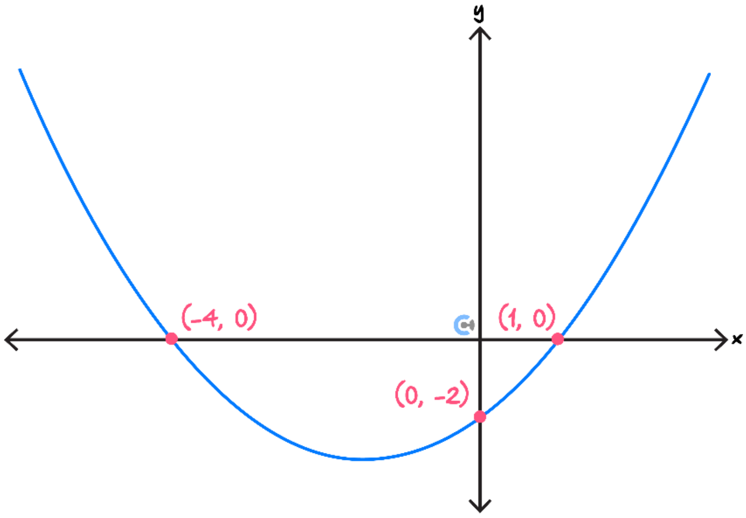
Sketch the graph of $y = 2(x+3)^2 - 8$ on the axis below, labelling axis intercepts and turning points with their coordinates.



Space for Personal Notes

Question 41

The graph of a parabola is shown below.



Find the rule of this parabola.

Since we have the axis intercepts of this parabola, we can express the equation in intercept form as,

$$y = a(x-1)(x+4)$$

We can solve for a since when x is equal to 0, y is equal to -2. Hence,

$$-2 = a(0-1)(0+4) \implies -2 = -4a \implies a = \frac{1}{2}$$

Hence our parabola has equation, $y = \frac{1}{2}(x-1)(x+4)$

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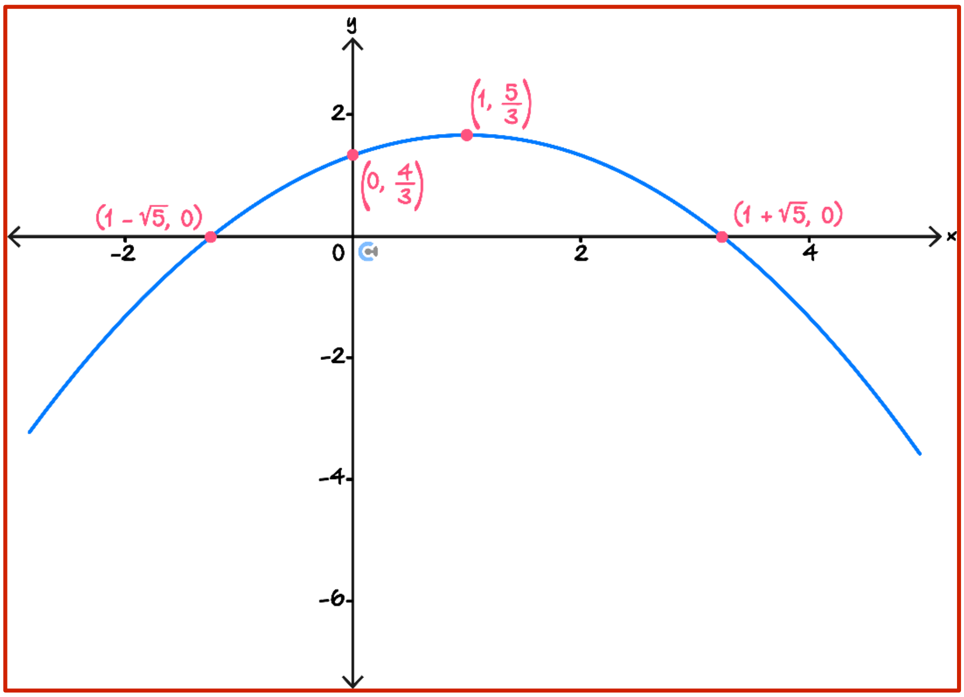
Question 42

Sketch the graph of $3y = 5 - (x-1)^2$ on the axis below, labelling axis intercepts and turning points with their coordinates.



The x-axis intercepts can be obtained by solving

$$5 - (x-1)^2 = 0 \implies x - 1 = \pm\sqrt{5} \implies x = 1 \pm \sqrt{5}.$$

The y-axis intercept can be obtained by evaluating y when $x = 0$. Hence,

$$3y = 5 - (0-1)^2 = 4 \implies y = \frac{4}{3}$$

The turning point can be read off, since,

$$y = \frac{-1}{3}(x-1)^2 + \frac{5}{3}$$

Hence our turning point is $\left(1, \frac{5}{3}\right)$.

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Sub-Section [1.3.4]: Solving Quadratic Inequalities and Hidden Quadratics

Question 43

a. Solve $x^2 > 1$ for x.

$x > 1$ or $x < -1$.

b. Solve $x(x-2) \leq 3$ for x.

We rearrange our inequality to get everything on one side. Thus,

$$x^2 - 2x - 3 \leq 0$$

We consider the graph of $x^2 - 2x - 3$. It will be a positive parabola, and have x-axis intercepts when,

$$x^2 - 2x - 3 = x^2 - 3x + x - 3 = (x-3)(x+1) = 0 \implies x = -1, 3$$

From here we see that $x^2 - 2x - 3 \leq 0$ if $x \in [-1, 3]$.

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Question 44

Solve $(x-1)^4-(x-1)^2=12$ for x.

We let $a=(x-1)^2$. After substituting this value into our equation we get the quadratic,

$$a^2-a-12=0$$

which we can solve as usual.
As $a^2-a-12=a^2-4a+3a-12=a(a-4)+3(a-4)=(a-4)(a+3)=0$, we see that $a=-3,4$.
Since $a=(x-1)^2\geq 0$, we reject $a=-3$, leaving us with,

$$(x-1)^2=4 \implies x-1=\pm 2 \implies x=-1,3$$

Question 45

Solve $x^2+6x+8\geq 2$ for x.

We first rearrange our inequality to get everything on one side. Thus,

$$x^2+6x+6\geq 0$$

We consider the graph of x^2+6x+6. It will be a positive parabola, and have x-axis intercepts when,

$$x^2+6x+6=0 \implies x=\frac{-6\pm\sqrt{36-24}}{2}=-3\pm\sqrt{3}$$

From here we see that $x^2+6x+8\geq 2$ if $x\leq -3-\sqrt{3}$ or if $x\geq -3+\sqrt{3}$.

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Question 46

For what values of x is $ax^2 + bx + c < d$, where $a, b, c, d \in R,\ a < 0$ and $c > d$?

We first rearrange our inequality to get everything on one side. Thus,

$$ax^2 + bx + c - d < 0$$

We consider the graph of $ax^2 + bx + c - d < 0$. It will be a negative parabola, and have x-axis intercepts when,

$$ax^2 + bx + c - d \implies x = \frac{-b \pm \sqrt{b^2 - 4a(c-d)}}{2a}$$

Since $c > d$ and $a < 0$, we know that $b^2 - 4a(c-d) > 0$, thus the above values of x are real numbers.

By the properties of a negative parabola, we see that $ax^2 + bx + c < d$ if,

$$x < \frac{-b - \sqrt{b^2 - 4a(c-d)}}{2a} \quad \text{or} \quad x > \frac{-b + \sqrt{b^2 - 4a(c-d)}}{2a}$$

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Section D: [1.4] – Quadratics Exam Skills (Checkpoints)

Sub-Section [1.4.1]: Find Turning Point Form Using Turning Points

Question 47

Find the turning point of the parabola $y = 2(x-1)^2 + 3$.

$$\boxed{(1,3)}$$

Question 48

Find the equation of a parabola that has a turning point at $(5,3)$ and has a y-axis intercept of 8.

From the turning point, $y = a(x-5)^2 + 3$.
From the y-axis intercept, we know that if $x = 0$, then $y = 8$, thus,

$$8 = a(0-5)^2 + 3 \implies 5 = 25a \implies a = \frac{1}{5}$$

Hence the equation of the parabola is $y = \frac{1}{5}(x-5)^2 + 3$

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Question 49

Find the turning point of the parabola $y = 2x^2 - 4x + 5$.

We complete the square to get,

$$y = 2(x-1)^2 + 3$$

Thus the parabola has a turning point of $(1, 3)$.

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Sub-Section [1.4.2]: Apply Quadratics to Model a Scenario

Question 50

A ball is thrown up into the air from a height of 1 metre. It reaches its maximum height of 2 metres after 1 second. The height in metres of the ball h, t seconds after the ball is launched is:

$$h(t) = a(t-1)^2 + 2$$

Find the value of a.

We know that $h(0) = 1$.
Thus $1 = a(0-1)^2 + 2 = a + 2 \implies a = -1$.

Question 51

A parabola-shaped bridge is used to cross a long river. The height of the bridge above the water level in metres, h, is a quadratic function of the horizontal distance of a point of a bridge from the starting river bank, x.

At the starting river bank, the height of the bridge is 2 metres above water level, and 5 metres away from the starting point ($x = 5$), the bridge is at its highest point, 6 metres above the water level ($h = 6$).

Relate x and h.

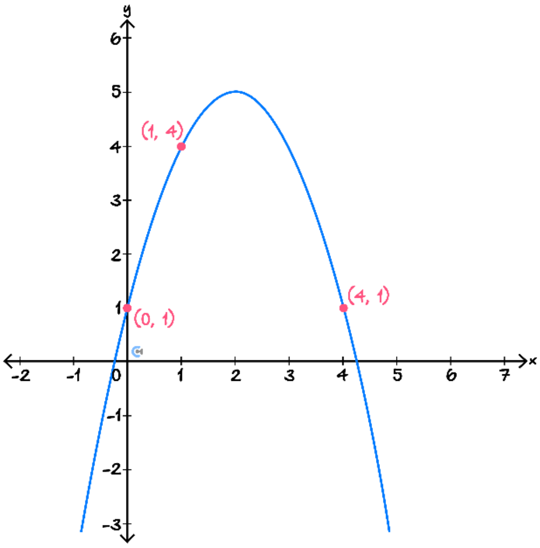
From the highest point we know that $h = a(x-5)^2 + 6$.
We can solve for a by using the fact that $h(0) = 2$, thus,

$$2 = 25a + 6 \implies a = -\frac{4}{25}$$

Thus, $h = -\frac{4}{25}(x-5)^2 + 6$

Question 52

A river passes through 3 points in a park as shown below:



Where the x-axis represents the position due east from the centre of the park, and the y-axis represents the position due north from the centre of the park. We can relate the north position (y) of the river to the east position (x) of the river through the equation:

$$y = ax^2 + bx + c$$

Find the values of a, b and c.

When $x = 0$ we know that $y = 0a + 0b + c = 1$. Hence $c = 1$.
For the other two values we can create a pair of simultaneous equations using the points $(1, 4)$ and $(4, 1)$.

$$4 = a + b + 1 \quad (1)$$
$$1 = 16a + 4b + 1 \quad (2)$$

We subtract $4 \times (1)$ from (2) to get, $-15 = 12a - 3 \implies -12 = 12a \implies a = -1$.
Substituting this back into (1) yields $4 = b \implies b = 4$

Sub-Section [1.4.3]: Apply Family of Functions to Find an Unknown of Function

Question 53

Consider the parabola $y = kx^2 - 6$. Find the value(s) of k such that the horizontal distance between x-axis intercepts of the parabola is less than 4.

We solve $y = 0$ to get $6 = kx^2 \implies x = \pm\sqrt{\frac{6}{k}}$.

Thus the horizontal distance between the x-axis intercepts is $2\sqrt{\frac{6}{k}}$.

We require this quantity to be < 4, thus,

$$2\sqrt{\frac{6}{k}} < 4$$
$$\implies \frac{24}{k} < 16$$
$$\implies k > \frac{3}{2}$$

Question 54

Let $y = x^2 + 4kx - 1$. Find the values of k such that $y \geq -2$ for all x.

We complete the square to get, $y = (x + 2k)^2 - 4k^2 - 1$.
Thus $y \geq -4k^2 - 1$.
For $y \geq -2$ for all x we simply require $-4k^2 - 1 \geq -2 \implies 4k^2 \leq 1 \implies \frac{-1}{2} < k < \frac{1}{2}$

Question 55

Find all values of k such that the equation $(x-k-1)^2-4=k$ has two real solutions for x, one positive and one negative.

The solutions to the equation $(x-k-1)^2-4=k$ are,

$$x=k+1\pm\sqrt{4+k}$$

Since $k+1+\sqrt{4+k}\geq k+1-\sqrt{4+k}$ we require,

$$k+1+\sqrt{4+k}>0 \quad \text{and} \quad k+1-\sqrt{4+k}<0$$

We rearrange the first inequality to become $k+4+\sqrt{4+k}-3>0$. Now we substitute $a=\sqrt{4+k}$ to get the quadratic inequality, $a^2+a-3>0$.

We solve the equality $a^2+a-3=0$ to get $a=\dfrac{1\pm\sqrt{13}}{2}$.

Since a^2+a-3 is a positive parabola, the solution to our inequality is $a>\dfrac{-1+\sqrt{13}}{2}$ or $a<\dfrac{-1-\sqrt{13}}{2}$.

However since $a>0$ our only solution is $a>\dfrac{-1+\sqrt{13}}{2}$, hence $4+k>\dfrac{14-2\sqrt{13}}{4} \implies k>\dfrac{-\sqrt{13}-1}{2}$.

We do something similar for $k+1-\sqrt{4+k}<0$, again substituting $a=\sqrt{4+k}$ to get the quadratic inequality, $a^2-a-3>0$.

After taking into account domain restrictions, the solution to this inequality is $0<a<\dfrac{1+\sqrt{13}}{2}$.

Hence $-4<k<\dfrac{\sqrt{13}-1}{2}$.

From here we combine these two restrictions to get $\dfrac{-\sqrt{13}-1}{2}k<\dfrac{\sqrt{13}-1}{2}$.

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Sub-Section [1.4.4]: Harder Quadratic Inequalities

Question 56

Solve $x(x+3) > 4$ for x.

We rearrange the equality to become $x^2+3x-4=(x-1)(x+4)>0$.
Since x^2+3x-4 is a positive parabola with roots of $-4, 1$ we see that it is greater than 0 if $x<-4$ or $x>1$.

Question 57

Solve $1+\frac{2}{x-2} \leq \frac{5}{(x-2)^2}$ for x.

We multiply both sides of our equation by $(x-2)^2$ to get, $(x-2)^2+2(x-2)-5 \leq 0$.
Substituting $a=x-2$ we get the inequality $a^2+2a-5 \leq 0$.
We first solve this as an equality to get $a=\dfrac{-2 \pm \sqrt{4-20}}{2}=-1 \pm \sqrt{6}$.
As a^2+2a-5 is a positive parabola with roots of $-1 \pm \sqrt{6}$ we see that it is ≤ 0 if $-1-\sqrt{6} \leq a \leq -1+\sqrt{6}$.
In terms of x, $1-\sqrt{6} \leq x \leq 1+\sqrt{6}$.
As in our original equation we divide by $x-2$, x cannot be 2 thus our solution is,

$$1-\sqrt{6} \leq x < 2 \quad \text{or} \quad 2 < x \leq 1+\sqrt{6}$$

Question 58

Solve $(x^2+2)^2-4 \geq 8x^2$ for x.

First we get everything in terms of x^2+2 to substitute $a = x^2+2$. Thus our inequality becomes,

$$(x^2+2)^2 - 8(x^2+2) + 12 \geq 0$$

After substituting $a = x^2+2$ we need to solve $a^2-8a+12 \geq 0$.
As $a^2-8a+12 = (a-6)(a-2)$ is a positive parabola with roots of 2 and 6, our inequality reduces to $a \leq 2$ or $a \geq 6$.
Since $a = x^2+2$, the only way $a \leq 2$ is if $x = 0$.
And $x^2+2 \geq 6 \implies x^2 \geq 4$, thus $x \leq -2$ or $x \geq 2$. Combining all cases yields,

$$x \leq -2 \quad \text{or} \quad x = 0 \quad \text{or} \quad x \geq 2$$

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Section E: [1.5] - Polynomials (Checkpoints)

Sub-Section [1.5.1]: Identify the Properties of Polynomials and Solve Long Division

Question 59

Consider the polynomial $f(x) = 3x^2 - 4x^4 + 1 - 2x$.

a. State the degree of $f(x)$.

$\boxed{4}$

b. State the leading coefficient of $f(x)$.

$\boxed{-4}$

c. State the constant term of $f(x)$.

$\boxed{1}$

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Question 60

Simplify the following using polynomial long division:

$$\frac{x^3 + 2x^2 - 5x - 6}{x - 2}$$

$\boxed{x^2 + 4x + 3}$

Question 61

The polynomial $P(x) = x^4 - 2x^2 - 5x + 3$ can be written in the form $P(x) = Q(x)(x - 2) + r$, where $r \in R$ and $Q(x)$ is a real valued polynomial. Find $Q(x)$ and r.

We divide $P(x)$ by $(x - 2)$ to get $Q(x)$. The remainder is r.
Thus $Q(x) = x^3 + 2x^2 + 2x - 1$ and $r = 1$.

Question 62

The polynomial $P(x) = 2x^4 + 3x^3 - 5x + 1$ can be written in the form $P(x) = Q(x)(x^2 - 2x + 3) + R(x)$, where $R(x)$ is a polynomial of degree 1 and $Q(x)$ is a polynomial.

a. State the degree of $Q(x)$.

Since the degree of $R(x)$ is 1, $Q(x)$ needs an x^2 term so that $P(x)$ has an x^4 term.
Similarly $Q(x)$ cannot have an x^n term for $n > 2$.
Hence the degree of $Q(x)$ is 2.

b. Find $Q(x)$ and $R(x)$.

$Q(x) = 2x^2 + 7x + 8$ and $R(x) = -10x - 23$.

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Sub-Section [1.5.2]: Apply Reminder and Factor Theorem to Find Reminders and Factors

Question 63

Find the remainder of the division $\frac{f(x)}{g(x)}$, where:

a. $f(x) = x^3 - 7x + 8$ and $g(x) = x + 3$.

$\boxed{\text{Our remainder is } f(-3) = -27 + 21 + 8 = 2.}$

b. $f(x) = 2x^3 - 6x^2 - 2x + 4$ and $g(x) = x - 1$.

$\boxed{\text{Our remainder is } f(1) = 2 - 6 - 2 + 4 = -2}$

c. $f(x) = -3x^3 + 8x^2 - 3x + 2$ and $g(x) = 3x + 1$.

$$\boxed{\text{Our remainder is } f\left(-\frac{1}{3}\right) = \frac{1}{9} + \frac{8}{9} + 1 + 2 = 4.}$$

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Question 64

For the polynomial $f(x) = ax^3 2x^2 - 3ax + 1$, we get a remainder of 5 when $f(x)$ is divided by $x + 2$. Find the value of a.

We have that $f(-2) = -8a + 8 + 6a + 1 = 9 - 2a = 5 \implies a = 2.$

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Question 65

Consider the expression:

$$f(x) = 2x^3 - ax^2 + b$$

Where a and b are non-zero constants.
It is known that $x + 1$ is a factor of $f(x)$ and that the remainder when $f(x)$ is divided by $x - 2$ is 3. Find the values of a and b.

Since $x + 1$ is a factor of $f(x)$ we know that $f(-1) = -2 - a + b = 0$.
Since remainder of $\dfrac{f(x)}{x-2}$ is 3, we know that $f(2) = 16 - 4a + b = 3$. We get the following pair of simultaneous equations.

$$-2 - a + b = 0$$
$$16 - 4a + b = 3$$

Solving these equations yields $a = 5$ and $b = 7$.

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Question 66

Find a cubic polynomial $f(x)$ which has the following properties:

- $f(x)$ has a leading coefficient of -2.
- $f(x)$ divided by $x^2 - 1$ leaves a remainder of 1.
- $x - 3$ is a factor of $f(x)$.

$f(x)$ will be of the form $f(x) = -2x^3 + ax^2 + bx + c$.
Since when $f(x)$ is divided by $x^2 - 1$ it leaves a remainder of 1 we know that $f(1) = f(-1) = 1$. We also know that $f(3) = 0$.
Thus we have the following equations.

$$\begin{aligned} -2 + a + b + c &= 1 \\ 2 + a - b + c &= 1 \\ -54 + 9a + 3b + c &= 0 \end{aligned}$$

By subtracting the first equation from the second we get $4 - 2b = 0 \implies b = 2$.
We thus know that $a + c = 1 \implies c = 1 - a$ and $-48 + 9a + c = 0$. From here we see that $-47 + 8a = 0 \implies a = \dfrac{47}{8}$.
Thus $c = 1 - \dfrac{47}{8} = -\dfrac{39}{8}$.
Hence our polynomial is $-2x^3 + \dfrac{47}{8}x^2 + 2x - \dfrac{39}{8}$

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Sub-Section [1.5.3]: Find Factored Form of Polynomials

Question 67

Factorise $x^3 - 2x^2 - x + 2$ as a product of three linear factors.

$$\boxed{(x-1)(x+1)(x-2)}$$

Question 68

Factorise $x^3 - 6x^2 + 3x + 10$ as a product of three linear factors.

$$\boxed{(x-5)(x+1)(x-2)}$$

Question 69

Factorise $2x^3 + \frac{25x^2}{3} + x - \frac{4}{3}$ as a product of three linear factors.

$$\boxed{\frac{1}{3}(2x+1)(3x-1)(x+4)}$$

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Question 70

Use the fact that $x^n - 1 = (1 + x + x^2 + \cdots + x^{n-1})(x - 1)$ to factorise $1 + x^2 + x^4 + x^6 + x^8$ as a product of two-degree four polynomials.

By our hint we know that $(1 + x^2 + x^4 + x^6 + x^8)(x^2 - 1) = x^{10} - 1$.
We also know that $x^{10} - 1 = (x^5 - 1)(x^5 + 1)$.
Again applying our fact we see that $x^5 - 1 = (x - 1)(1 + x + x^2 + x^3 + x^4)$ and that $x^5 + 1 = -((-x)^5 - 1) = -(-x - 1)(1 - x + x^2 - x^3 + x^4) = (x + 1)(1 - x + x^2 - x^3 + x^4)$.
Hence,

$$\begin{aligned} x^{10} - 1 &= (x - 1)(1 + x + x^2 + x^3 + x^4)(x + 1)(1 - x + x^2 - x^3 + x^4) \\ &= (x^2 - 1)(1 + x + x^2 + x^3 + x^4)(1 - x + x^2 - x^3 + x^4) \end{aligned}$$

Dividing both sides by $x^2 - 1$ we get that,

$$1 + x^2 + x^4 + x^6 + x^8 = (1 + x + x^2 + x^3 + x^4)(1 - x + x^2 - x^3 + x^4).$$

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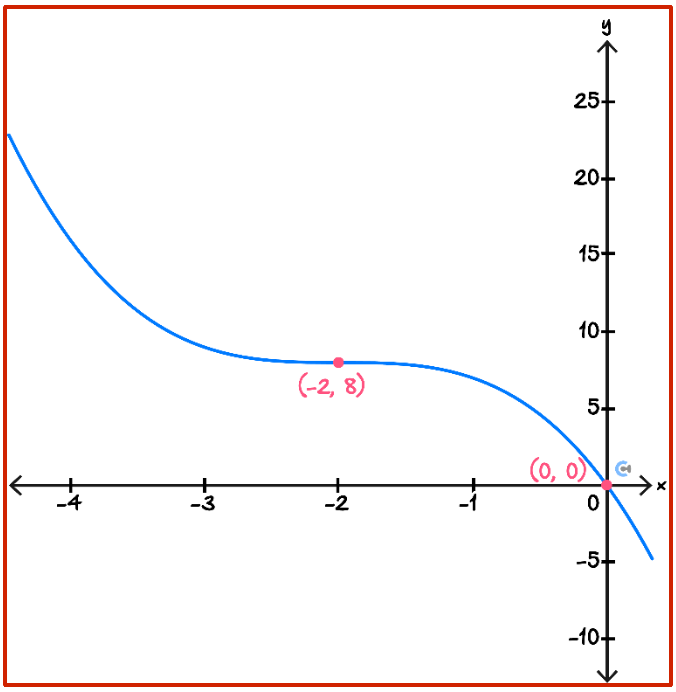
Sub-Section [1.5.4]: Graph Factored and Unfactored Polynomials

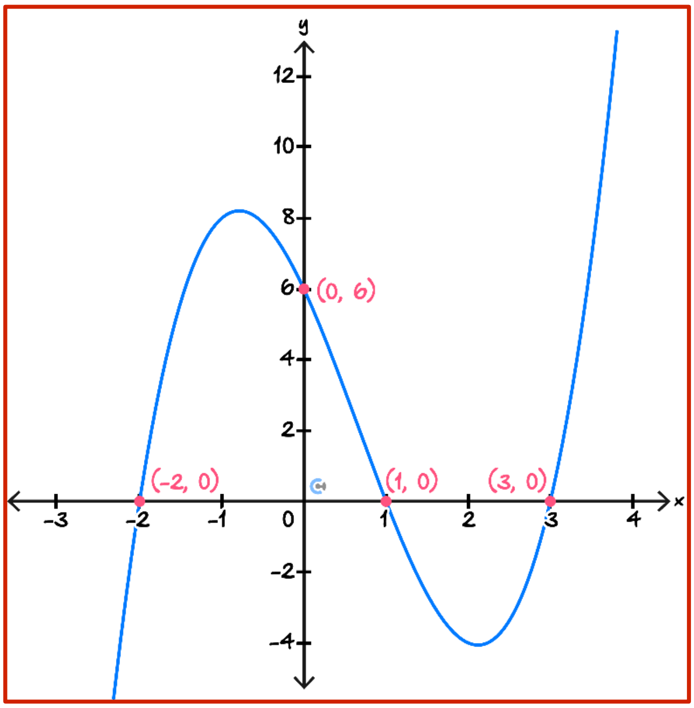
Question 71

Sketch the graphs of each of the functions on the axes provided.

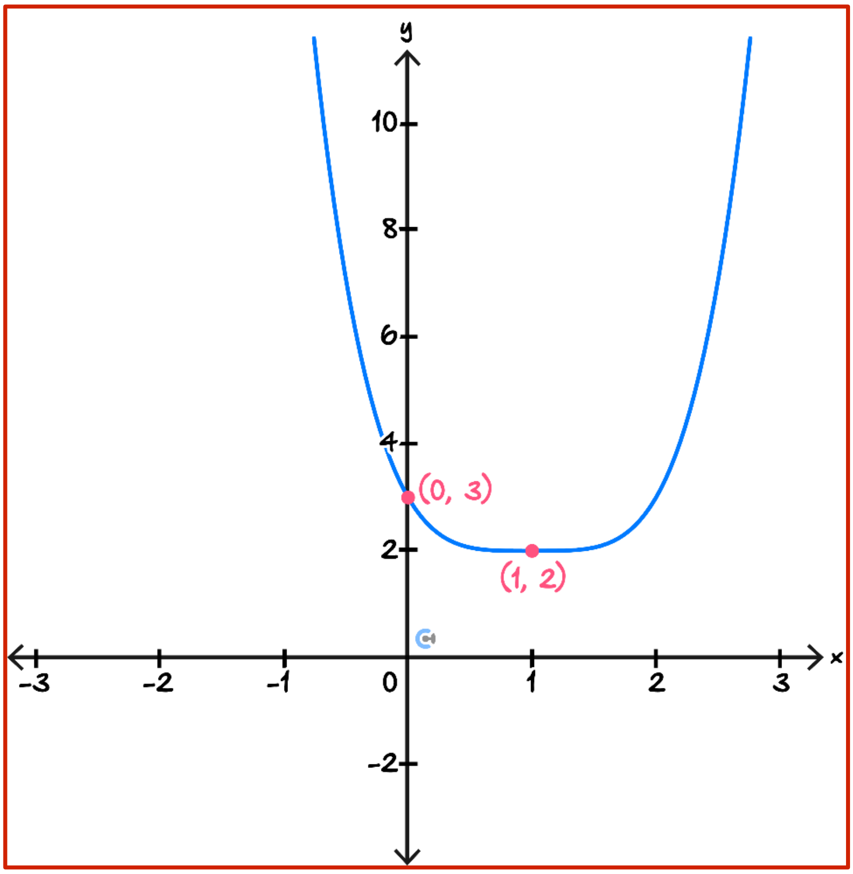
a. $y = 8 - (x+2)^3$



b. $y = (x-1)(x+2)(x-3)$



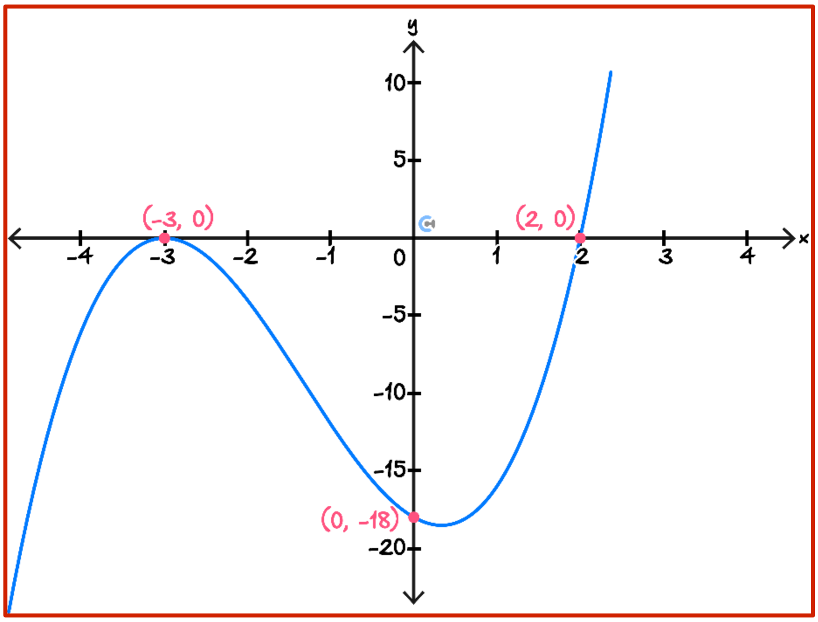
c. $y = 2 + (x-1)^4$



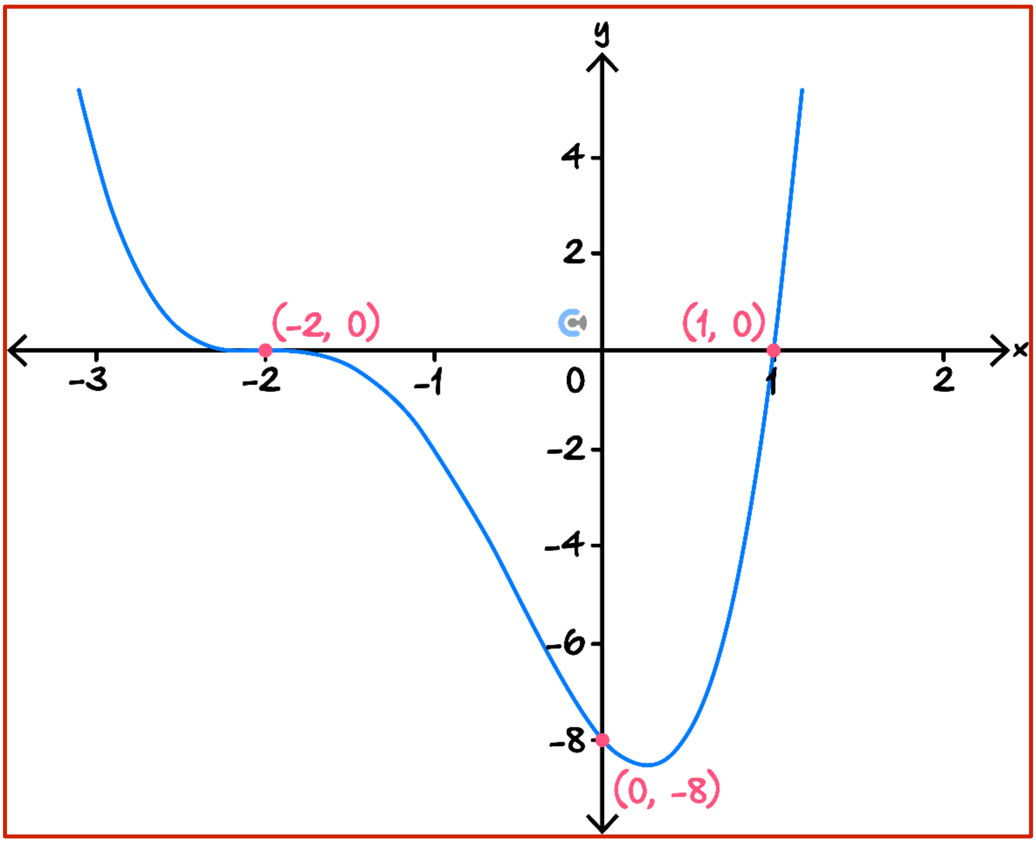
Question 72

Sketch the graphs of each of the functions on the axes provided.

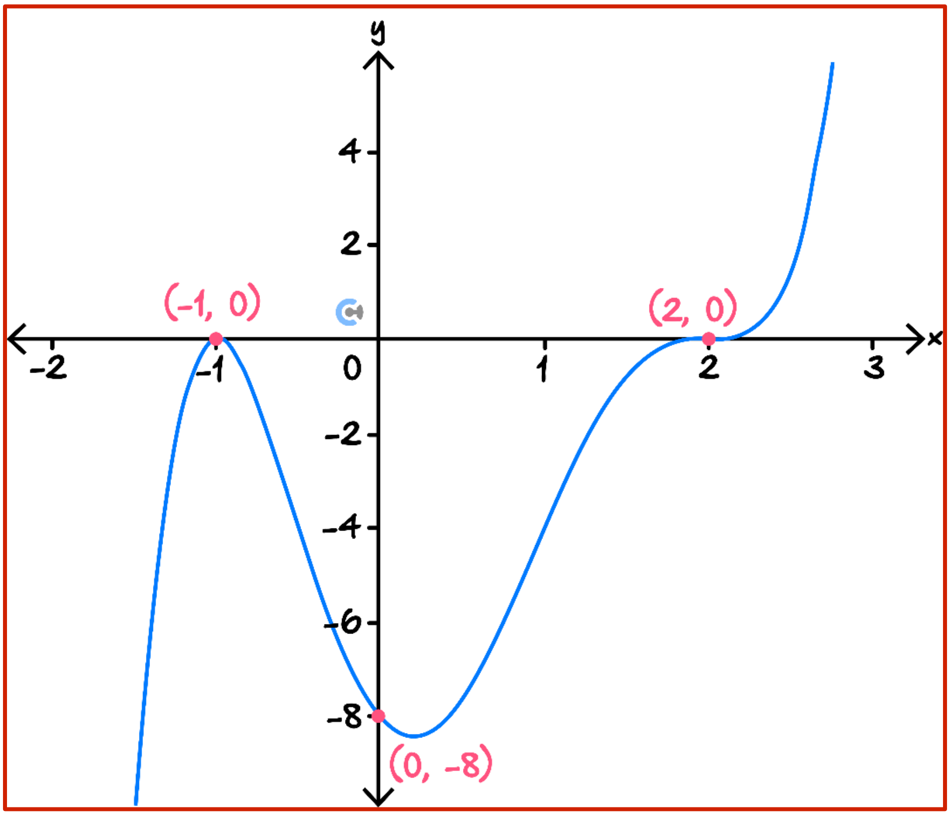
a. $y = (x+3)^2(x-2)$



b. $y = (x-1)(x+2)^3$



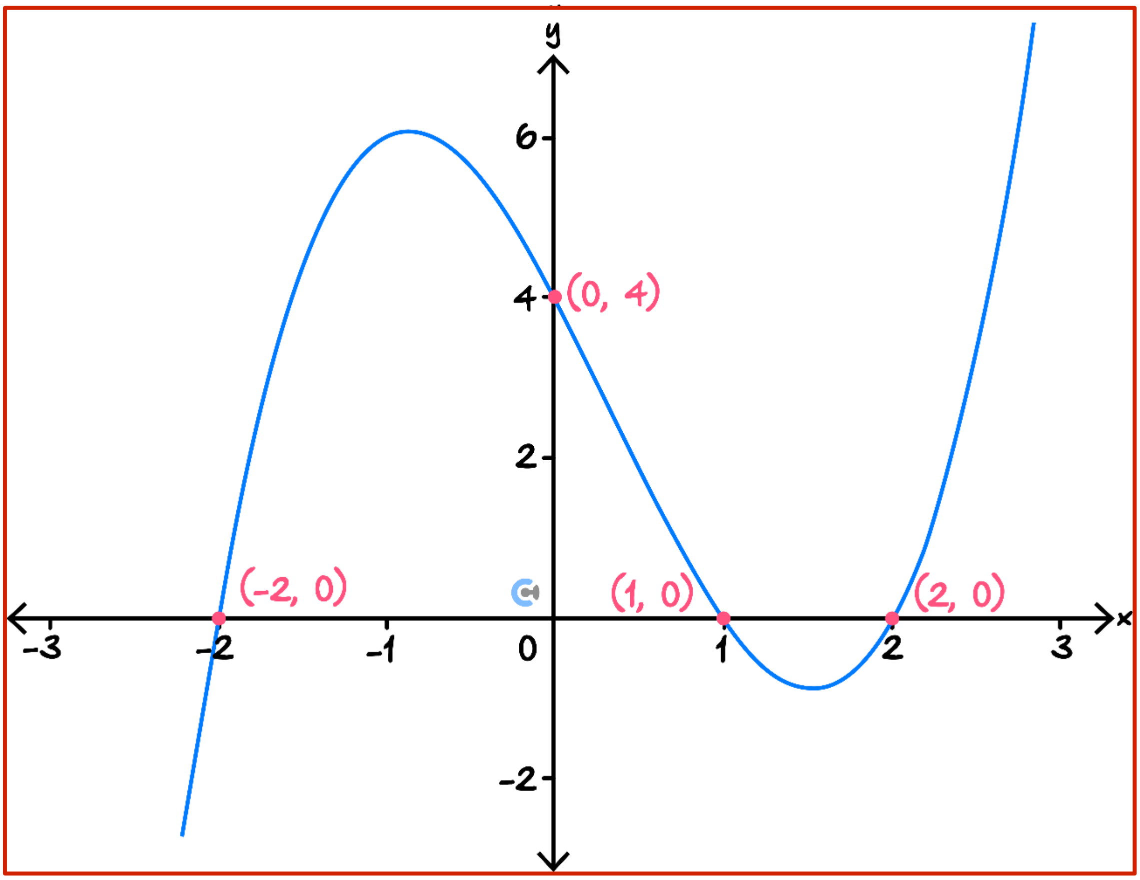
c. $y = (x+1)^2(x-3)^3$



Question 73

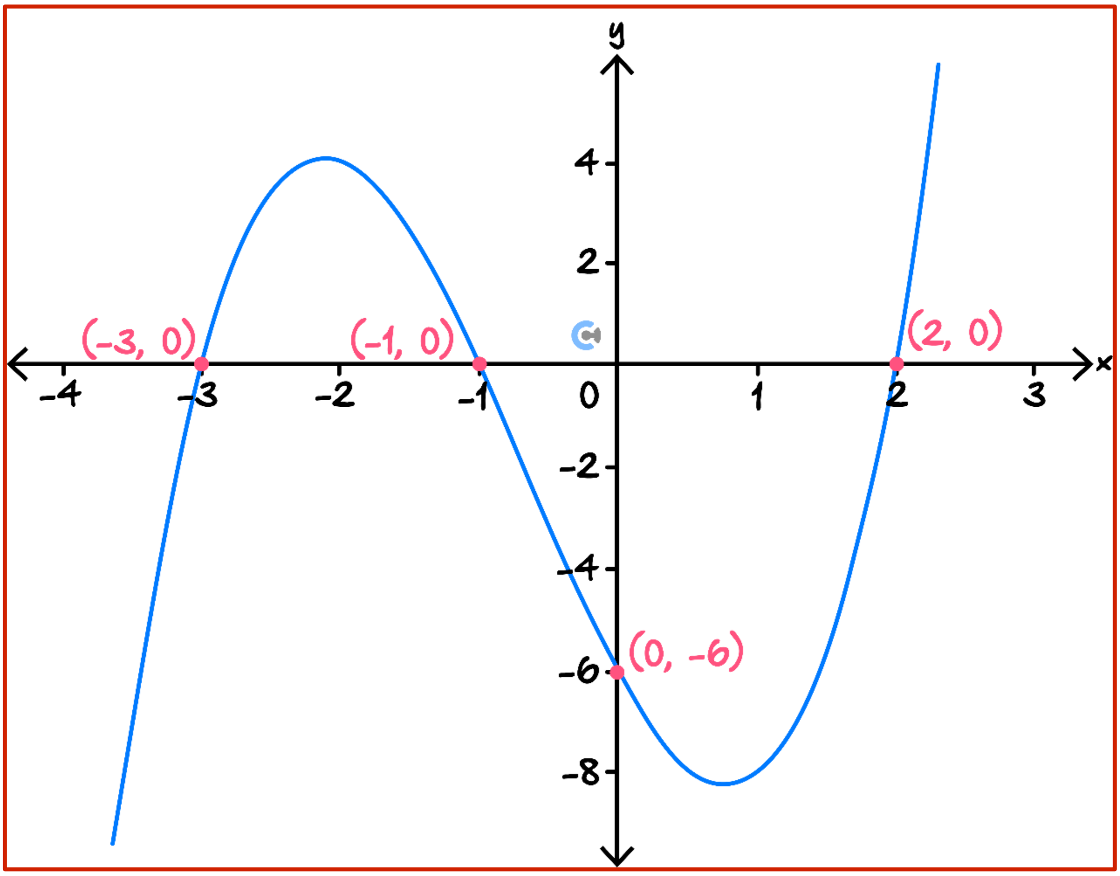
Factorise and hence, sketch the graphs of each of the functions on the axes provided.

a. $y = x^3 - x^2 - 4x + 4$



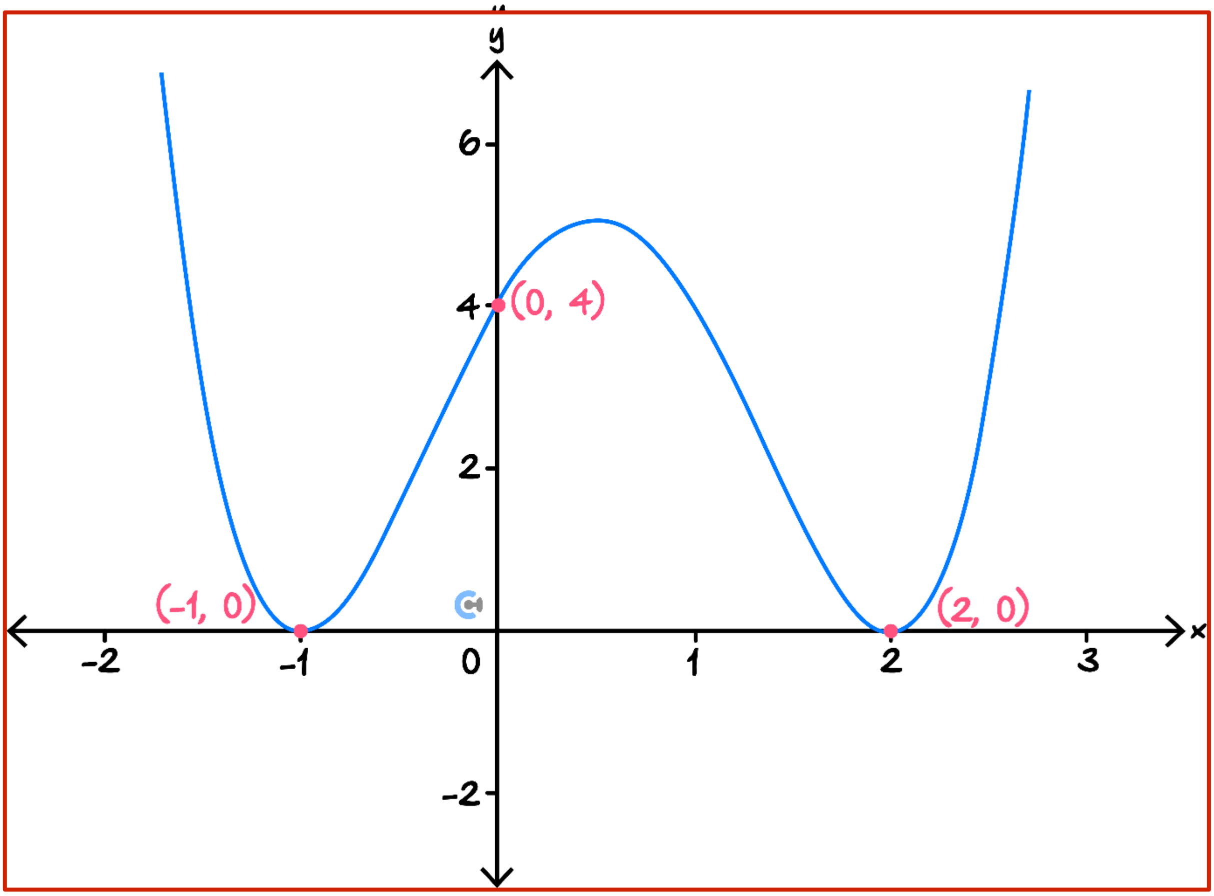
Solution Pending

b. $y = x^3 + 2x^2 - 5x - 6$



Solution Pending

c. $y = x^4 - 2x^3 - 3x^2 + 4x + 4$



Solution Pending

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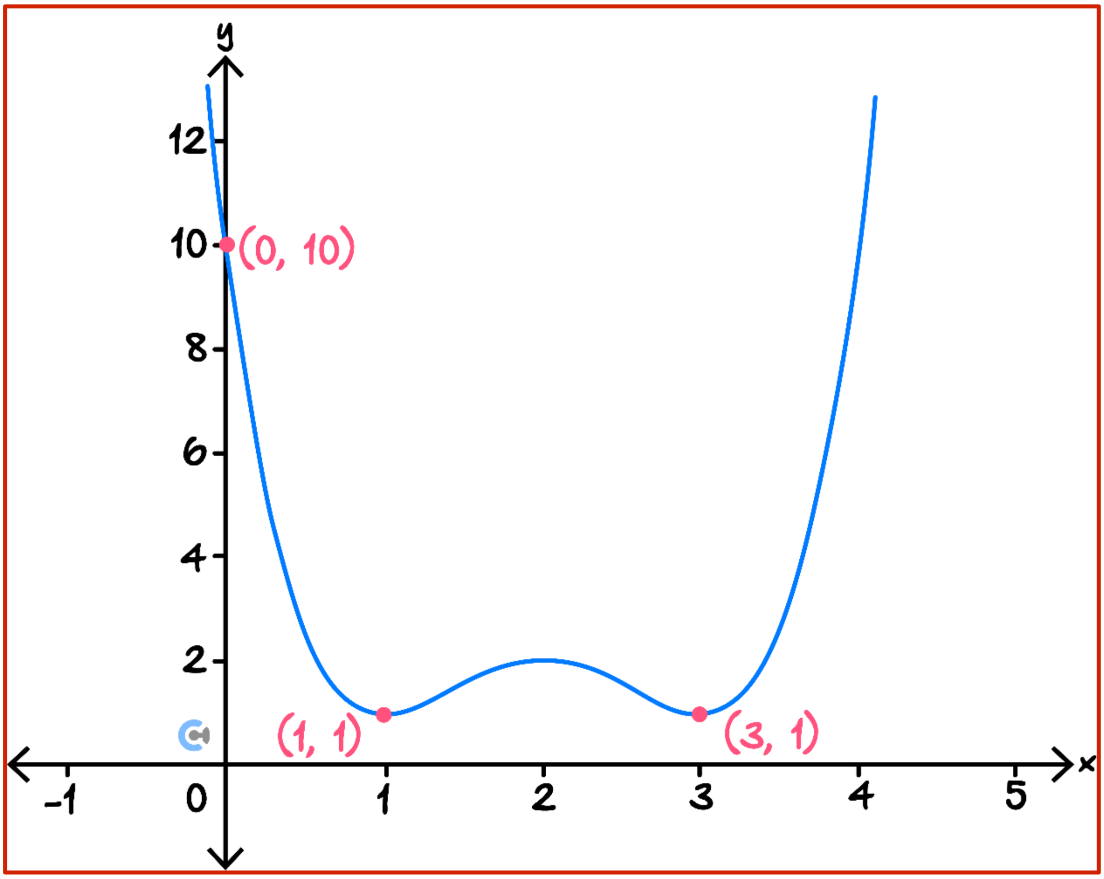
Question 74

Sketch the graph of $y = x^4 - 8x^3 + 22x^2 - 24x + 10$ on the axis below.

Hint: Factorise $x^4 - 8x^3 + 22x^2 - 24x + 9$ instead.



Solution Pending

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Section F: [1.6] - Polynomials Exam Skills (Checkpoints)

Sub-Section [1.6.1]: Solve Polynomial Inequalities

Question 75

Solve the following inequalities for x:

a. $x(x-1)(x+2) \leq 0$.

$\boxed{x \leq -2 \text{ or } 0 \leq x \leq 1}$

b. $(x-2)(x+1)(x+3) > 0$.

$\boxed{-3 < x < -1 \text{ or } x > 2}$

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Question 76

Solve the following inequalities for x:

a. $(x-5)(x^2+x-2) \leq 0$.

$\boxed{x \leq 2 \text{ or } 1 \leq x \leq 5}$

b. $(1-x)(x^2-4x+4) \geq 0$.

$\boxed{x \leq 1 \text{ or } x = 2}$

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Question 77

Solve the following inequalities for x:

a. $x^3 - 5x^2 - 8x + 12 > 0$.

Factor as $(x-1)(x+2)(x-6)$.
Therefore, $-2 < x < 1$ or $x > 6$.

b. $-x^3 + 4x^2 + x - 4 \leq 0.$

Factor as $-(x-1)(x+1)(x-4)$
Therefore $-1 \leq x \leq 1$ or $x \geq 4$.

Question 78

Solve the inequality $4x^5 - 16x^4 + 13x^3 - 3x^2 > 4x^3 - 16x^2 + 13x - 3$.

We rewrite our inequality as $4x^5 - 16x^4 + 9x^3 + 13x^2 - 13x + 3 > 0$ and realise that $x^2 - 1$ is a factor of the left hand side.
Hence we factor the left side as $(2x-1)^2(x-3)(x^2-1)$.
Therefore $-1 < x < \frac{1}{2}$ or $\frac{1}{2} < x < 1$ or $x > 3$.

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Sub-Section [1.6.2]: Solve Number of Solution Problems

Question 79

Find the values of k, for which the equation $x(x^2 + 4) = 4kx^2$ has:

a. 1 solution.

Only solution is $x = 0$ if the discriminant of the quadratic $x^2 - 4kx + 4$ is less than 0. Therefore $-1 < k < 1$

b. 2 solutions.

$k = \pm 1$

c. 3 solutions.

$k < -1$ or $k > 1$

Question 80

Find the values of k, for which the equation $kx^9 + 2x^6 + x^3 = 0$ has:

a. 1 solution.

We observe that x^3 is a one to one and onto function.
Thus, the number of solutions to $kx^9 + 2x^6 + x^3 = 0$ is simply the number of solutions to $kx^3 + 2x^2 + x = x(kx^2 + 2x + 1) = 0$.
Only solution is $x = 0$ if the discriminant of the quadratic $kx^2 + 2x + 1$ is less than 0.
Therefore, $k > 1$.

b. 2 solutions.

$\boxed{k = 1 \text{ or } k = 0}$

c. 3 solutions.

$\boxed{k < 0 \text{ or } 0 < k < 1}$

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Question 81

Find the values of k, for which the equation $x(x-2k-2)(x^2+kx+4) = -x^2-kx-4$ has:

a. 4 solutions.

We can simplify our equation to be $(x^2+2x(k-1)+1)(x^2+kx+4)=0$
We require the discriminant for both quadratics to be greater than zero.

$$\Delta 1 = 4(k+1)^2 - 4$$
$$\Delta 2 = k^2 - 16$$

Therefore, $k < -2$ or $k > 0$ and $k < -4$ or $k > 4$.
So, 4 solutions if $k < -4$ or $k > 4$.

b. 3 solutions.

One discriminant is zero and the other is greater than zero.
Therefore $k = -4, 4$

c. 2 solutions.

One discriminant is less than zero and the other discriminant is greater than zero.
Therefore, $-4 < k < 0$ or $2 < k < 4$.

d. 1 solution.

One discriminant equals zero and the other is less than zero.
Therefore, $k = 0$ or $k = -2$.

e. No solutions.

Both discriminants are less than zero.
Therefore, $-2 < k < 0$.

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Question 82

Consider the polynomial $P(x) = x^3 + ax + b$.

Show that if $\Delta = -4a^3 - 27b^2 = 0$, that $P(x) = 0$ has less than 3 solutions.

Hint: If r_1, r_2, r_3 are the roots of $P(x)$, show that $\Delta = (r_1 - r_2)^2(r_2 - r_3)^2(r_3 - r_1)^2$.

Please use a calculator.

If $P(x) = 0$ has 3 distinct solutions, we can factorise $P(x) = (x - r_1)(x - r_2)(x - r_3)$, where r_1, r_2, r_3 are all different.
By expanding our factorised form, we can compare x^2 coefficients to see that $r_1 + r_2 + r_3 = 0$, hence $P(x) = (x - r_1)(x - r_2)(x + r_1 + r_2)$.
Since $r_3 = -r_2 - r_1$, we can express $(r_1 - r_2)^2(r_2 - r_3)^2(r_3 - r_1)^2$ in terms of r_2 and r_1. Similarly we can express a and b, and thus Δ in terms of r_2 and r_1. The following equations show that our expressions are equal.

$$\begin{aligned}
\Delta &= -4a^3 - 27b^2 \\
&= -4(-r_1^2 - r_1 r_2 - r_2^2)^3 - 27(r_1^2 r_2 + r_1 r_2^2)^2 \\
&= 4r_1^6 + 12r_2 r_1^5 - 3r_2^2 r_1^4 - 26r_2^3 r_1^3 - 3r_2^4 r_1^2 + 12r_2^5 r_1 + 4r_2^6 \\
&= (r_1 - r_2)^2(2r_1 + r_2)^2(r_1 + 2r_2)^2 \\
&= (r_1 - r_2)^2(r_2 - r_3)^2(r_3 - r_1)^2
\end{aligned}$$

Thus if $\Delta = 0$ we see that either $r_1 = r_2$ or $r_2 = r_3$ or $r_3 = r_1$, hence $P(x) = 0$ has less than 3 solutions.

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Sub-Section [1.6.3]: Apply Bisection Method to Approximate x-Intercepts

Question 83 CAS-Active.

Use the bisection method to find the approximate real solution to the equation $x^3 + 2x^2 - 5x + 3 = 0$. Use the interval $[-4, -3]$ for the first iteration and a maximum error of 0.1. Give your approximation correct to two decimal places.

Our answer is the midpoint of the first interval that has width < 0.2
$x \approx -3.61$.

```
ResourceFunction["BisectionMethodFindRoot"][x^3 + 2 x^2 - 5 x + 3,
 {x, -4, -3}, 4, 8, "Steps"]
```

steps	a	f[a]	b	f[b]
1	-4.000	-9.	-3.000	9.
2	-4.000	-9.	-3.500	2.125
3	-3.750	-2.85938	-3.500	2.125
4	-3.625	-0.228516	-3.500	2.125
5	-3.625	-0.228516	-3.563	0.982178
6	-3.625	-0.228516	-3.594	0.385406
7	-3.625	-0.228516	-3.609	0.0806007
8	-3.617	-0.0734172	-3.609	0.0806007

Question 84 CAS-Active.

Use the bisection method to find the approximate real solution to the equation $x\log_2(x) + 3x = 4$. Use the interval $[0.1, 2]$ for the first iteration and a maximum error of 0.01. Give your approximation correct to two decimal places.

Our answer is the midpoint of the first interval that has width < 0.02
$x \approx 1.22$

```
In[38]:= ResourceFunction["BisectionMethodFindRoot"][x * Log[2, x] + 3 x - 4,
 {x, 0.1, 2}, 8, 9, "Steps"]
```

Out[38]=

steps	a	f[a]	b	f[b]
1	0.10000000	-4.03219	2.0000000	4.
2	1.0500000	-0.776091	2.0000000	4.
3	1.0500000	-0.776091	1.5250000	1.50343
4	1.0500000	-0.776091	1.2875000	0.331887
5	1.1687500	-0.230821	1.2875000	0.331887
6	1.1687500	-0.230821	1.2281250	0.0484618
7	1.1984375	-0.0917099	1.2281250	0.0484618
8	1.2132813	-0.0217551	1.2281250	0.0484618
9	1.2132813	-0.0217551	1.2207031	0.0133208

Question 85 CAS-Active.

Use the bisection method to approximate π correct to three decimal places.

We choose a function f such that $f(0) = \pi$.
Such a function can be $f(x) = \sin(x)$
Our answer is the midpoint of the first interval that has width < 0.001
Since $3 < \pi < 4$ we can choose an interval of $[3, 4]$.
$x \approx 3.141$

```
ResourceFunction["BisectionMethodFindRoot"][Sin[x], {x, 3, 4}, 6, 12, "Steps"]
```

steps	a	f[a]	b	f[b]
1	4.00000	-0.756802	3.00000	0.14112
2	3.50000	-0.350783	3.00000	0.14112
3	3.25000	-0.108195	3.00000	0.14112
4	3.25000	-0.108195	3.12500	0.0165919
5	3.18750	-0.0458912	3.12500	0.0165919
6	3.15625	-0.0146568	3.12500	0.0165919
7	3.15625	-0.0146568	3.14063	0.000967653
8	3.14844	-0.00684479	3.14063	0.000967653
9	3.14453	-0.00293859	3.14063	0.000967653
10	3.14258	-0.000985471	3.14063	0.000967653
11	3.14160	-8.90891×10^{-6}	3.14063	0.000967653
12	3.14160	-8.90891×10^{-6}	3.14111	0.000479372

Question 86

Explain why you cannot use the bisection method to approximate the solution to the equation $x^4 - 2x^2 + 1 = 0$.

Because $f(x) = x^4 - 2x^2 + 1 = (x^2 - 1)^2 > 0$ we will not be able to find an initial interval $[a, b]$ such that $f(a)f(b) < 0$.

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Section G: [1.1 - 1.6] - Exam 1 Overall

Question 87

Let the coordinates of the point X be (a, b). Find the coordinates of X', which is the point on X reflected across the lines $x = 1$ and $y = -3$. Give your answer in terms of a and b.

$$\boxed{(2-a, -b-6)}$$

Question 88

Find the equation of the line that is parallel to $y = -3x - 4$ and passes through the point $(7, 5)$.

$$\boxed{y = 26 - 3x}$$

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Question 89

Solve the simultaneous linear equations:

$$\frac{2}{3}x + \frac{1}{2}y = 4,$$

$$\frac{5}{4}x - \frac{5}{4}y = -\frac{5}{4}.$$

$$\boxed{x = 3 \text{ and } y = 4}$$

Question 90

Consider the functions $f(x) = 2x + 3$ and $g(x) = (x + 2)^2$.

a. Find the vertical distance between f and g, when $x = 2$.

$$\boxed{g(2) - f(2) = 16 - 7 = 9}$$

b. Find the horizontal distance between f and g, when $y = 4$.

$f(x) = 4 \implies x = \frac{1}{2}$ and $g(x) = 4 \implies x = 0$

Therefore horizontal distance $= \frac{1}{2} - 0 = \frac{1}{2}$

c. Find the distance between the point $(2, 4)$ and $g(x)$, when $x = 14$.

$g(14) = (14+2)^2 = 16^2 = 256$

$\Rightarrow$ Between $(2,4)$ and $(14,256)$

$\text{dist} = \sqrt{(14-2)^2 + (256-4)^2}$

$= \sqrt{12^2 + 252^2} = 12\sqrt{442}$

Question 91

Consider the simultaneous linear equations:

$$\frac{m}{3}x - y = m,$$

$$4x + my = -7,$$

Where m is a real constant.

a. Find the values of m for which there is a unique solution to the simultaneous equations.

$m \in \mathbb{R}$

b. If possible, determine the value(s) of m for which there are infinitely many solutions.

No value of m for which there are infinitely many solutions.

c. If possible, determine the value(s) of m for which there are no solutions.

No value of m for which there are no solutions.

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Question 92

Cam is standing at the point $(1, 6)$ when a bus goes past him. The bus' path is described by the line $2y - 3x = 4$. Find the shortest distance between Cam and the bus.

$$\boxed{\frac{5}{\sqrt{13}} = \frac{5\sqrt{13}}{13}}$$

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Question 93 (4 marks)

a. For what values of x is $x^2 - 7x + 12 > 0$? (2 marks)

$\boxed{x < 3 \text{ or } x > 4}$

b. For what values of x is $1 - \frac{1}{x} - \frac{12}{x^2} > 0$? (2 marks)

$\boxed{x < -3 \text{ or } x > 4}$

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Question 94 (3 marks)

The sum of the age of a son and his father is 35 years and the product is 150. Find their ages.

$\boxed{5 \text{ and } 30.}$

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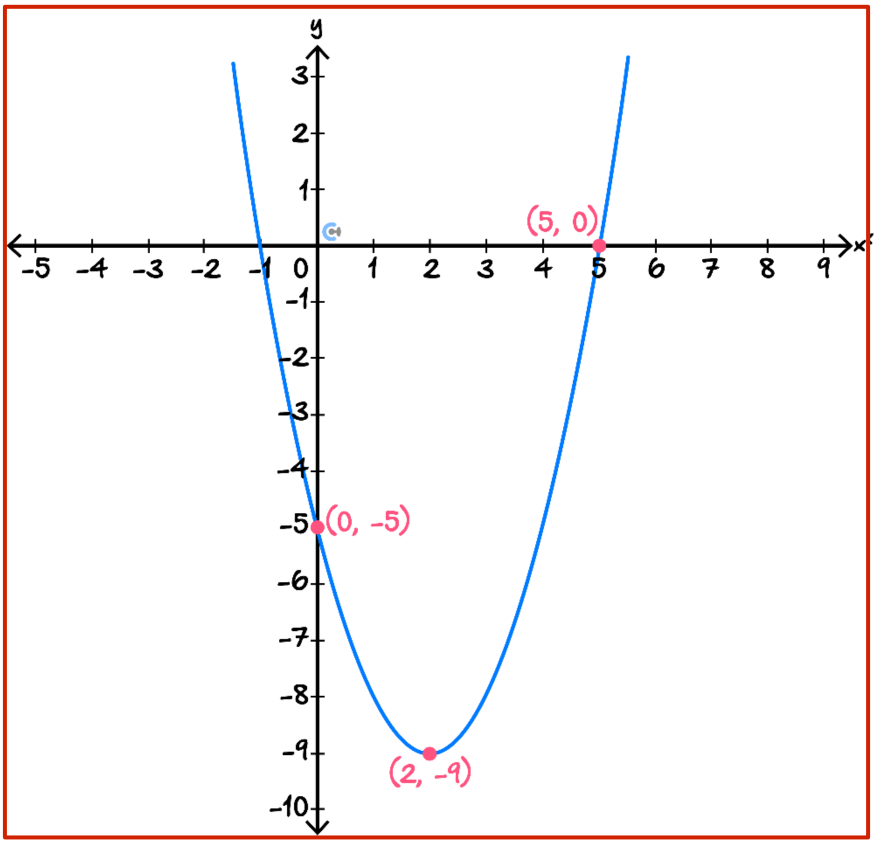
Question 95 (4 marks)

Consider the function $f(x) = x^2 - 4x - 5$.

a. Solve the equation $f(x) = 0$. (1 mark)

$$\boxed{x = -1, 5}$$

b. Sketch the graph of $y = f(x)$ on the axes below. Label the turning point and all axes intercept with coordinates. (2 marks)



c. Hence, find the value(s) of x such that $f(x) + 5 < 0$. (1 mark)

$$\boxed{0 < x < 4}$$

Question 96 (2 marks)

Solve the inequality $x^2 - 6x - 7 \leq 0$.

$$\boxed{-1 \leq x \leq 7}$$

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Question 97 (3 marks)

Consider the function $f(x) = kx^2 - 4x + 6$, where k is a real number. Find all possible values of k if $f(x)$ is always greater than 1.

$$\boxed{k > \frac{4}{5}}$$

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Question 98 (5 marks)

Consider the function $f(x) = x^2 - kx - 4$, where k is a real number.

a. Show that the graph $y = f(x)$ always has two x-intercepts. (1 mark)

Consider the discriminant for $x^2 - kx - 4 = 0$

$\Delta = k^2 + 16 > 0$

Therefore, must have two x-intercepts.

b. Find the values of k such that the distance between the two x-intercepts is less than 6. (3 marks)

$-2\sqrt{5} < k < 2\sqrt{5}$

c. Find the minimum possible distance between the two x-intercepts. (1 mark)

Minimum distance of 4 when $k = 0$.

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Question 99

Consider the polynomial $f(x) = x^3 - 7x + 6$.

a. Show that $f(1) = 0$.

$f(1) = 1^3 - 7 \times 1 + 6 = 0$

b. Solve $f(x) = 0$ for x.

Since $f(1) = 0$ we know that $x - 1$ is a factor of f.
We can then factorise f as such:

$$f(x) = x^3 - 7x + 6 = (x-1)(x^2 + x - 6) = (x-1)(x-2)(x-3)$$

Hence $x = -3, 1, 2$.

c. Hence, solve $f(x) \geq 0$ for x.

$-3 \leq x \leq 1$ or $x \geq 2$.

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Question 100

For what values of k does the equation $k(x^3 + x^2) = x$ have exactly one solution.

We can rewrite our equation to be $x(kx^2 + kx - 1) = 0$.
We see that if $k \neq 0$ we have a product of a quadratic and x, hence we require the quadratic to have no solution.
Hence, the discriminant of the quadratic $\Delta = k^2 + 4k < 0$.
From the graph of $k^2 + 4k$ we see that it is less than 0 if $-4 < k < 0$.
Now if $k = 0$ our equation turns to $x = 0$ which obviously has one solution.
Thus, our equation has exactly one solution of $-4 < k \leq 0$.

Question 101

Consider the polynomial $f(x) = x^3 - 3x^2 + x + 1$.

a. Fully factorise $f(x)$ into linear factors.

Since $f(1) = 0$ we know that $x - 1$ is a factor of f.
Hence $f(x) = (x-1)(x^2 - 2x - 1)$.
We can solve $x^2 - 2x - 1 = 0$ to extract the other linear factors.
The other linear factors are, $1 \pm \sqrt{2}$.
Hence $f(x) = (x-1)(x-1-\sqrt{2})(x-1+\sqrt{2})$

b. A bisection method is used to solve $f(x) = 0$ with the first interval being [2,3].
Use the fact that $\sqrt{2} \approx 1.4$ to write down the next 3 intervals.

The only root we can approximate with this method is $1 + \sqrt{2} \approx 2.4$.
Thus all of our intervals will have to contain 2.4.
Hence our first interval is $[2, 2.5]$, our second interval is $[2.25, 2.5]$ and our third interval is $[2.375, 2.5]$.

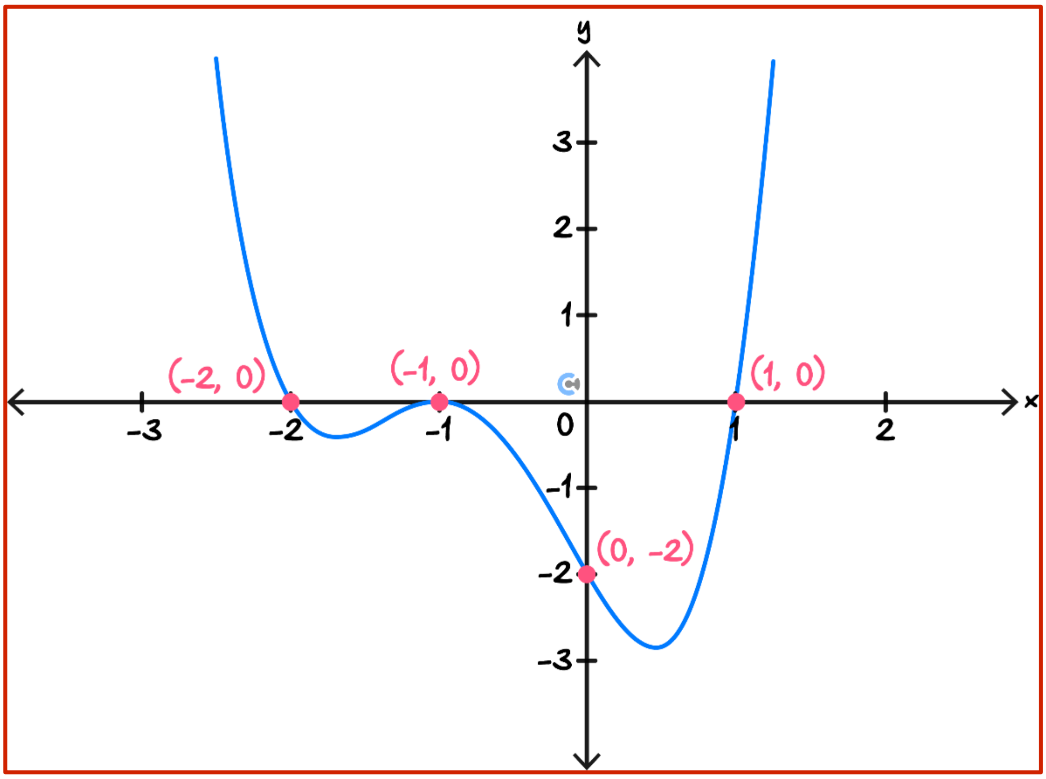
Question 102

Let $f(x) = x^4 + 3x^3 + x^2 - 3x - 2$.

a. Show that $x^2 - 1$ is a factor of f.

Since $f(1) = 1 + 3 + 1 - 3 - 2 = 0$ we know that $x - 1$ is a factor of f.
Since $f(-1) = 1 - 3 + 1 + 3 - 2 = 0$ we know that $x + 1$ is a factor of f.
Hence $x^2 - 1 = (x - 1)(x + 1)$ must be a factor of f.

b. Sketch the graph of $y = f(x)$ on the axis below. Label all axis intercepts with their coordinates. Note that some turning points occur at $(-1.69, -0.40)$ and $(0.44, -2.83)$.



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Question 103 (5 marks)

Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum.

a. Find the coordinates of P. (3 marks)

$OP = \sqrt{x^2 + (2x-4)^2}$
$d = \sqrt{5x^2 - 16x + 16}$.
We want to minimise d.
Can do this by finding the smallest value of $f(x) = 5x^2 - 16x + 16$.

$$\begin{aligned} f(x) &= 5\left(x - \frac{8}{5}\right)^2 + \left(16 - 5 \times \frac{64}{25}\right) \\ &= 5\left(x - \frac{8}{5}\right)^2 + \frac{80}{5} - \frac{64}{5} \\ &= 5\left(x - \frac{8}{5}\right)^2 + \frac{16}{5} \end{aligned}$$

so $f(x)$ has a minimum value of $\frac{16}{5}$ when $x = \frac{8}{5}$.
$y = 2 \times \frac{8}{5} - \frac{20}{5} = -\frac{4}{5}$.
So P is the point $\left(\frac{8}{5}, -\frac{4}{5}\right)$.

b. Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers. (2 marks)

$$d = \sqrt{\left(0 - \frac{8}{5}\right)^2 + \left(0 + \frac{4}{5}\right)^2}$$
$$= \frac{4\sqrt{5}}{5}$$

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Question 104

Consider the simultaneous linear equations:

$$kx - 3y = k + 3$$

$$4x + (k + 7)y = 1$$

Where k is a real constant.

a. Find the value of k for which there are infinitely many solutions.

Marks	0	1	2	3	Average
%	43	22	10	25	1.2

The following are three different methods that can be used to identify and justify $k = -4$.

Method 1: Using gradient, m, and y-intercept, c.

$m_1 = \frac{k}{3}, c_1 = -\left(\frac{k+3}{3}\right)$

$m_2 = \frac{-4}{k+7}, c_2 = \frac{1}{k+7}$

from the gradient $k(k+7) = -12$ so

$k = -4$ or $k = -3$

from the intercepts $k = -3 \Rightarrow c_1 = 0$ and $c_2 = \frac{1}{4}$ so there are no solutions in this case

$k = -4 \Rightarrow c_1 = c_2 = \frac{1}{3}$ so there are infinitely many solutions in this case

Answer: $k = -4$

Method 2: Using ratios

Comparison of coefficients as common ratios (all ratios would require the same value of k to have infinite solutions).

$\frac{k}{4} = \frac{-3}{k+7} = \frac{k+3}{1}$

Using $\frac{k}{4} = \frac{-3}{k+7}$

$k^2 + 7k = -12 \Rightarrow k = -4$ or $k = -3$

Using $\frac{-3}{k+7} = \frac{k+3}{1}$

$k^2 + 10k + 21 = -3 \Rightarrow k = -4$ or $k = -6$

Using $\frac{k}{4} = \frac{k+3}{1} \Rightarrow k = -4$

$k = -4$ satisfies all 3 ratios and the lines will be identical.

Answer: $k = -4$

Method 3: Using determinant

determinant $= k(k+7) + 12$

determinant $= 0$ for infinitely many solutions or no solutions

$k^2 + 7k + 12 = (k+4)(k+3) = 0$

$\Rightarrow k = -4$ or $k = -3$

when $k = -3$ the two equations become

$-3x - 3y = 0$ and $4x + 4y = 1$ so there are no solutions in this case

when $k = -4$ the two equations become

$-4x - 3y = -1$ and $4x + 3y = 1$ so there are infinitely many solutions in this case

Answer: $k = -4$

Many students who tried to solve simultaneous equations rarely succeeded. Other variations of these methods were seen. Whichever method was chosen, students had to justify their selection of k.

b. Find the values of k for which there is a unique solution.

Marks	0	1	Average
%	67	33	0.4

For unique solutions

$k^2 - 7k + 12 \neq 0$

$\therefore k \in R \setminus \{-4, -3\}$

Most students were aware of the two values of k that led to either no or infinite solutions. The most successful approach was using the determinant. Many students chose a value of k that was not selected as the answer to **part a**.

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Question 105 (4 marks)

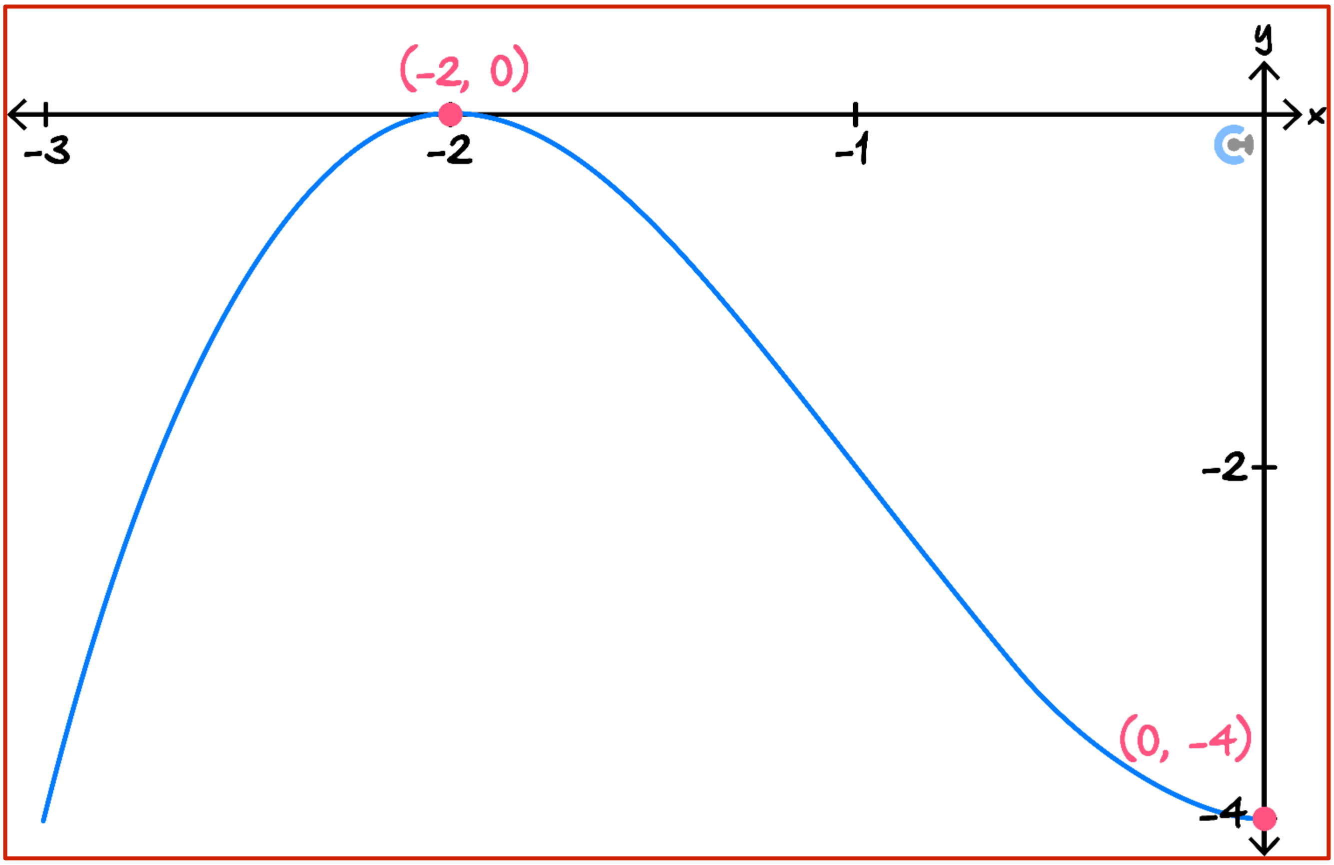
Let $f: [-3,0] \to R, f(x) = (x+2)^2(x-1)$.

a. Show that $(x+2)^2(x-1) = x^3 + 3x^2 - 4$.

$(x+2)^2(x-1)$
$= (x^2+4x+4)(x-1)$
$= x^3 + 4x^2 + 4x - x^2 - 4x - 4$
$= x^3 + 3x^2 - 4$

This question was answered well, although some students either did not fully expand the cubic or made notational errors by omitting the brackets on the quadratic. It should be noted that $x^2 + 4x + 4(x-1)$ is not equivalent to $x^3 + 4x^2 + 4x - x^2 - 4x - 4$.

b. Sketch the graph of f on the axes below. Label the axis intercept and any stationary points with their coordinates.



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