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VCE Mathematical Methods ½
Functions & Relations I [0.7]
Workshop Solutions

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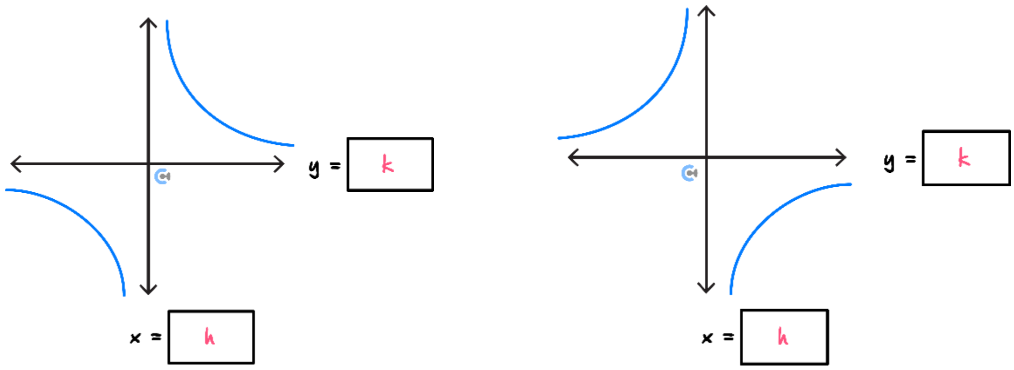
Section A: Recap

Rectangular Hyperbola

$$y = \frac{a}{x-h} + k$$



where $a > 0$ where $a < 0$

- Steps

1. Find the horizontal and vertical asymptotes and plot them on the axis.
2. Find the x- and y-intercepts and plot on the axes (if they exist).
3. Identify the shape of the graph by considering any reflections, and sketch the curve.

Finding the Equation of a Hyperbola from its Graph

- We generally need three facts (h, k, and a) about the hyperbola.

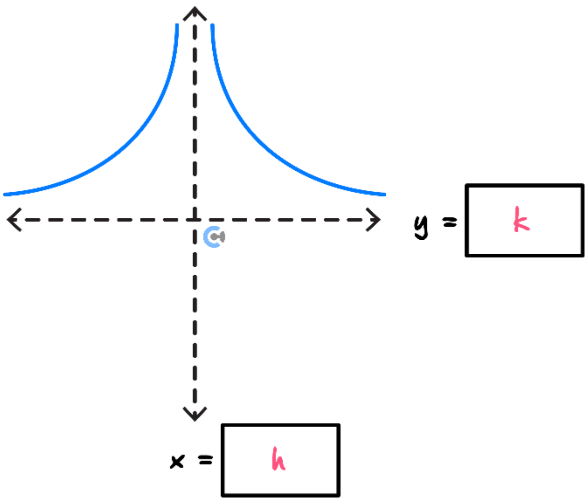
$$y = \frac{a}{x-h} + k$$

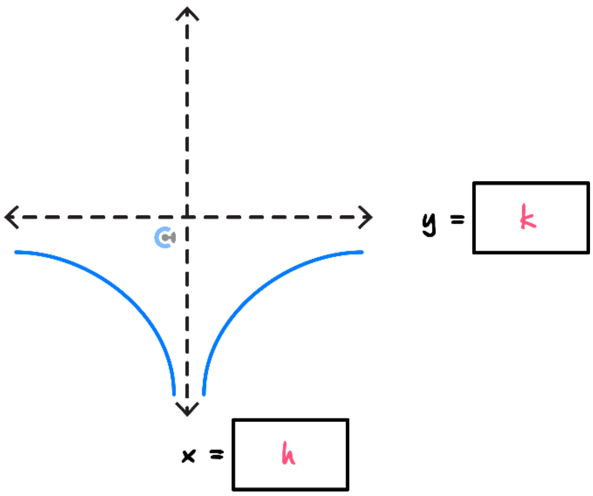
- Steps
  - Look for the asymptotes.
  - Sub in a point to find the value of a.

Definition

Truncus

$$y = \frac{a}{(x-h)^2} + k$$





$where\ a > 0$ $where\ a < 0$

- Steps

1. Find the horizontal and vertical asymptotes and plot them on the axis.
2. Find the x- and y-intercepts and plot on the axes (if they exist).
3. Identify the shape of the graph by considering any reflections and sketch the curve.

Finding the Equation of a Truncus from its Graph

- We generally need three facts (h, k, and a) about the truncus.

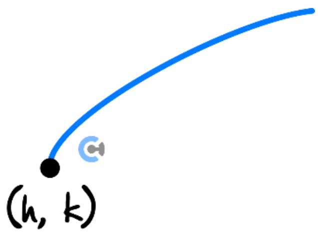
$$y = \frac{a}{(x-h)^2} + k$$

- Steps
  - Look for the asymptotes.
  - Sub in a point to solve the value of a.

Definition

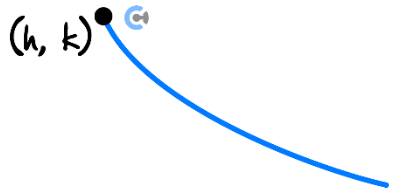
Square Root Functions

$$y = a\sqrt{b(x-h)} + k$$



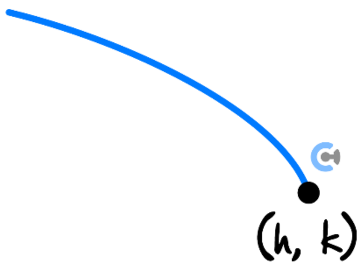
where:

$a > 0$ and $b > 0$.



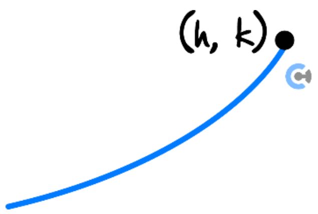
where:

$a < 0$ and $b > 0$.



where:

$a > 0$ and $b < 0$.



where:

$a < 0$ and $b < 0$.

- **Steps for sketching roots**

1. Find the starting point (h, k).
2. Find the x- and y-intercepts and plot on the axes (if they exist).
3. Identify the shape of the graph by considering any reflections and sketch the curve.

Space for Personal Notes

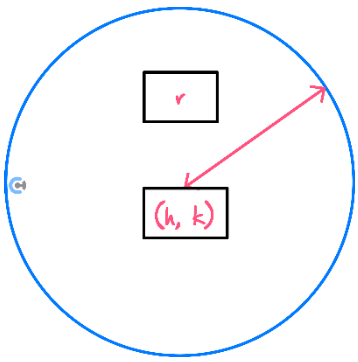
Finding the Equation of a Root Function from its Graph

- We generally need three facts about the root function.

$$y = a\sqrt{\pm(x-h)} + k$$

- Steps
  - Look for the starting point (h, k).
  - Sub in a point to solve the value of a.

Circles



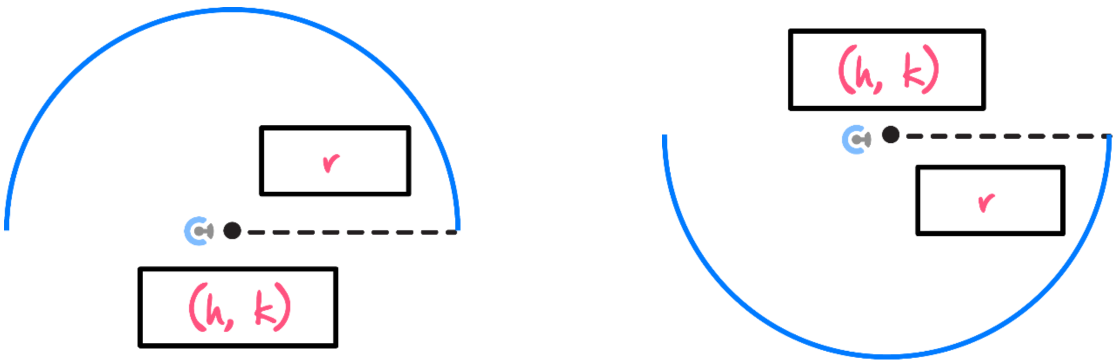
$$(x-h)^2 + (y-k)^2 = r^2$$

$$\text{where } r > 0$$

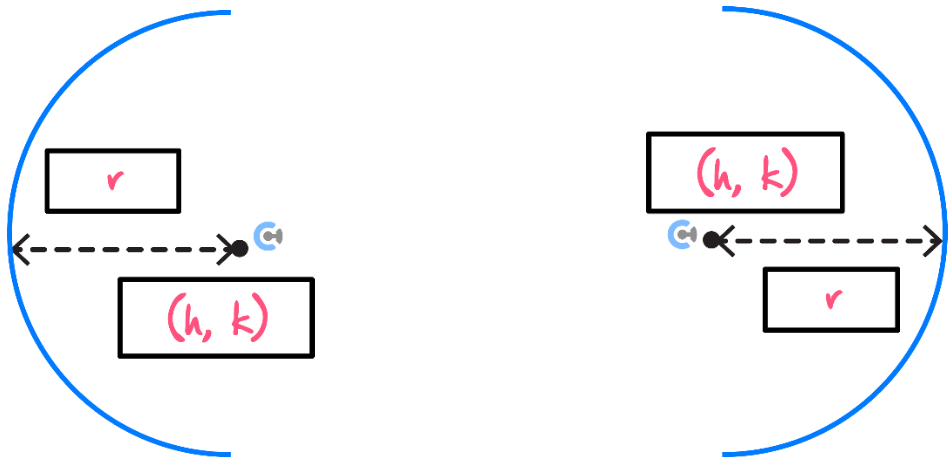
- Centre: (h, k)
- Radius: r
- Steps
  1. Find the centre of the circle.
  2. Find the radius of the circle.
  3. Find axes intercepts (if they exist).
  4. Identify the shape of the graph and sketch the curve.

Definition

Semicircles



$$y = \pm\sqrt{r^2 - (x-h)^2} + k$$



$$x = \pm\sqrt{r^2 - (y-k)^2} + h$$

- **Steps**

  1. Find the centre of the semicircle.
  2. Find the radius of the circle.
  3. Find axes intercepts if they exist.
  4. Identify the shape of the graph and sketch the curve.

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Finding the Equation of a Root Function from its Graph

- We need generally three facts about the circles/semicircles.

$$(x-h)^2+(y-k)^2=r^2$$

$$y=\pm\sqrt{r^2-(x-h)^2}+k$$

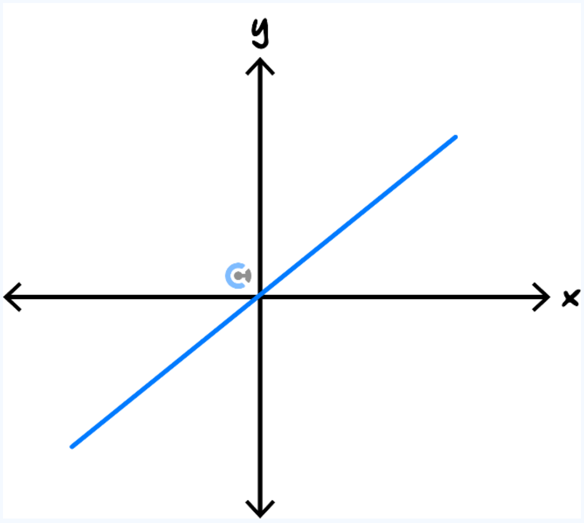
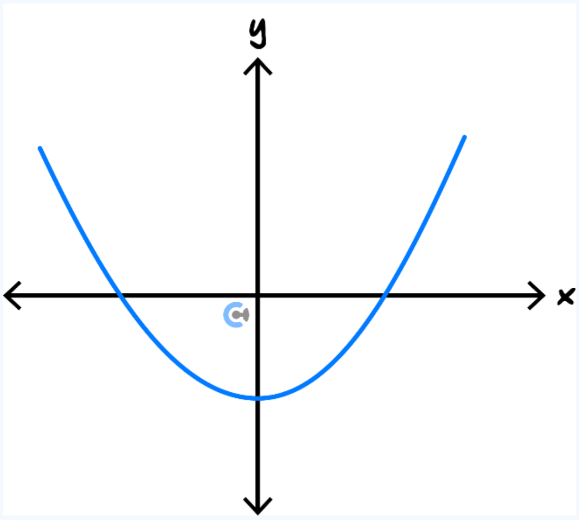
$$x=\pm\sqrt{r^2-(y-k)^2}+h$$

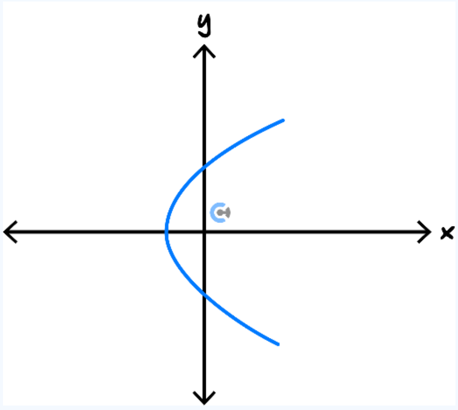
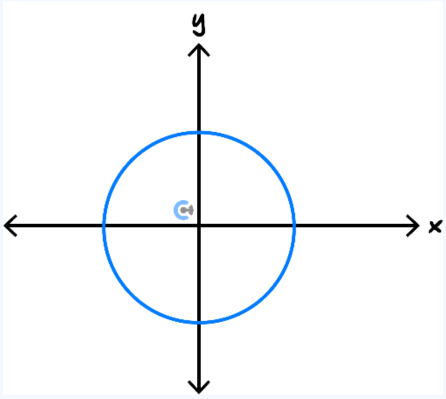
- **Steps**

1. Identify the centre, (h, k).
2. Identify the radius, r.

Types of Relations

- There are four types of relations:

One to One	Many to One
	
One x to One y.	Many x's to One y.

One to Many	Many to Many
	
One x to Many y's.	Many x's to Many y's.

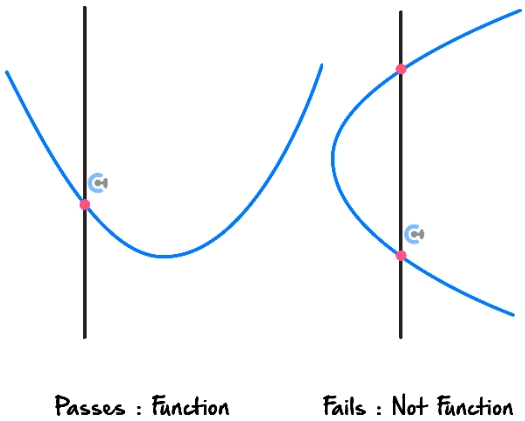
Functions

$$y = f(x)$$

- Functions are relations which make one y-value at any given x-value.

Vertical Line Test

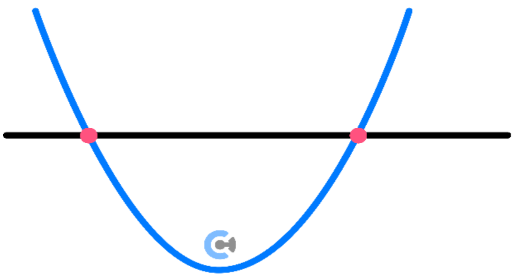
- **Definition**: Tells apart between functions and non-function relations.



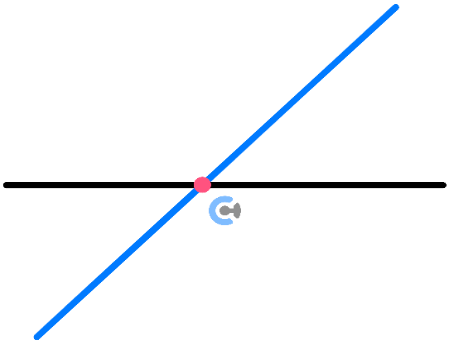
Every function only intersects a vertical line once.

Horizontal Line Test

- **Definition**: Tells apart between many-to-one and one-to-one functions. (And relations.)



Fails: Many to one



Passes: One to one

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Section B: Warmup

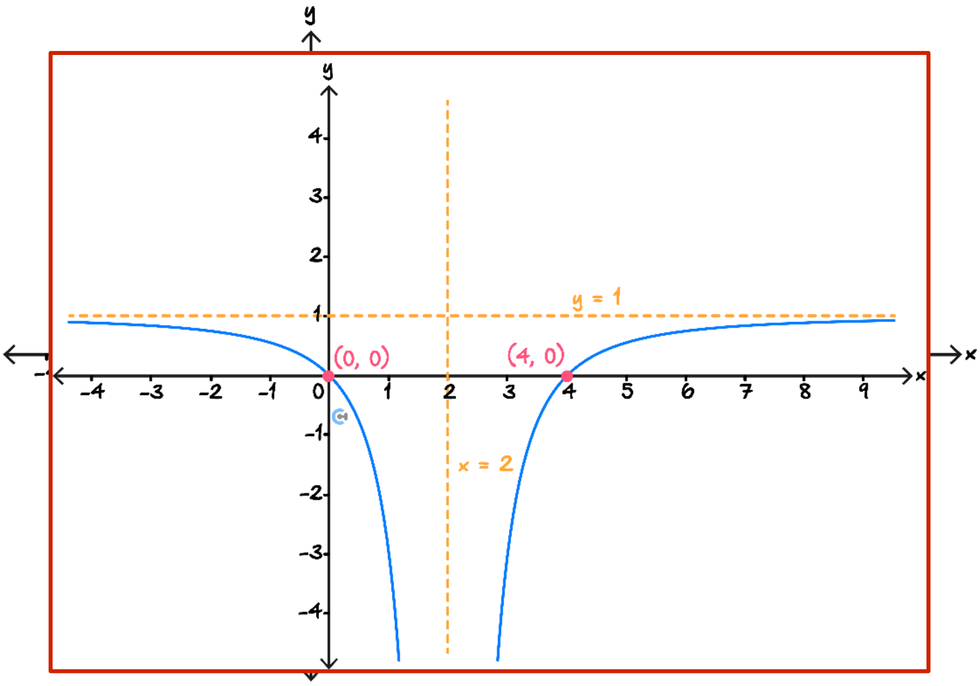
Question 1

Let $f: R\backslash\{2\} \to R, f(x) = \frac{-4}{(x-2)^2} + 1$.

a. Evaluate $f(1)$.

$\boxed{f(1) = -3}$

b. Sketch the graph of f on the axes below, labelling all asymptotes with their equations.



Consider the function $g(x) = \sqrt{5 - x} - 1$.

c. Find the x and y intercepts of $y = g(x)$.

x-int: $(4,0)$
y-int: $(0, \sqrt{5} - 1)$

d. State any points of intersection of the graphs of f and g.

The previous part and a rough sketch allow us to say that the only intersection is at $(4,0)$.

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Section C: Exam 1 (21 Marks)

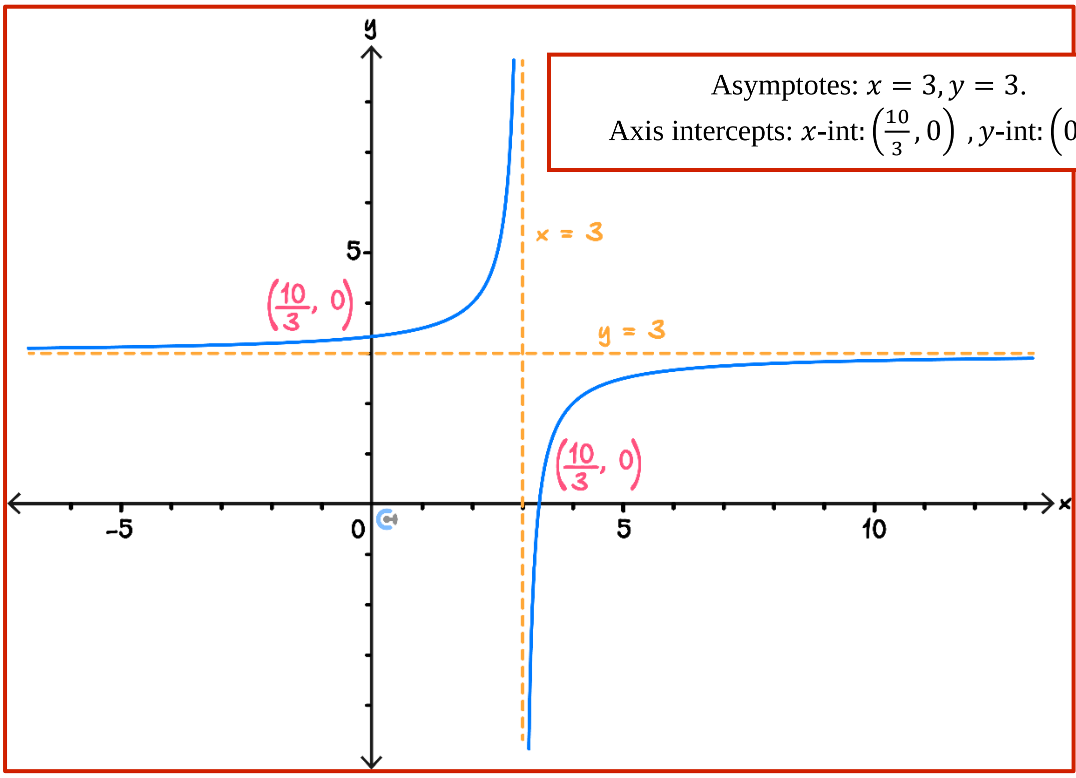
Question 2 (5 marks)

Let $f: D \to R, f(x) = \frac{3x-10}{x-3}$.

a. Express f in the form $a + \frac{b}{x-3}$ stating the values of a and b. (1 mark)

$a = 3$ and $b = -1$

b. Sketch the graph of $y = 3 - \frac{1}{x-3}$ on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates. (3 marks)



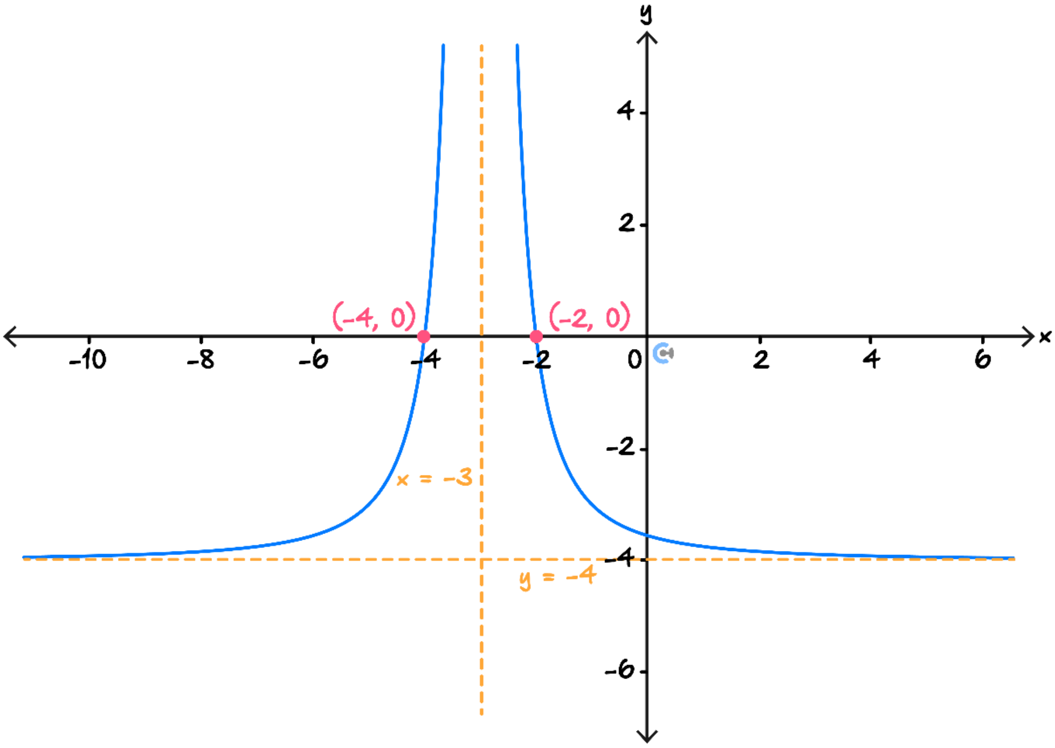
Asymptotes: $x = 3, y = 3$.
Axis intercepts: x-int: $\left(\frac{10}{3}, 0\right)$, y-int: $\left(0, \frac{10}{3}\right)$

c. Find the values of x for which $3 - \frac{1}{x-3} \leq 1$. (1 mark)

$3 < x \leq \frac{7}{2}$

Question 3 (2 marks)

Part of the graph of the function with the equation $y = \frac{a}{(x+b)^2} + c$ is shown below. Find the values of a, b, c. Show your working.



$$\boxed{\begin{gathered} y = \frac{4}{(x+3)^2} - 4 \\ a = 4, b = 3, c = -4 \end{gathered}}$$

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Question 4 (6 marks)

The function defined by $y = a\sqrt{x-h} + k$, where a, h and k are non-zero integers, has a y-intercept at $\left(0, -4\sqrt{5}+2\right)$ and has a starting point at $(-5,2)$.

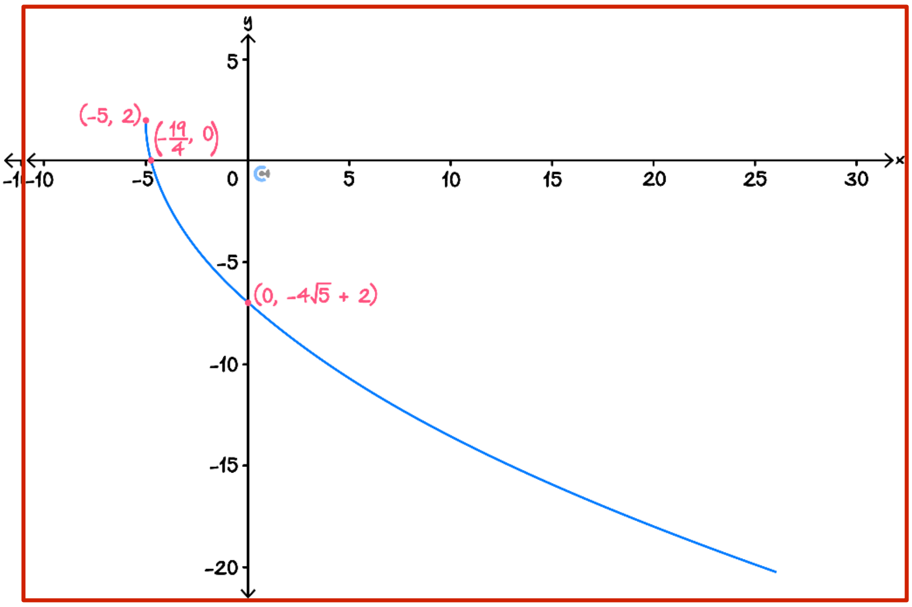
a. Determine the values of a, h, and k. (2 marks)

$a = -4, h = -5$, and $k = 2$.

b. Find the coordinates of the x-intercept. (2 marks)

$\left(-\frac{19}{4}, 0\right)$

c. Sketch the graph of the function on the axis below, labelling all key features. NOTE: $\sqrt{5} \approx 2.24$. (2 marks)



Starting point at $(-5,2)$ and x-intercept $\left(-\frac{19}{4}, 0\right)$, and y-intercept $\left(0, -4\sqrt{5}+2\right)$

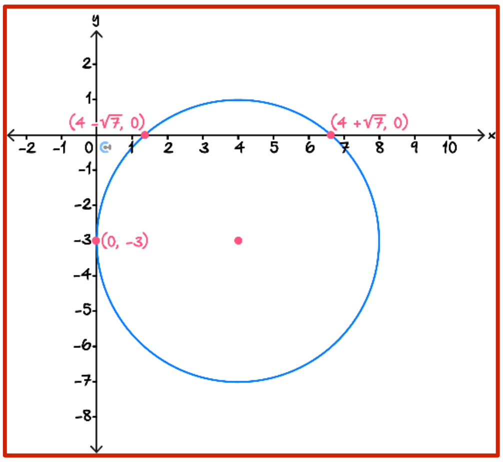
Question 5 (8 marks)

For the relation defined by $(x-4)^2+(y+3)^2=16$,

a. Find the radius and centre of the equation. (2 marks)

$r=4$
Centre $=(4,-3)$

b. Graph the relation, labelling all the coordinates of the axial intercepts. (3 marks)



c. State the domain and the range of the relation. (1 mark)

Domain $=[0,8]$
Range $=[-7,1]$

d. Identify the equation of the semicircle derived from this relation, given that it passes through $(2,2\sqrt{3}-3)$ and is considered to be a function. (2 marks)

The desired semi-circle is the top half of the circle:

$$y=\sqrt{16-(x-4)^2}-3$$

Section D: Exam 2 (26 Marks)

Question 6 (1 mark)

The graph of the function $f: D \to R, f(x) = \frac{3x+2}{5-x}$, where D is the maximal domain, has asymptotes:

A. $x = -5, y = \frac{3}{2}$

B. $x = -3, y = 5$

C. $x = 5, y = 3$

D. $x = 5, y = -3$

Question 7 (1 mark)

Find the equations of the asymptotes of the graph with the rule $y = \frac{-4x+14}{x-3}$.

A. $x = -3, y = -4$

B. $x = 3, y = 2$

C. $x = 3, y = -4$

D. $x = -4, y = 3$

Question 8 (1 mark)

Which of the following is a one-to-one function?

A. $f(x) = x^2 + 1$

B. $x^2 + y^2 = 4$

C. $f(x) = \sqrt{2x} + 1$

D. $f(x) = \sqrt{x^2 + 1}$

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Question 9 (1 mark)

The relation $x^2 + y^2 - 4x + 8y - 5 = 0, 2 \leq x \leq 7$ describes which of the following?

A. Circle with radius 25 and centre $(-2,4)$.

B. Circle with radius 5 and centre $(2,-4)$.

C. Right semicircle with radius 25 and centre $(2,-4)$.

D. Right semicircle with radius 5 and centre $(2,-4)$.

Question 10 (1 mark)

The function $y = 4\sqrt{1-x} - 4$ passes through the coordinates:

A. $(-3,4)$

B. $(0,-4)$

C. $(1,0)$

D. $(0,1)$

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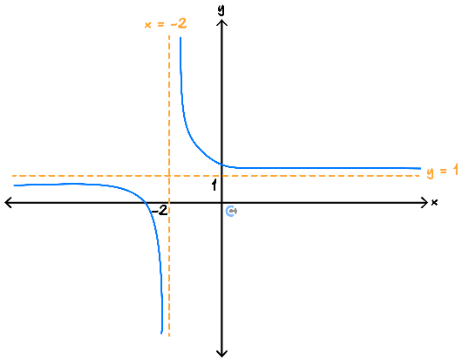
Question 11 (1 mark)



The rule for the hyperbola shown in the graph above could be:

A. $y = \frac{1}{x-2} - 1$

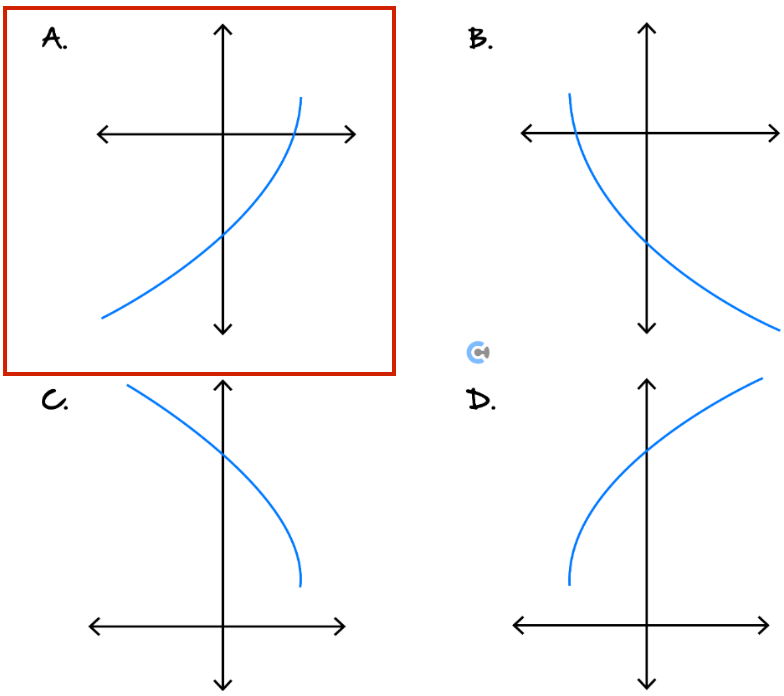
B. $y = \frac{1}{x-2} + 1$

C. $y = \frac{1}{x+2} - 1$

D. $y = \frac{1}{x+2} + 1$

Question 12 (1 mark)

Which of the following graphs could represent the function given by $y = -2\sqrt{3-2x} + 1$?



Question 13 (8 marks)

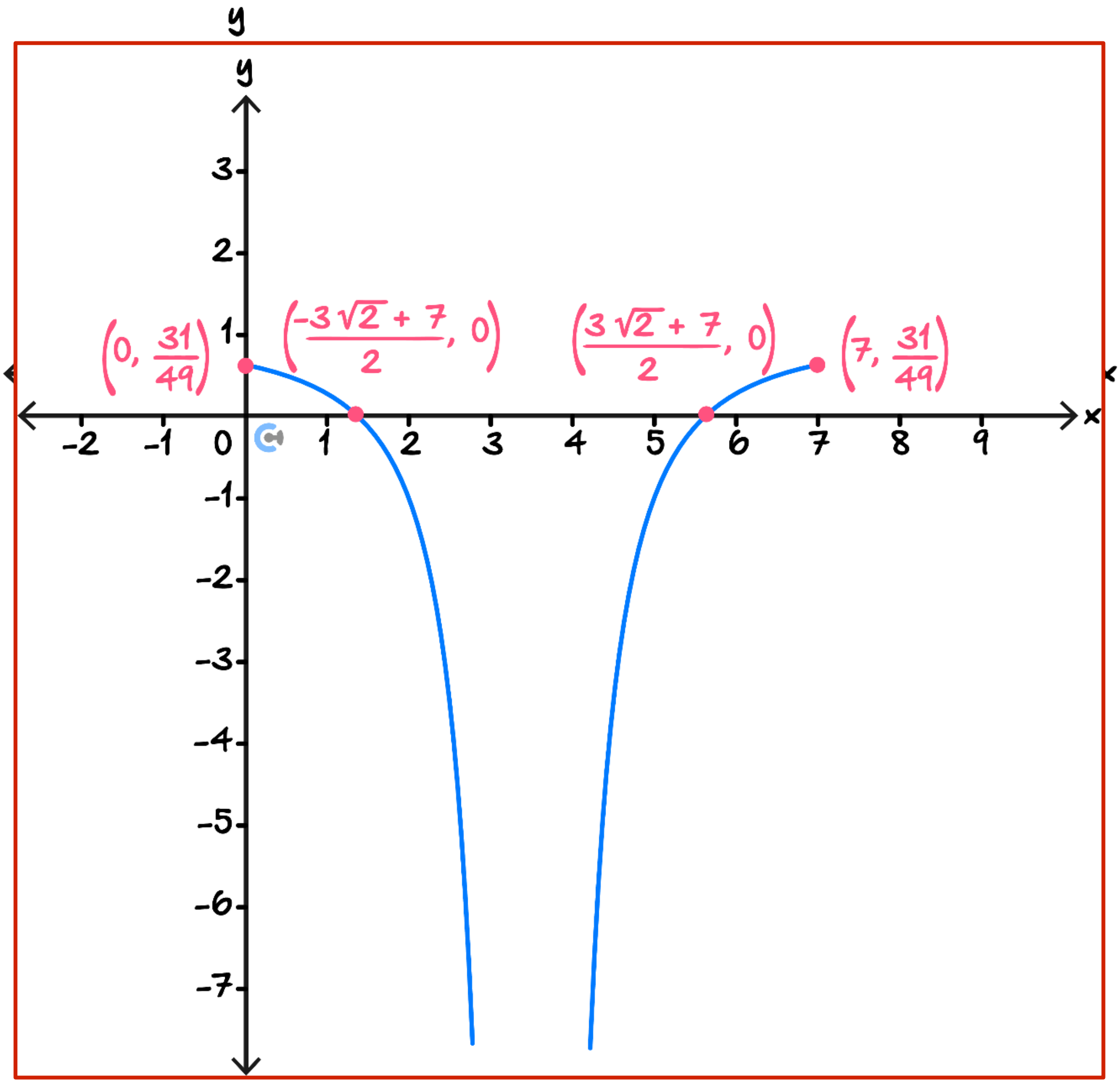
The cross-section of a water cleft is modelled with the equation $h = -\frac{18}{(2x-7)^2} + 1, 0 \leq x \leq 7$ where h is the height in metres from the water's surface, and x is the horizontal distance from the warning sign (origin).

a. Find the x-intercept of the function and hence find the width of the cleft at the water surface. (2 marks)

$$x = \frac{\pm 3\sqrt{2} + 7}{2}$$

Width = $3\sqrt{2}$

b. Sketch the graph of $h = -\frac{18}{(2x-7)^2} + 1$ for $0 \leq x \leq 7$, labelling all axial intercepts and endpoints. (2 marks)



c. What's the width of the cleft at:

i. 0.5 metres above the water surface? (2 marks)

$f(x) = 0, x = ½, 13/2$
Width $= 6\ m$

ii. 3.5 metres below the water surface? (2 marks)

$f(x) = -3.5, x = 5/2, 9/2$
Width $= 2\ m$

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Question 14 (11 marks)

James has a slide which is modelled by the function $s(x) = a\sqrt{x-h} + k$ where x is the horizontal distance in metres from the starting point and s is the height above the ground. The slide starts 10 m off the ground when $x = 0$ and touches the ground 16 metres horizontally away from the starting point.

a.

i. State the coordinates of the start and finish of the slide. (1 mark)

Start: (0,10)
End: (16,0)

ii. Find the values of a, h, k. (2 marks)

From the start point $h = 0$ then $a = -\frac{5}{2}, k = 10$.

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In[48]:= s[x_] := a √x + k
In[49]:= Solve[s[0] == 10 && s[16] == 0]
Out[49]= {{a → -5/2, k → 10}}
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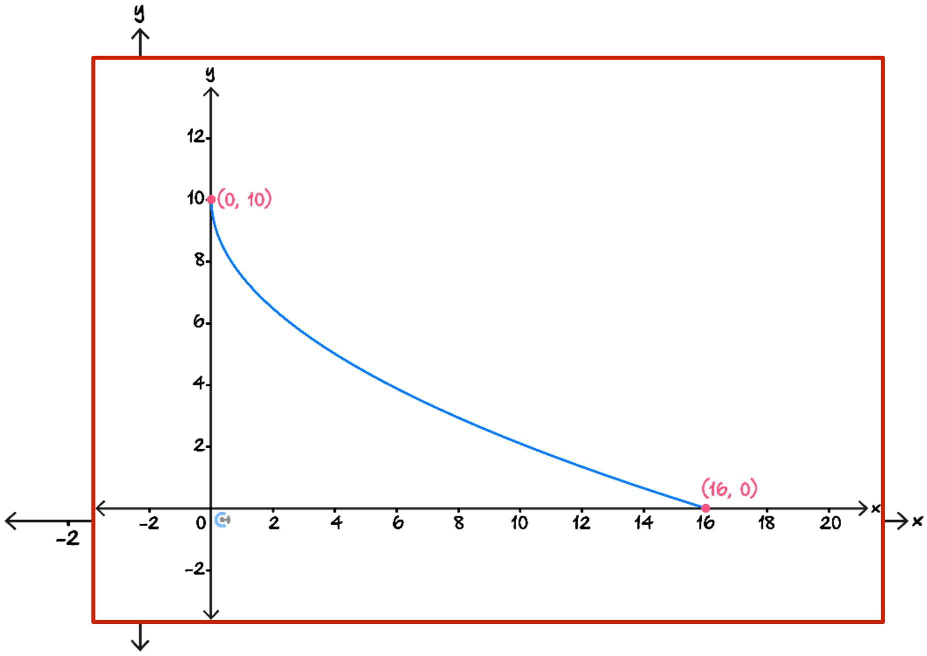
b. Hence, sketch the graph of $y = s(x)$ that describes the slide on the axis below from its start to finish, labelling all key features. (2 marks)



c. What's the distance from the starting point to the end of the slide? (2 marks)

$$\text{Distance} = \sqrt{16^2 + 10^2} = 2\sqrt{89}$$

James decides to put a metal rod under the slide to help its structural rigidity. The rod has an equation $y = -x + b$.

d.

i. Find the value of b.

HINT: The metal rod will hit/ intersect with the slide once. (2 marks)

In[59]:= Solve$\left[-x + b == -5/2\sqrt{x} + 10\right]$

Out[59]= $\left\{\left\{x \to \frac{1}{8}\left(-55 + 8b - 5\sqrt{-135 + 16b}\right)\right\}, \left\{x \to \frac{1}{8}\left(-55 + 8b + 5\sqrt{-135 + 16b}\right)\right\}\right\}$

In[60]:= Solve$[-135 + 16b == 0, b]$

Out[60]= $\left\{\left\{b \to \frac{135}{16}\right\}\right\}$

ii. Find the intersection point between the slide and the metal rod. (2 marks)

In[62]:= Solve$\left[-x + 135/16 == -5/2\sqrt{x} + 10 \;\&\&\; y == -x + 135/16\right]$

Out[62]= $\left\{\left\{x \to \frac{25}{16}, y \to \frac{55}{8}\right\}\right\}$

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Section E: Extension Exam 1 (15 Marks)

Question 15 (4 marks)

A hyperbola of the form $y = \frac{a}{x-h} + k$ passes through the point $(1, -2), (3, 4)$, and $(5, 2)$.

Determine the role of the hyperbola.

Solution: Let $f(x) = \dfrac{a}{x-h} + k$. Then

$$f(1) = \frac{a}{1-h} + k = -2 \implies a = -(1-h)(k+2) = -2 + 2h - k + hk \qquad (1)$$
$$f(3) = \frac{a}{3-h} + k = 4 \implies a = (4-k)(3-h) = 12 - 4h - 3k + hk \qquad (2)$$
$$f(5) = \frac{a}{5-h} + k = 2 \implies a = (2-k)(5-h) = 10 - 2h - 5k + hk \qquad (3)$$

From (1) and (2)

$$\begin{aligned} -2 + 2h - k + hk &= 12 - 4h - 3k + hk \\ 14 - 6h - 2k &= 0 \\ k &= 7 - 3h \end{aligned}$$

(2) − (3) gives

$$\begin{aligned} 2 - 2h + 2k &= 0 \\ 2 - 2h + 2(7 - 3h) &= 0 \\ 16 - 8h &= 0 \\ h &= 2 \end{aligned}$$

then $k = 1$ and $a = (2-1)(5-2) = 3$.

So $y = \dfrac{3}{x-2} + 1$

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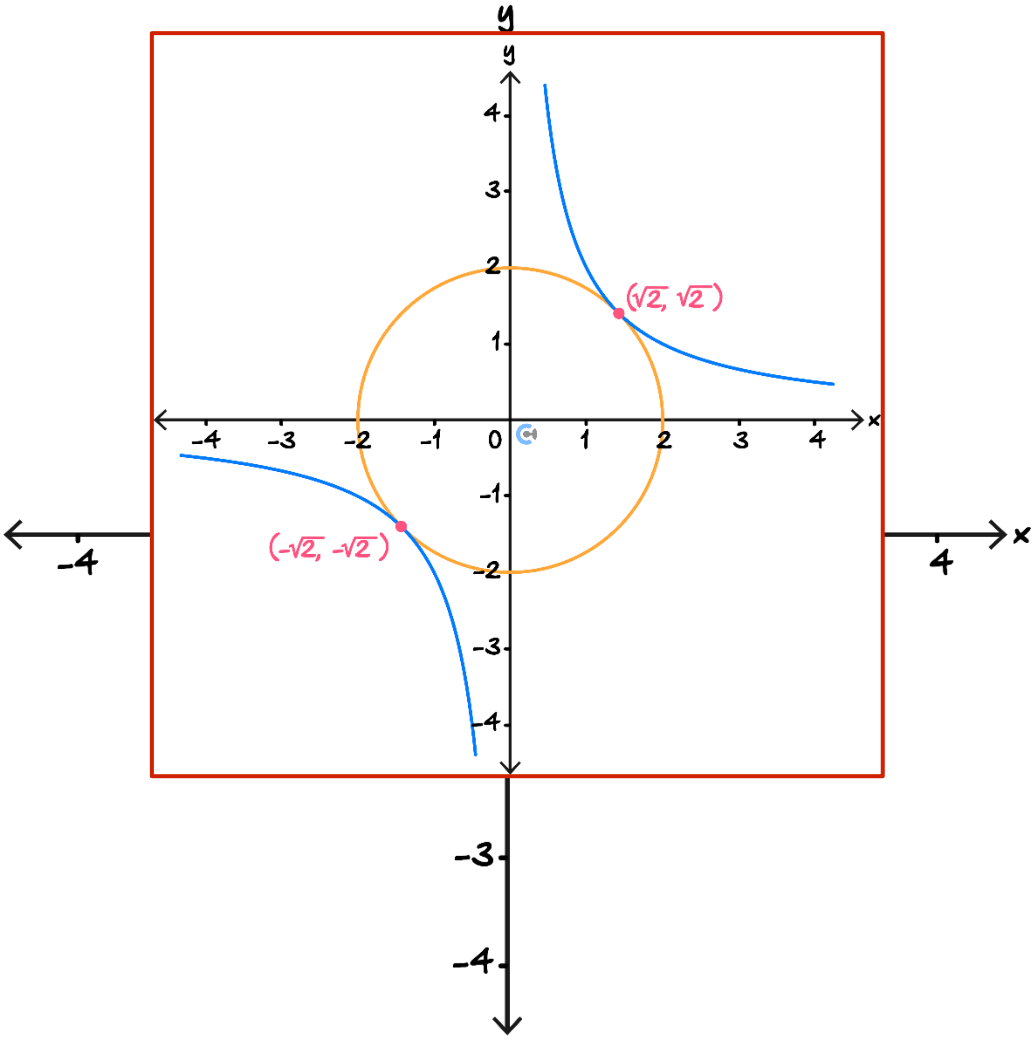
Question 16 (11 marks)

In this question, we will investigate the intersections of hyperbolas and circles. Throughout this question assume for the constants a and r that $a \neq 0$ and $r > 0$ unless otherwise specified.

a. On the axes below sketch the graph of $x^2 + y^2 = 4$ and $y = \frac{2}{x}$. Label all points of intersection of the two graphs. (4 marks)



Solution: Sub in $y = \frac{2}{x}$ into the circle equation.

$$\begin{aligned} x^2 + \frac{4}{x^2} &= 4 \\ x^4 - 4x^2 + 4 &= 0 \\ (x^2 - 2)^2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

b. List all the possibilities for the number of intersections that a circle of the form $x^2 + y^2 = r^2$ and a hyperbola of the form $y = \frac{a}{x}$ may have. (1 mark)

Solution: $0, 2$ or 4. Note that 3 intersections is not possible due to the symmetry.

c. A circle of the form $x^2 + y^2 = r^2$ and a parabola of the form $y = \frac{a}{x}$ intersect each other exactly twice. Find the possible values of a in terms of r.

TIP: Check that your answer works with **part a.** (4 marks)

Solution: Sub in $y = \frac{a}{x}$ into the circle equation.

$$x^2 + \frac{a^2}{x^2} = r^2$$
$$x^4 - r^2x^2 + a^2 = 0$$
$$\left(x^2 - \frac{1}{2}r^2\right)^2 = \frac{1}{4}r^4 - a^2$$

Now observe: If $\frac{1}{4}r^4 - a^2 < 0$ then there are no solutions for x.
If $\frac{1}{4}r^4 - a^2 = 0$ then $x^2 = \frac{1}{2}r^2 \implies x = \pm\frac{1}{\sqrt{2}}r$. So two solutions which is what we want.
Therefore we require $\frac{1}{4}r^4 - a^2 = 0 \implies r^4 = 4a^2 \implies r^2 = \pm 2a$.
Therefore

$$a = \pm\frac{1}{2}r^2$$

The gradient of a circle $x^2 + y^2 = r^2$ at any point (p, q) is $-\frac{p}{q}$ and the gradient of the hyperbola $y = \frac{a}{x}$ at any point (p, q) is $-\frac{a}{p^2}$.

d. When the hyperbola intersects the circle exactly twice, the hyperbola, and circle always have the same gradient at the point of intersection. Determine this gradient, assuming that $a > 0$. (2 marks)

Solution: From previous part intersections occur at $\left(\pm\frac{1}{\sqrt{2}}r, \pm\frac{1}{\sqrt{2}}r\right)$. Therefore

$$\frac{p}{q} = 1$$

and so the gradient is -1.
Note: If $a < 0$ then the gradient will be 1.

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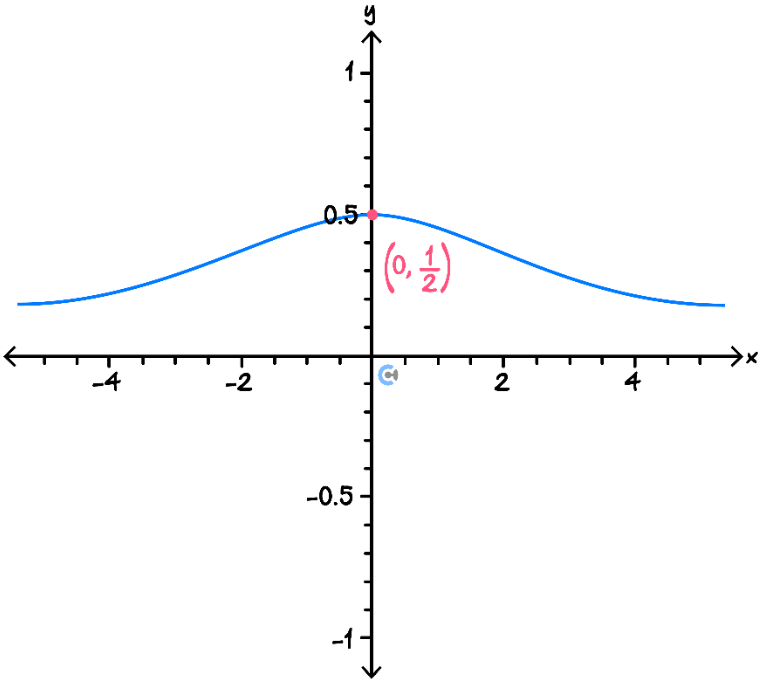
Section F: Extension Exam 2 (12 Marks)

Question 17 (1 mark)



A possible equation for the graph shown above is:

A. $y = \frac{1}{4-x^2}$

B. $y = \frac{1}{4+x^2}$

C. $y = \frac{1}{2+x^2}$

D. $y = \frac{1}{\sqrt{4+x^2}}$

Question 18 (1 mark)

The equation of a circle with centre $(-2.5, 1.5)$ and radius 4 is given by:

A. $(x - 2.5)^2 + (y + 1.5)^2 = 16$

B. $(x + 2.5)^2 - (y - 1.5)^2 = 16$

C. $(2x + 5)^2 + (2y - 3)^2 = 64$

D. $(2x + 5)^2 + (2y - 1.5)^2 = 16$

Question 19 (1 mark)

The features of the graph of the function with rule $f(x) = \frac{x^2-4x+3}{x^2-x-6}$ include:

A. Asymptotes at $x = 1$ and $x = -2$.

B. Asymptotes at $x = 3$ and $x = -2$.

C. An asymptote at $x = -2$ and a point of discontinuity at $x = 3$.

D. An asymptote at $x = 3$ and a point of discontinuity at $x = -2$.

Question 20 (9 marks)

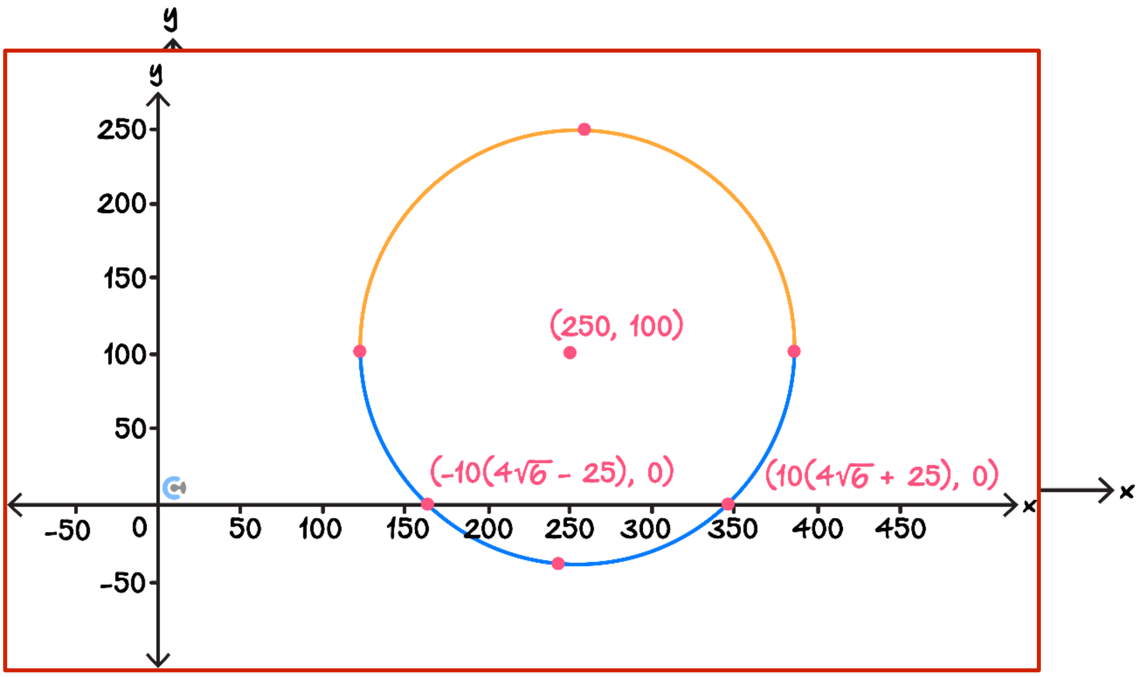
Contour Park has the shape of a circle with a radius of 140 m and its centre is located 250 m east, 100 m north of Contour station.

Let x be the horizontal distance from the station in metres and y be the vertical distance from the station in metres.

a. Find the equation for the boundary of Contour Park. (1 mark)

$$\boxed{(x-250)^2 + (y-100)^2 = 19600}$$

b. Sketch the boundary, labelling all axial intercepts. (2 marks)



Angad realises that there are dangerous bulldog ants living in the park and decides to put up a warning fence around the **top** half of the park. This warning fence is always a distance of 5 metres away from the park boundary.

c. What is the semi-circle equation which describes the curve of the fence around the top half of the park? (2 marks)

Same centre as park boundary but radius has been increased by 5 metres. Therefore,

$$(x-250)^2+(y-100)^2=145^2$$

So semi-circle:

$$y=\sqrt{145^2-(x-250)^2}+100$$

The colony of bulldog ants is located at $(200,150)$ and bulldog ants distribute themselves no more than 80 m from their colony.

d. Specify the **range** of x-values for which the bulldog ants could be outside of the warning fence. (2 marks)

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In[80]:= fence[x_] := √(145^2 - (x - 250)^2) + 100
In[81]:= ants[x_] := √(80^2 - (x - 200)^2) + 150
In[82]:= Reduce[ants[x] > fence[x]]
Out[82]= -5/8 (-243 + √2263) < x < 5/8 (243 + √2263)
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$$-\frac{5}{8}\left(-243+\sqrt{2263}\right)<x<\frac{5}{8}\left(243+\sqrt{2263}\right)$$

e. Determine the minimum distance, k metres, that the warning fence must always be away from the park boundary in order to ensure that the bulldog ants do not cross the warning fence. Give your answer correct to two decimal places. (2 marks)

The distance d between the park's center $(250,100)$ and the ants' colony $(200,150)$ is:

$$d = \sqrt{(250-200)^2 + (100-150)^2} = \sqrt{50^2 + (-50)^2} = \sqrt{5000} \approx 70.71$$

Now, for the bulldog ants not to cross the warning fence:

$$140 + k \geq 70.71 + 80$$

$$k \geq 70.71 + 80 - 140 = 10.71$$

Hence, the minimum distance k that the warning fence must be away from the park boundary to ensure the ants do not cross is 10.71 metres.

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VCE Mathematical Methods ½

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