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VCE Mathematical Methods ½
Functions & Relations II [0.8]
Workshop Solutions

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Section A: Recap

Set Operators

- Intersection: "AND".

$$A \cap B = \textit{What values are in set A AND in set B.}$$

- Union: "OR".

$$A \cup B = \textit{What values are in set A OR in set B.}$$

- Set difference: "Except".

$$A \setminus B = \textit{What values are in set A except those also in set B.}$$

Interval Notation

- Parentheses (non-inclusive):

$$x \in (a, b) \Rightarrow a < x < b$$

- Square brackets [inclusive]:

$$x \in [a, b] \Rightarrow a \leq x \leq b$$

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Definition

Maximal Domain

- The maximal domain is the biggest possible domain for a rule without committing a mathematical crime.
- In Methods, we need to consider 3 important rules:

$$\sqrt{z}, \qquad z \geq 0$$

$$\log(z), \qquad z > 0$$

$$\frac{1}{z}, \qquad z \neq 0$$

Range

- The range is the possible value for the output of a function.

Definition

Functional Notation

$$f: Domain \to Codomain, f(x) = Rule$$

- Codomain is simply all the values the function works within.
- Codomain is **not** the same as range.

Definition

Piecewise (Hybrid) Functions

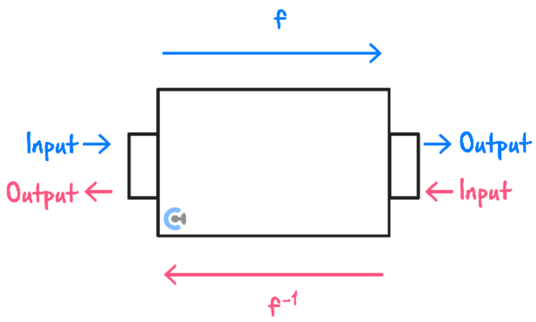
- Series of functions.

$$h(x) = \begin{cases} f(x), & Domain_1 \\ g(x), & Domain_2 \end{cases}$$

- $Domain_1$ and $Domain_2$ represent the x-values for which the two functions are defined.
- The two domains do not have to join!

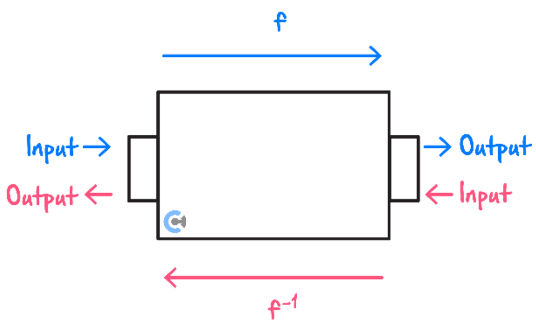
Inverse Relation

- **Definition**: Inverse is a relation that does the opposite.

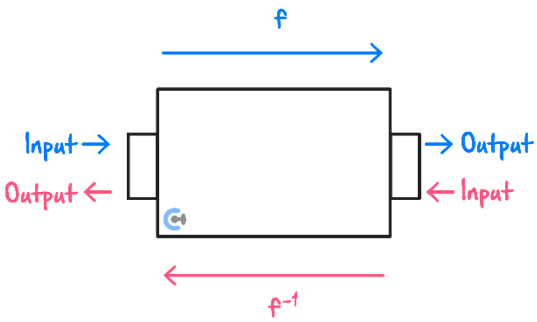


Solving for an Inverse Relation

- Swap x and y.



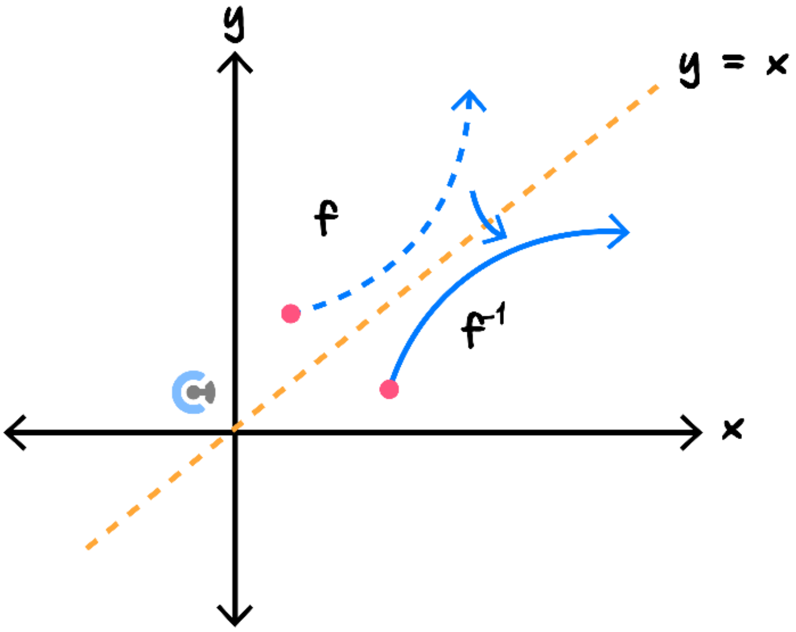
Domain and Range of Inverse Functions



$$\textbf{Dom } f^{-1} = \textbf{Ran } f$$

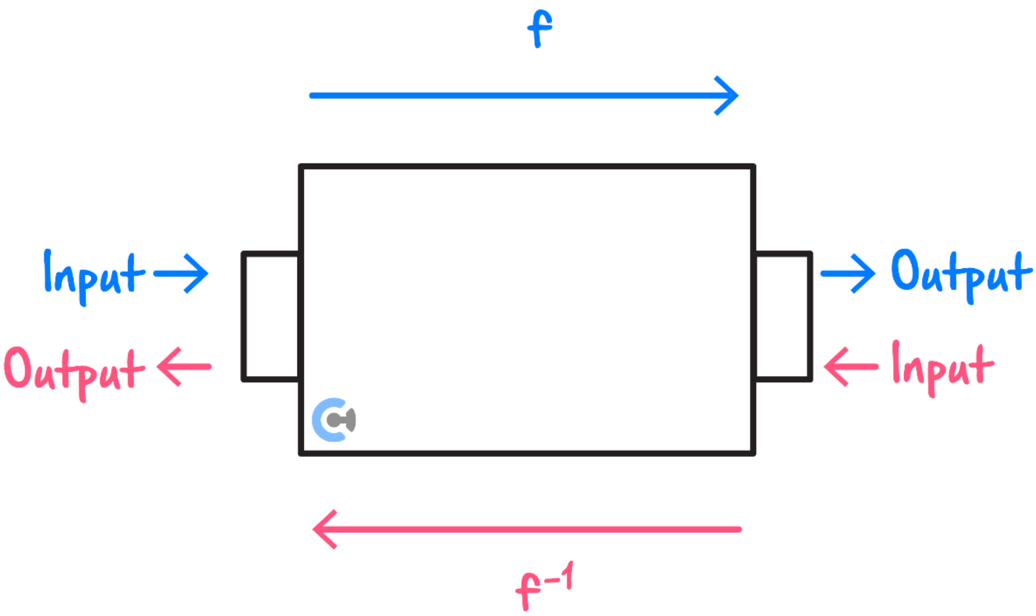
$$\textbf{Ran } f^{-1} = \textbf{Dom } f$$

Symmetry of Inverse Functions



- Inverse functions are always symmetrical around $y = x$.

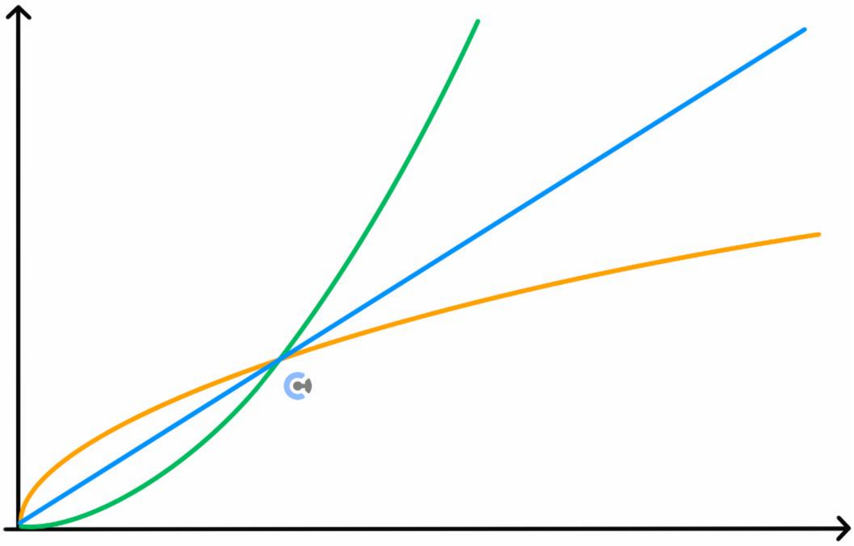
Validity of Inverse Functions



- Requirement for Inverse Function:

$$f \text{ needs to be } 1{:}1.$$

Intersection between a Function and its Inverse



$f(x) = x$ OR $f^{-1}(x) = x$

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Section B: Warm Up

Question 1

For the sets $A = [-3,5]$ and $B = (-6,2]$, expressing the following in interval notation:

a. $A \cap B$.

$\boxed{[-3, 2]}$

b. $A \cup B$.

$\boxed{(-6, 5]}$

c. $A \backslash B$.

$\boxed{(2, 5]}$

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Question 2

a. Find the maximal domain of $f(x) = \sqrt{(x-3)(x+1)}$ expressing your answer in interval notation.

$\boxed{x \in (-\infty, 1] \cup [3, \infty)}$

b. Find the range of the function $f: [-2, 3) \to \mathbb{R}, f(x) = x^2 - 4$.

$\boxed{[-4, 5)}$

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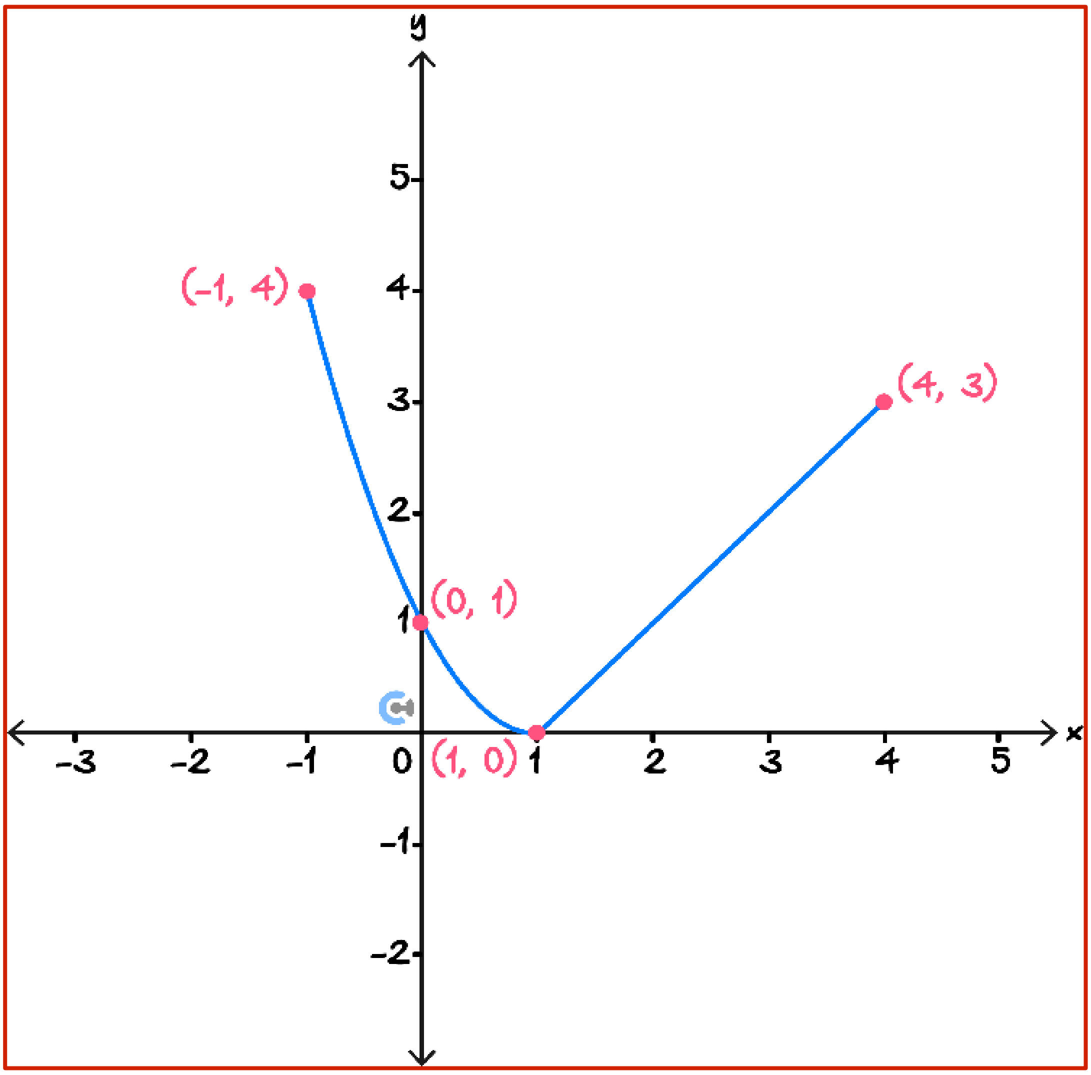
Question 3

Consider the hybrid function f, where:

$$f(x) = \begin{cases} (x-1)^2, & -1 \leq x \leq 1 \\ x-1, & 1 < x < 4 \end{cases}$$

a. Sketch the graph of $y = f(x)$ on the axes below. Label endpoints and axes intercept with coordinates.



b. Hence, state the range of f.

$\boxed{[0, 4)}$

c. State how many solutions $f(x) = 3$ has.

$\boxed{2.}$

Question 4

Consider the function $f: [-2, 4] \to \mathbb{R}, f(x) = 2x - 2$.

a. Find the rule for the inverse function, f^{-1}.

$$\boxed{f^{-1}(x) = \frac{x+2}{2}}$$

b. State the domain and range of f^{-1}.

Solution: dom f^{-1} = ran $f = [-6, 6]$
ran f^{-1} = dom $f = [-2, 4]$

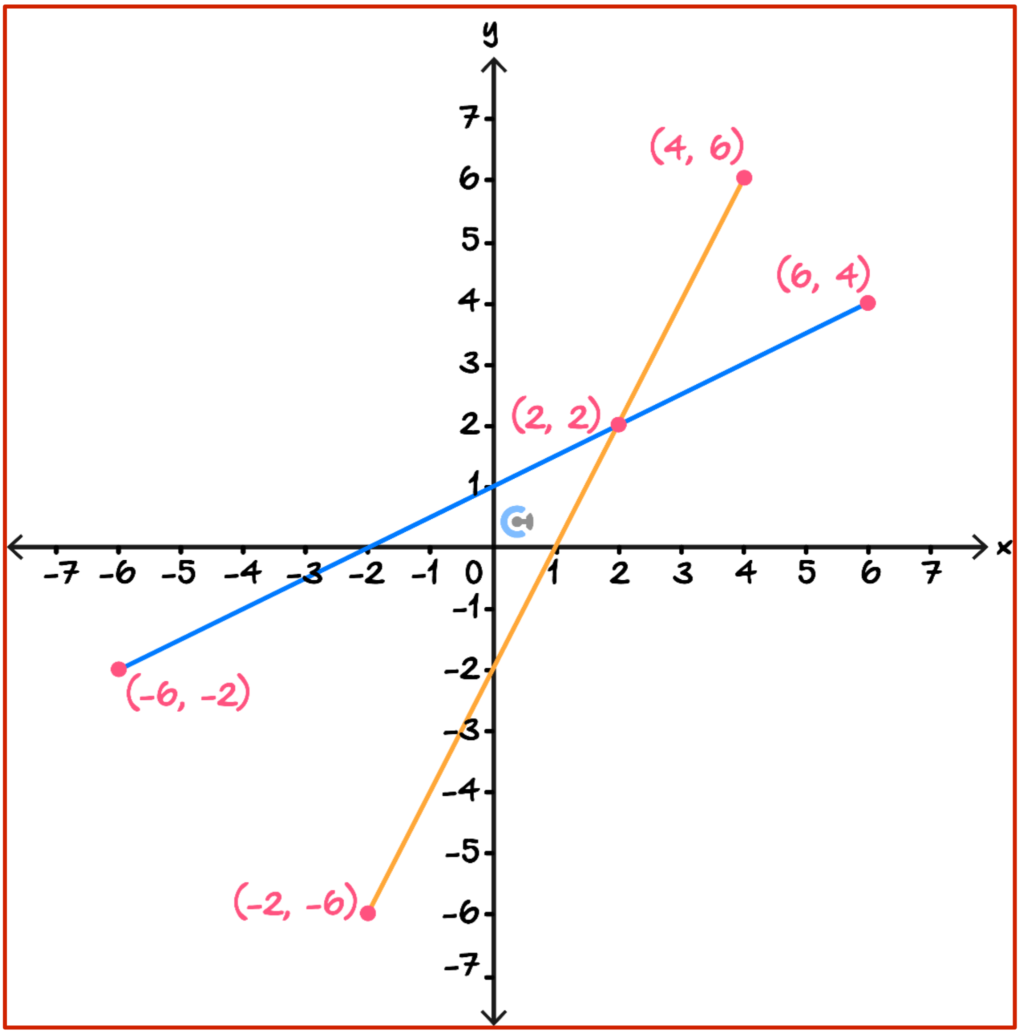
c. Hence, fully define the inverse function using functional notation.

$$\boxed{f^{-1} : [-6, 6] \to \mathbb{R},\ f^{-1}(x) = \frac{1}{2}x + 1}$$

d. Find the point of intersection between f and f^{-1}.

Solve $2x - 2 = x \implies x = 2$. Point of intersection is $(2, 2)$

e. Sketch the graphs of f and f^{-1} on the axes below.



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Section C: Exam 1 Questions (21 Marks)

INSTRUCTION: 21 Marks. 27 Minutes Writing.

Question 5 (2 marks)

Consider the function $f: (-\infty, a] \cup [b, \infty) \to R, f(x) = \sqrt{x^2 - 3}$.

Find all possible values of a and b such that $f(x)$ is defined.

$\boxed{a \leq -\sqrt{3} \text{ and } b \geq \sqrt{3}.}$

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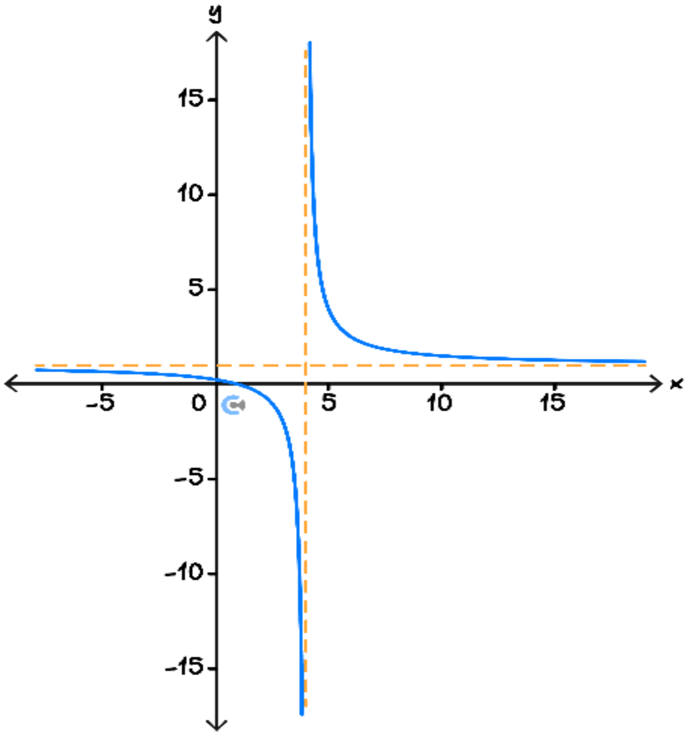
Question 6 (4 marks)

The figure below shows a graph of $y = \frac{3}{x-4} + 1, x \neq 4$.



a. State the equations for the horizontal and vertical asymptotes of the curve, marked as dotted lines in the figure. (1 mark)

Horizontal: $y = 1$. Vertical: $x = 4$.

The function f is defined as $f: (1, \infty)\backslash\{4\} \to \mathbb{R}, f(x) = \frac{3}{x-4} + 1$.

b. State the range of $f(x)$. (1 mark)

$(-\infty, 0] \cup (1, \infty)$.

c. Obtain an expression for $f^{-1}(x)$. (1 mark)

$$f^{-1}(x) = \frac{3}{x-1} + 4 = \frac{4x-1}{x-1}$$

d. State the domain and range of $f^{-1}(x)$. (1 mark)

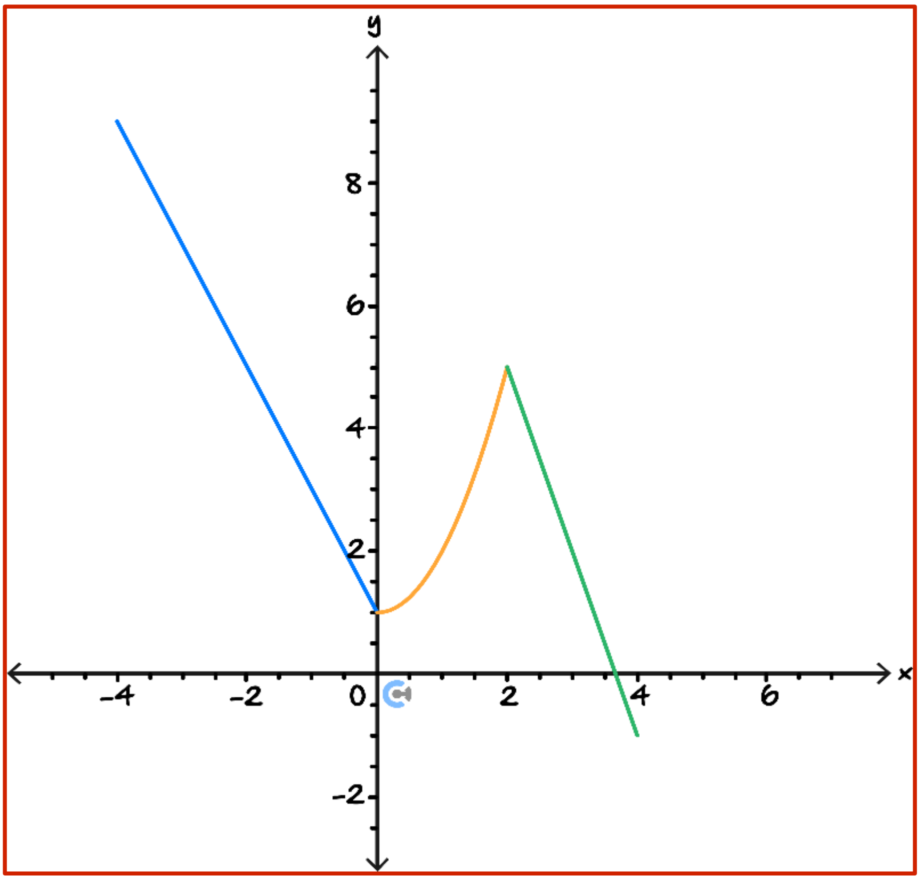
Domain: $(-\infty, 0] \cup (1, \infty)$ and Range: $(1, \infty) \backslash \{4\}$.

Question 7 (8 marks)

A function f has the definition:

$$f(x) = \begin{cases} -2x + 1, & -4 < x < 0 \\ x^2 + 1, & 0 \le x < 2 \\ 11 - 3x, & 2 \le x \le 4 \end{cases}$$

a. Draw the graph of $y = f(x)$ on the axes below. (3 marks)



b. Explain why f does not have an inverse function. (1 mark)

f is not a $1:1$ one function. It does not pass the horizontal line test.

c. How many solutions are there to $f(x) = 0$? (1 mark)

$\boxed{1}$

d. State the range of $f(x)$. (1 mark)

$\boxed{\text{Ran: } [-1,9)}$

e. Solve the equation $f(x) = 4$ for x. (2 marks)

$\boxed{x = \frac{-3}{2}, \sqrt{3}, \frac{7}{3}}$

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Question 8 (4 marks)

Consider the function $f(x) = \sqrt{3x-7} + 4$, where f is defined over its maximal domain.

a. Find the domain and the rule for the inverse function f^{-1}. (2 marks)

$$\boxed{f^{-1}(x) = \frac{1}{3}\left((x-4)^2 + 7\right), dom: x \geq 4}$$

b. Find an intersection between $f(x)$ and $f^{-1}(x)$. (2 marks)

$$\boxed{\left(\frac{\sqrt{29}+11}{2}, \frac{\sqrt{29}+11}{2}\right)}$$

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Question 9 (3 marks)

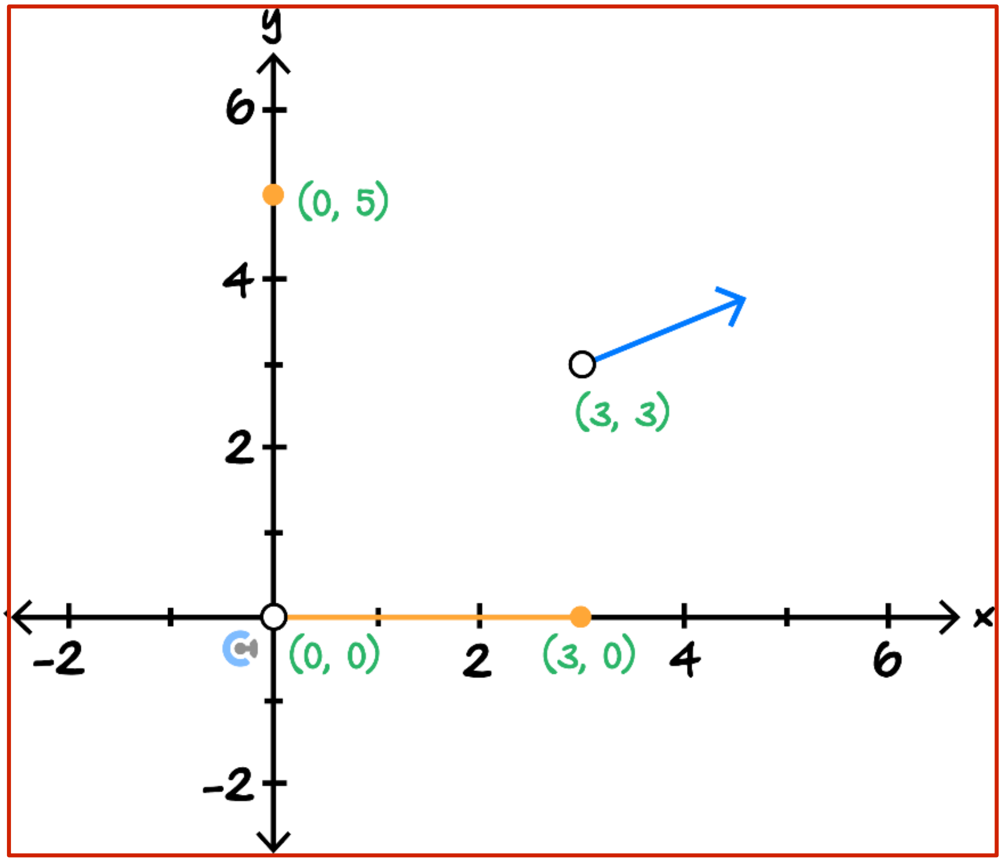
In an effort to reduce the time her children spend in the shower, a mother introduced a penalty scheme with fines to be paid from the children's pocket money according to the following:

If someone spends more than 3 minutes in the shower, the fine in dollars is equal to the shower time in minutes; if someone spends up to and including 3 minutes in the shower, there is no fine. If someone chooses not to shower at all, there is a fine of $5 because that child won't be nice to be near.

Define appropriate symbols, express the penalty scheme as a mathematical rule in hybrid form, and sketch the graph that represents it.

Let the time in the shower be t minutes and the dollar amount be the fine C.

The rule is $C = \begin{cases} 5, & t = 0 \\ 0, & 0 < t \leq 3 \\ t, & t > 3 \end{cases}$



Section D: Tech Active Exam Skills

Calculator Commands: Finding Maximal Domain

Mathematica

```
FunctionDomain[func, x]
```

TI-Nspire

- Type up the domain (or find it under the book button).

domain(func,x)

Casio Classpad

- Sketch the function and analyse.

Calculator Commands: Defining Hybrid Functions on CAS

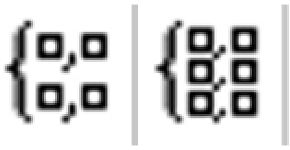
Mathematica

- Piecewise

Piecewise[{{val_1, $cond_1$}, {val_2, $cond_2$}, …}]

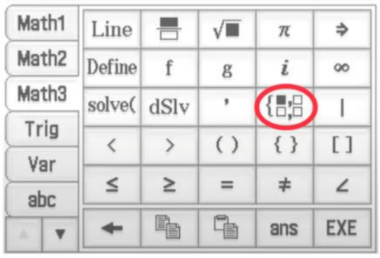
Represents a piecewise function with values val_i in the regions defined by the conditions $cond_i$.

TI-Nspire



$$\begin{cases} func1, dom1 \\ func2, dom2 \end{cases}$$

Casio Classpad



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Calculator Commands: Finding the Equation of a Polynomial that Passes Through Points

- Given n points we can find a degree $n-1$ polynomial that passes through all of these points.
- **Example:** Find the equation of the quadratic function that passes through the points $(0,6)$, $(2,2)$, and $(3,3)$.
- **TI:**

Define $f(x)=a\cdot x^2+b\cdot x+c$	*Done*
solve$(f(0)=6$ and $f(2)=2$ and $f(3)=3,a,b,c)$	$a=1$ and $b=^-4$ and $c=6$
$f(x)\|a=1$ and $b=^-4$ and $c=6$	$x^2-4\cdot x+6$

- **Casio:**

```
define f(x) = a*x^2 + b*x + c
                                        done
{f(0)=6
{f(2)=2
{f(3)=3 |a,b,c
                                        {a=1,b=-4,c=6}
f(x)|{a=1,b=-4,c=6}
                                        x^2-4·x+6
```

- **Mathematica:**

```
In[9]:= f[x_] := a x^2 + b x + c

In[10]:= Solve[f[0] == 6 && f[2] == 2 && f[3] == 3]
Out[10]= {{a → 1, b → -4, c → 6}}

In[11]:= f[x] /. {a → 1, b → -4, c → 6}
Out[11]= 6 - 4 x + x^2
```

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Section E: Exam 2 Questions (21 Marks)

INSTRUCTION: 21 Marks. 27 Minutes Writing.

Question 10 (1 mark)

The domain of the inverse of $\{(3,-2),(4,-7),(6,-9),(7,-11)\}$ is D. Which of the following statements is true?

A. D is $\{x: -3 < x < 7\}$

B. D is $\{x: 3 < x < 7\}$

C. D is $\{-11,-9,-7,-2\}$

D. D is $\{3,4,6,7\}$

Question 11 (1 mark)

Which of the following does not have an inverse function?

A. $f: R \to R, f(x) = 4x - 1$

B. $f: [2,\infty) \to R, f(x) = 2(x-2)^2$

C. $g: [-4,4] \to R, g(x) = \sqrt{16 - x^2}$

D. $f: R \to R, f(x) = \frac{4}{x-3} + 1$

Question 12 (1 mark)

The data $(1,-5)$, $(2,4)$, $(4,6.25)$ can be modelled by the equation $y = a - \frac{b}{x^2}$. The values of a and b respectively are:

A. 7 and 12.

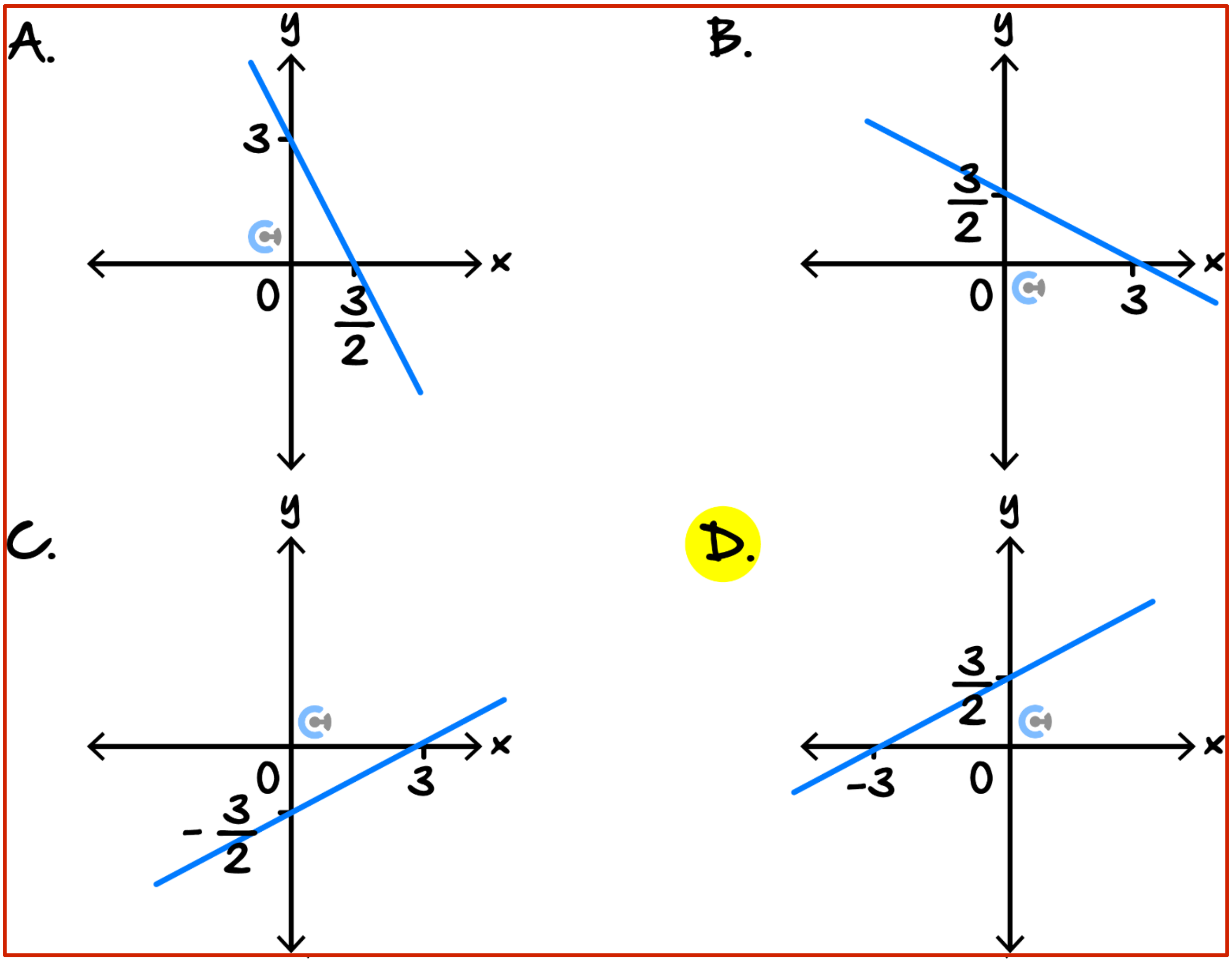
B. 12 and 7.

C. -7 and -12.

D. 7 and -12.

Question 13 (1 mark)

A sketch of the inverse of $f(x) = 2x - 3$ is:



Question 14 (1 mark)

The domain and the range for the graph with the equation $5 - y = -\frac{5}{(x-5)^2}$ respectively are:

A. $\{x: x \in \mathbb{R}\backslash\{5\}\}$ and $\{y: y < 5\}$.

B. $\{x: x \in \mathbb{R}\backslash\{5\}\}$ and $\{y: y > 5\}$.

C. $\{x: x \in \mathbb{R}\backslash\{5\}\}$ and $\{y: y < -5\}$.

D. $\{x: x \in \mathbb{R}\backslash\{5\}\}$ and $\{y: y > -5\}$.

Question 15 (1 mark)

Which of the following has an inverse which is a function?

A. $x^2 + y^2 = 4$

B. $y = \frac{12}{2x-1} - 3$

C. $y = \sqrt{6 - x^2}$

D. $y = 1$

Question 16 (1 mark)

The maximal domain of $y = \frac{-3x+6}{\sqrt{4x-7}}$ is:

A. $\mathbb{R}\backslash\left\{\frac{7}{4}\right\}$.

B. $\mathbb{R}\backslash\left\{\frac{7}{4}, 2\right\}$.

C. $\left[\frac{7}{4}, \infty\right)$.

D. $\left(\frac{7}{4}, \infty\right)$.

Question 17 (1 mark)

The graph of $y = x^2 - ax$ has a range of $[-4, \infty)$, where a is a positive constant. The value of a is:

A. 1.

B. 2.

C. 8.

D. 4.

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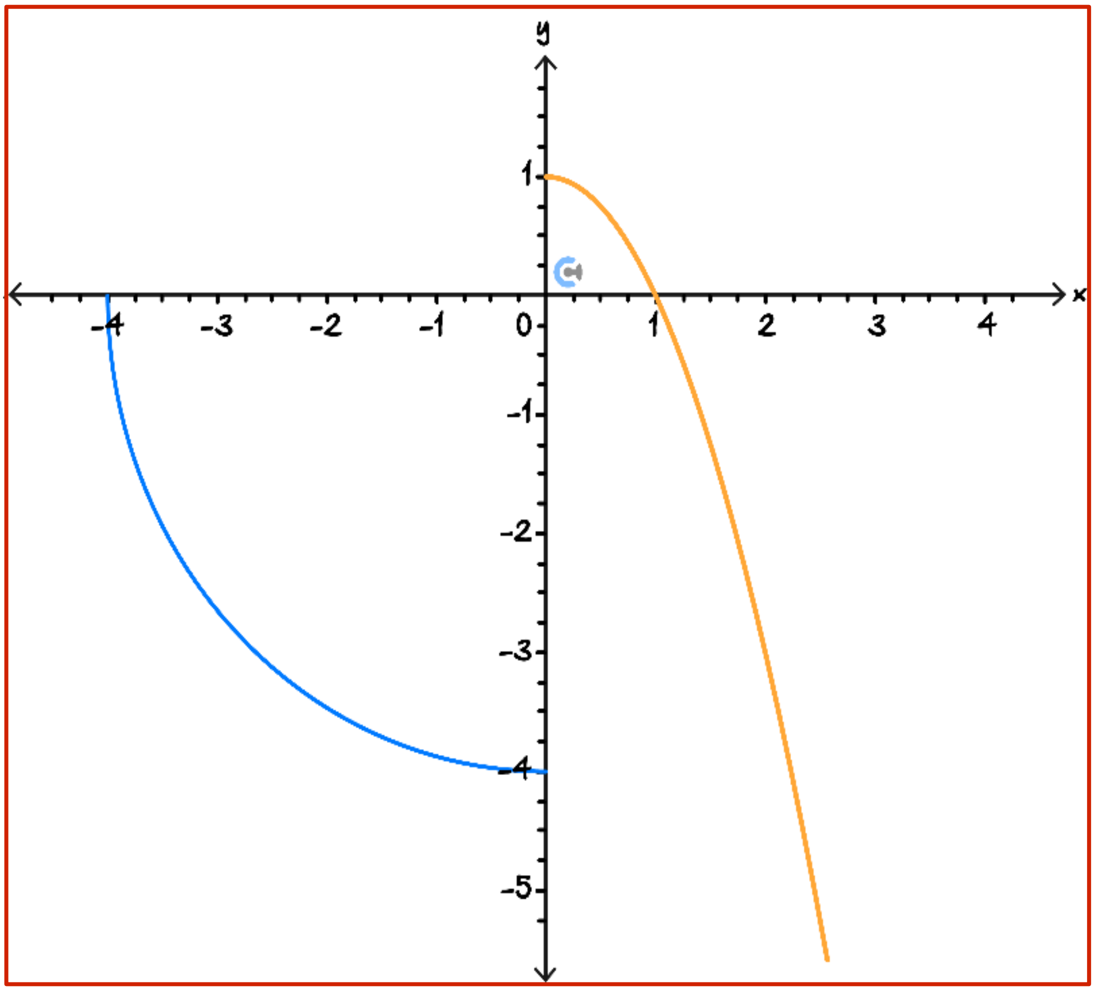
Question 18 (8 marks)

a. Sketch the graph of $y = g(x)$. (2 marks)

$$g(x) = \begin{cases} -\sqrt{16 - x^2}, & -4 \le x \le 0 \\ -x^2 + 1, & x > 0 \end{cases}$$



b. State the range of g. (1 mark)

Ran: (-infinity,1]

c. Solve for x:

i. $g(x) = 0$. (1 mark)

X= -4, 1

ii. $g(x) = -1$. (1 mark)

x = $-\sqrt{15}, \sqrt{2}$

d. Find k if $g(x) = k$ has:

i. 0 solutions. (1 mark)

$k \geq 1$

ii. 1 solution. (1 mark)

$k \in (-4, \infty) \cup (0,1)$

iii. 2 solutions. (1 mark)

$k \in [-4,0]$

Question 19 (5 marks)

If a rock falls from a height of 80 metres towards the surface of the Earth, the height, H (in metres) after t seconds is approximately $H(t) = 80 - \frac{7}{12}t^2$.

a. In general, quadratic functions are not one-to-one. However, the function H is one-to-one under its implied domain. Why? (1 mark)

Quadratic equations are U-shaped and if you draw a horizontal line across the graph, the line will intersect the graph twice. This would make the function fail what we call the "Horizontal Line" test. In this case, this problem is dealing with a real-world situation. The real world happens in Quadrant I. So, it is assumed that the domains (the x-values which are t-values here) are restricted to $t \geq 0$. With this restriction, the function H is one-to-one.

b. Find the inverse of H, stating its domain and range given the scenario of the question. (3 marks)

$$H^{-1}(t) = 2\sqrt{\frac{3}{7}} \times \sqrt{80 - \mathrm{t}} = \frac{2\sqrt{21(80-t)}}{7}$$

$$\operatorname{ran}(H^{-1}) = \operatorname{dom}(H) = 0 \leq t \leq \frac{8\sqrt{105}}{7}$$

$$\operatorname{dom}(H^{-1}) = \operatorname{ran}(H) = 0 \leq h \leq 80$$

c. Find how long it will take for the rock to fall 60 metres to 2 decimal places. (1 mark)

$$H(t) = 60$$

$$t = \frac{4\sqrt{105}}{7}$$

$$t = 5.86s$$

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Section F: Extension Exam 1 (11 Marks)

INSTRUCTION: 11 Marks. 11 Minutes Writing.

Question 20 (11 marks)

Consider the function $f(x) = \sqrt{6x - x^2} + 1$.

a. Write $f(x)$ in the form $\sqrt{r^2 - (x-h)^2} + k$, and state the values of **positive** integers, r, h, and k. (1 mark)

$f(x) = \sqrt{9 - (x-3)^2} + 1$, and so $r = 3, h = 3$ and $k = 1$.

The function f has its domain restricted to $[a, 6]$ so that the inverse function f^{-1} exists.

b. State the smallest possible value of a. (1 mark)

$a = 3$

c. Hence, define the inverse function f^{-1}. (3 marks)

Solution: ran $f = [1,4] = \text{dom } f^{-1}$ and dom $f = [3,6] = \text{ran } f^{-1}$.
$x = \sqrt{9-(y-3)^2} + 1 \implies (x-1)^2 = 9 - (y-3)^2 \implies (y-3)^2 = 9 - (x-1)^2 \implies$
$y = \sqrt{9-(x-1)^2} + 3$ by considering the range.

$$f^{-1} : [1,4] \to \mathbb{R},\ f^{-1}(x) = \sqrt{9-(x-1)^2} + 3$$

d. Find the point of intersection between f and f^{-1}. (2 marks)

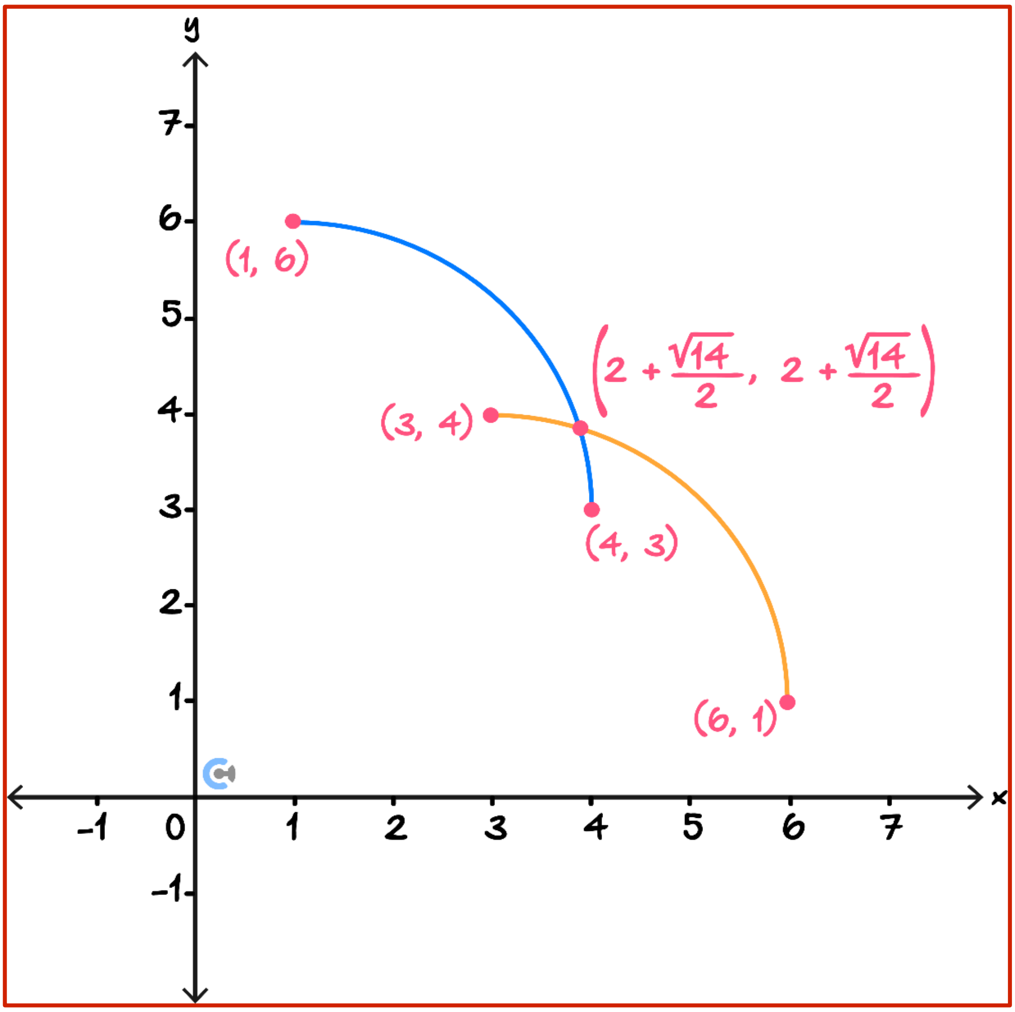
Solution: Solve $x = \sqrt{6x - x^2} + 1$.

$$\begin{aligned}(x-1)^2 &= 6x - x^2\\ x^2 - 2x + 1 &= 6x - x^2\\ 2x^2 - 8x + 1 &= 0\end{aligned}$$

$$x = \frac{8 \pm \sqrt{64-8}}{4} = \frac{8 \pm 2\sqrt{14}}{4} = 2 \pm \frac{\sqrt{14}}{2}$$

Only point of intersection is $\left(2 + \frac{\sqrt{14}}{2}, 2 + \frac{\sqrt{14}}{2}\right)$

e. Sketch the graphs of f and f^{-1} on the axes below. Label all endpoints and points of intersection with coordinates. (2 marks)



f. Consider all functions of the form $g: [0, r] \to \mathbb{R}, g(x) = \sqrt{r^2 - x^2}$ where $r > 0$. State the x-values for all points of intersection of g and g^{-1}. (2 marks)

Note that semi-circles of this form are their own inverse. Thus $x \in [0, r]$.

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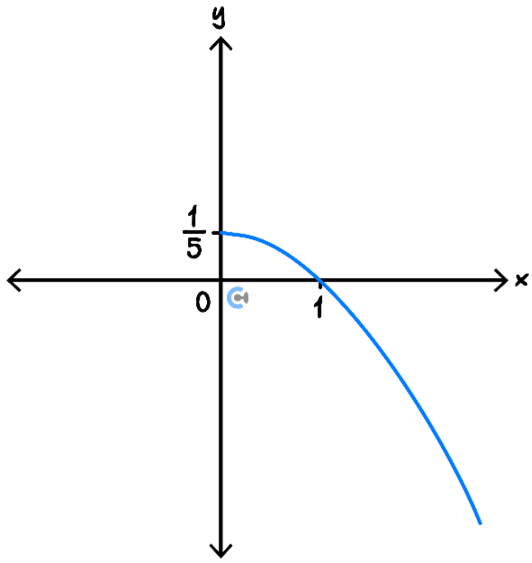
Section G: Extension Exam 2 (13 Marks)

INSTRUCTION: 13 Marks. 13 Minutes Writing.

Question 21 (1 mark)



The graph above represents the inverse of:

A. $f(x) = \sqrt{5 - x}$.

B. $f(x) = \frac{1}{5}\sqrt{1 - x}$.

C. $f(x) = \sqrt{1 - 5x}$.

D. $f(x) = \sqrt{5x - 1}$.

Question 22 (1 mark)

Which set of ordered pairs represents a function?

A. $\{(1,7), (2,6), (4,3), (4,4), (12,6)\}$

B. $\{(2,4), (2,5), (4,6), (4,7), (4,8)\}$

C. $\{(0,4), (1,4), (2,4), (3,4), (4,4)\}$

D. $\{(0,2), (0,3), (2,4), (3,5), (4,6)\}$

Question 23 (1 mark)

The maximal domain of

$$y = \frac{4x+3}{\sqrt{x^2-2x-8}}$$

is:

A. $x \in [-2,4]$.

B. $x \in (-\infty,-2] \cup [4,\infty)$.

C. $x \in R \backslash [-2,4]$.

D. $x \in R \backslash (-2,4)$.

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Question 24 (10 marks)

Let $f(x) = \sqrt{x^2 + 8x - 20}$.

a. Determine the maximal domain of f. (1 mark)

$\boxed{x \in (-\infty, -10] \cup [2, \infty)}$

Let $g: [2, \infty) \to \mathbb{R}, g(x) = f(x)$.

b. What type of function is g? (1 mark)

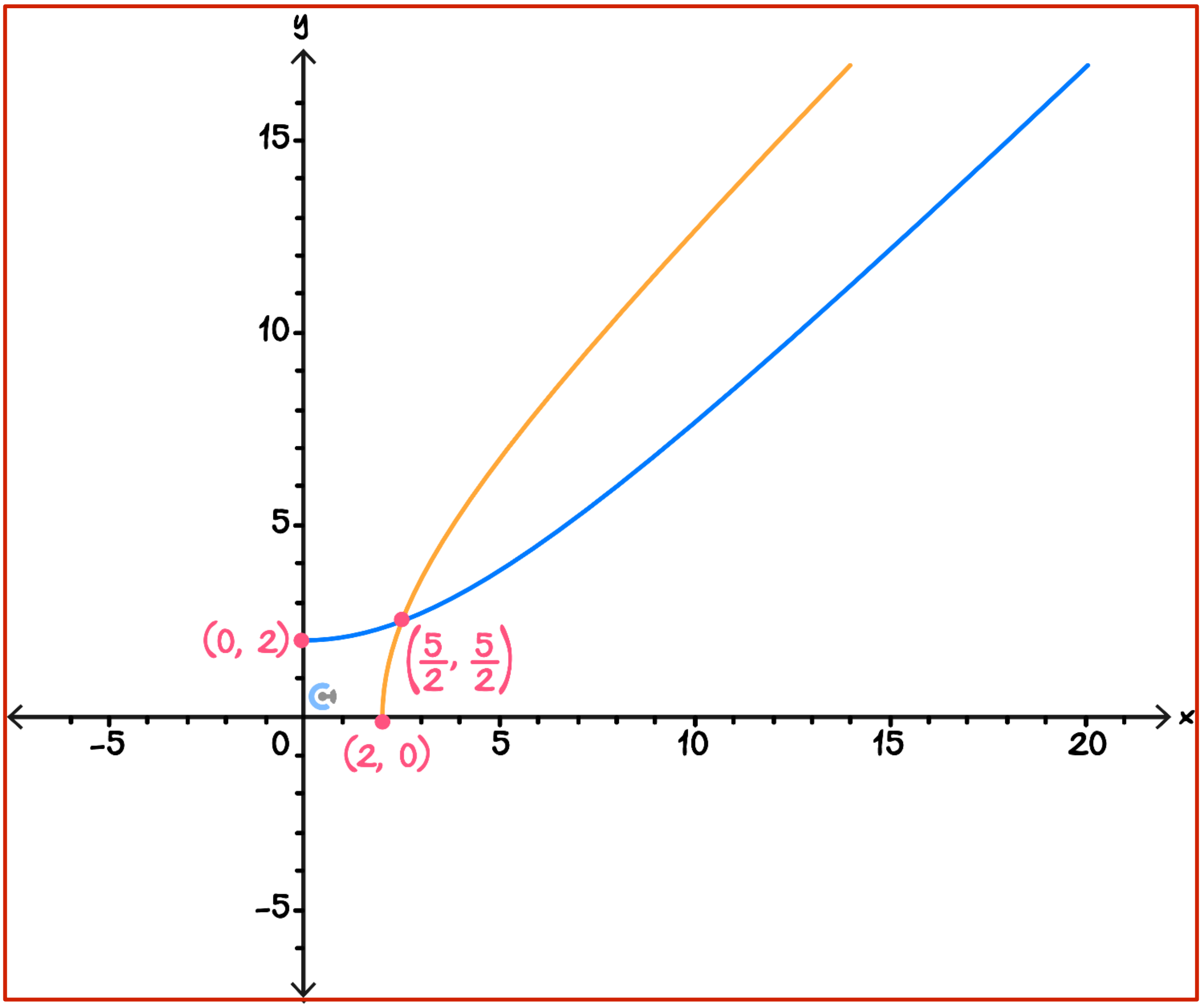
$\boxed{g \text{ is a one-to-one function.}}$

c. Define g^{-1}, the inverse function of g. (2 marks)

Solution: Solve $x = \sqrt{y^2 + 8y - 20} \implies x^2 = (y+4)^2 - 36 \implies (y+4)^2 = x^2 + 36$
Now dom $g = [2, \infty)$ and ran $g = [0, \infty)$, therefore

$$g^{-1} : [0, \infty) \to \mathbb{R},\ g(x) = \sqrt{x^2 + 36} - 4$$

d. Sketch the graph of g and g^{-1} on the axes below. Label all axes intercepts and points of intersection with coordinates. (3 marks)



e. Now, let $h: [2, \infty) \to \mathbb{R}: [h^{-1}(x) = f(x) + k$, where k is a real number. Determine the values of k for which h and h^{-1} have a point of intersection. (3 marks)

Solution: We solve $h(x) = x \implies x = \dfrac{k^2 + 20}{2k + 8}$
Then dom $h^{-1} = [k, \infty)$ and dom $h = [2, \infty)$
The x-coordinate of the intersection must be within both of these domains.

$$\frac{k^2 + 20}{2k + 8} \geq k \implies k \leq -10 \text{ or } -4 < k \leq 2$$
$$\frac{k^2 + 20}{2k + 8} \geq 2 \implies k > -4$$

Thus $-4 < k \leq 2$ for intersection.

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VCE Mathematical Methods ½

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- After school weekdays and all-day weekends.

1-on-1 Video Consults	Text-Based Support
• Book via bit.ly/contour-methods-consult-2025 (or QR code below). • One active booking at a time (must attend before booking the next).	• Message +61 440 138 726 with questions. • Save the contact as "Contour Methods".

Booking Link for Consults
bit.ly/contour-methods-consult-2025

Number for Text-Based Support
+61 440 138 726