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VCE Mathematical Methods ½
Polynomials Exam Skills [0.6]
Workshop

Error Logbook:

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Section A: Recap

Degree of Polynomial Functions

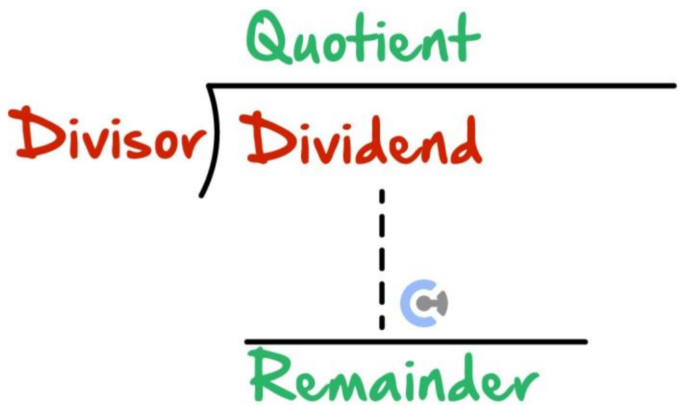
$$\textit{Degree} = \textit{Highest Power of the Polynomial}$$

Roots of Polynomial Functions

$$\textit{Roots} = x\textit{-intercept}$$

Polynomial Long Division

- Division of polynomials:



$$\frac{\textit{Dividend}}{\textit{Divisor}} = \textit{Quotient} + \frac{\textit{Remainder}}{\textit{Divisor}}$$

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<u>Remainder Theorem</u>

- **Definition**:
  - Finds the remainder of long division without the need of long division,

$$\textit{when } P(x) \textit{ is divided by } (x-\alpha), \textit{the remainder is } P(\alpha).$$

- **Steps**
  1. Find x-values which make the divisor equal to 0.
  2. Substitute it into the dividend function.

<u>Factor Theorem</u>

- For every x-intercept, there is a factor:

$$\textit{If } P(\alpha) = 0 \textit{ then}, (x-\alpha) \textit{ is a factor of } P(x).$$

<u>Factorising Polynomials</u>

- The steps are:
  - Find a single root by trial and error.
    - (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

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Definition

Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$\textit{Potential root} = \pm\frac{\textit{Factors of constant term } a_0}{\textit{Factors of leading coefficient } a_n}$$

- If the roots are rational numbers, the roots can only be $\pm\frac{\text{Factors of constant term } a_0}{\text{Factors of leading coefficient } a_n}$.

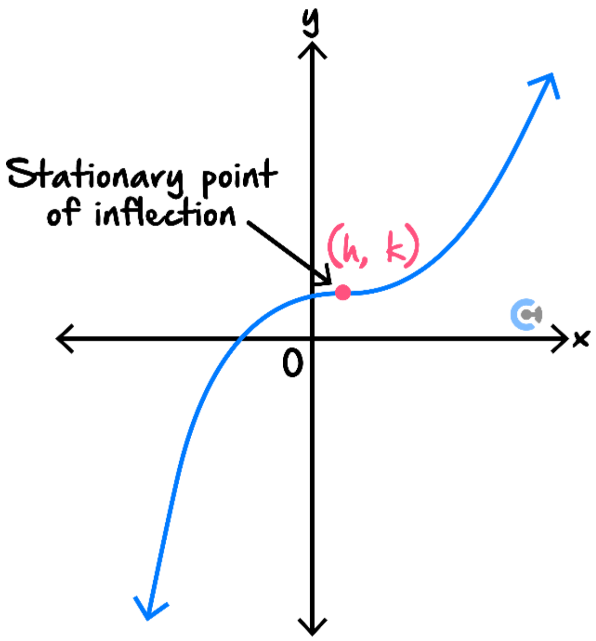
Sum and Difference of Cubes

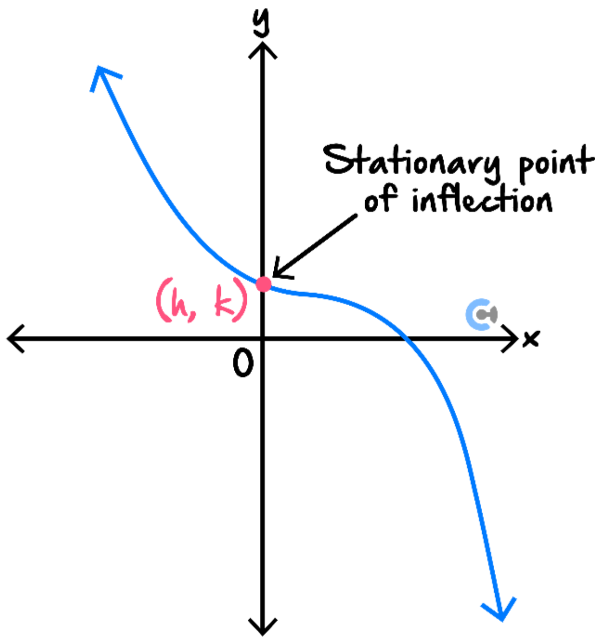
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Graphs of $a(x - h)^n + k$, where n is an Odd Positive Integer

- All graphs look like a "cubic".

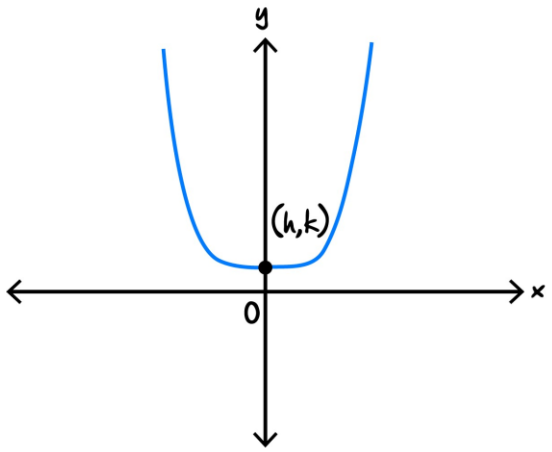


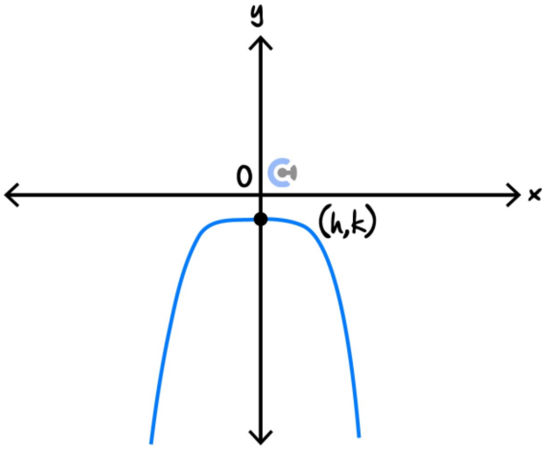


- The point (h, k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

<u>Graphs of $a(x-h)^n+k$, where n is an Even Positive Integer</u>

- All graphs look like a "quadratic".

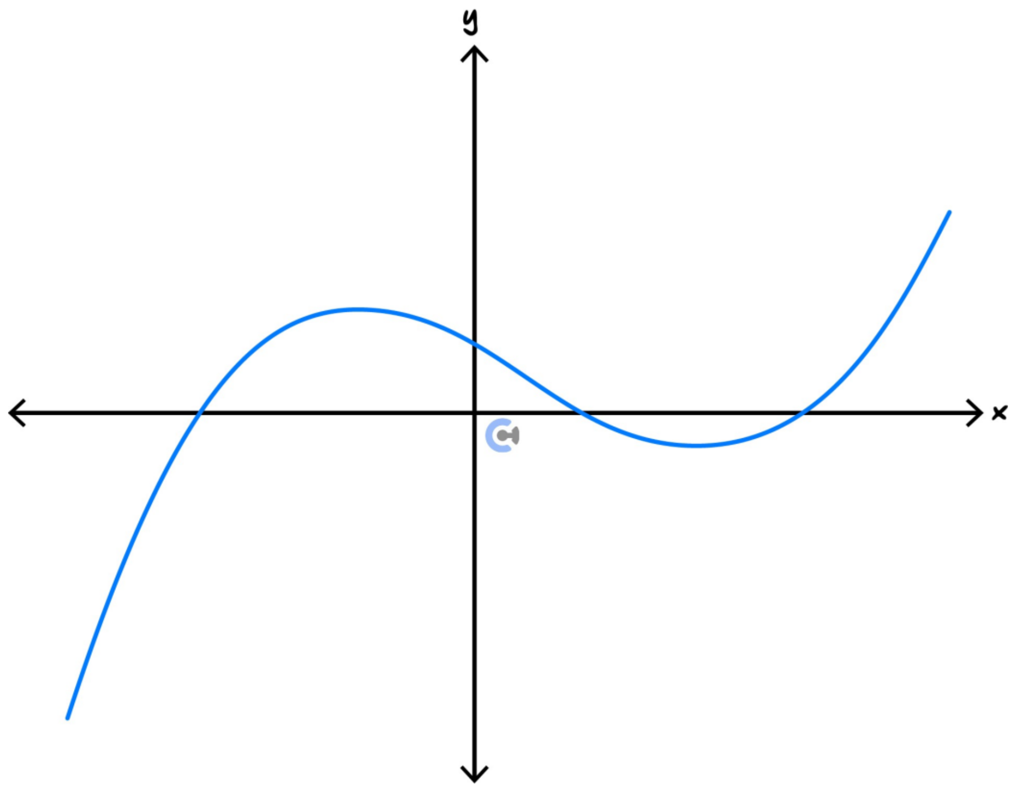




- The point (h,k) gives us the turning point.

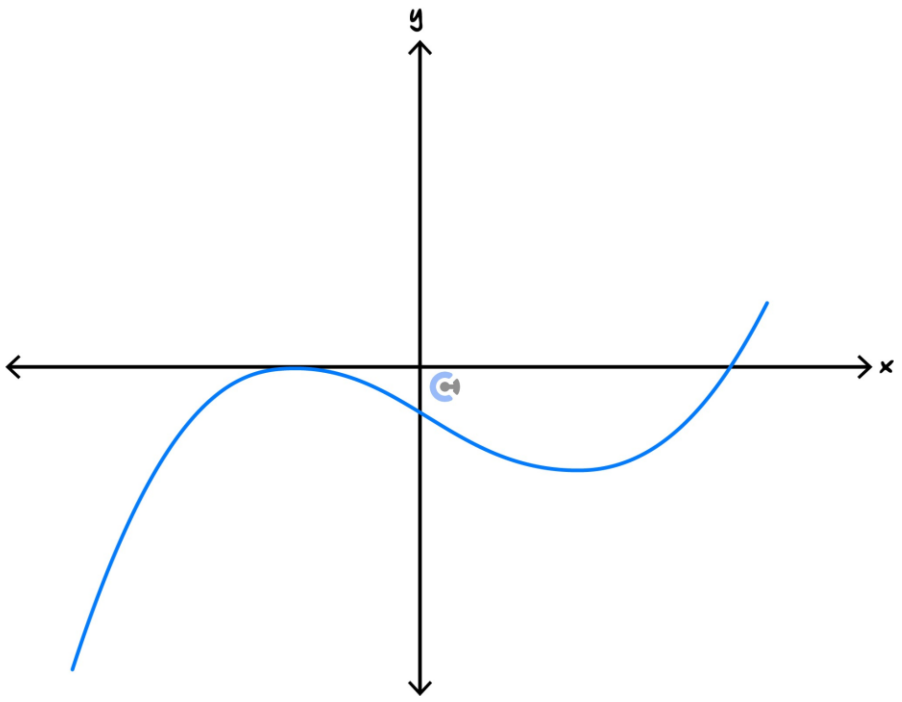
<u>Graphs of Factorised Polynomials</u>

- **All non-repeated linear factors correspond to x-intercepts of the graph.**



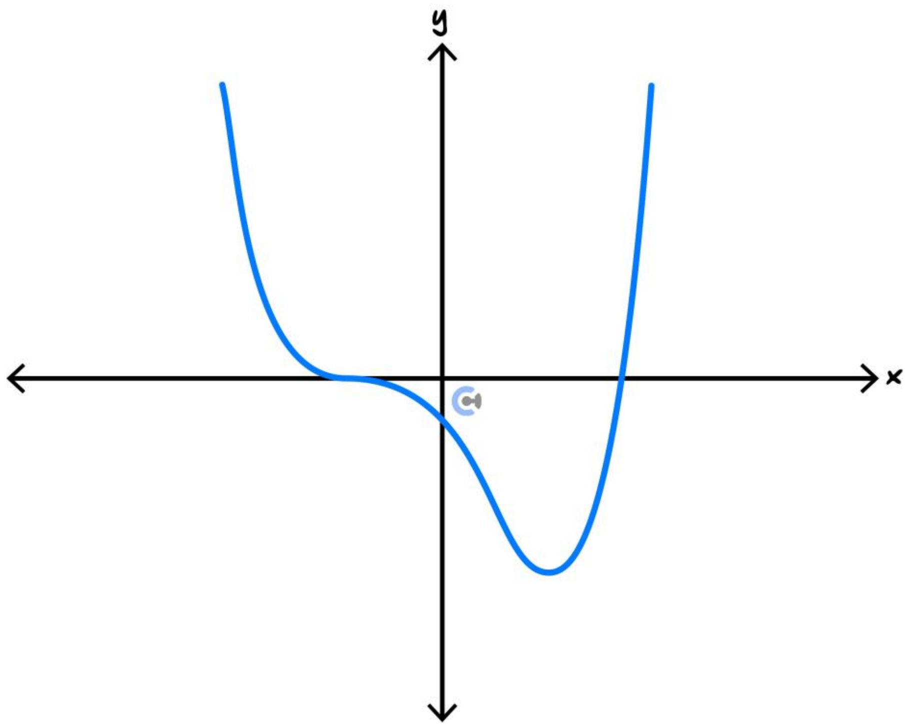
- E.g., $f(x)=(x-a)(x-b)(x-c)$ results in x-intercepts at $(a,0)$, $(b,0)$, and $(c,0)$.

- **All squared linear factors correspond to x-intercepts and T.P. of the graph.**



- E.g., $f(x) = (x-a)^2(x-b)$ will have an x-intercept $(a, 0)$ which is also a local minimum/maximum.

- **All cubed linear factors correspond to x-intercepts and SPI of the graph.**



- E.g., $f(x) = (x-a)^3(x-b)$ has an x-intercept $(a, 0)$ which is also a stationary point of inflection.

Definition

Steps to Graphing Factorised Polynomials

- **Steps:**

1. Plot x-intercepts.
2. Determine whether the polynomial is positive or negative.
3. Use the repeated factors to deduce the shape.
    - Non - Repeated: Only x-intercept.
    - Even Repeated: x-intercept and a turning point.
    - Odd Repeated: x-intercept and a stationary point of inflection.

Definition

Solving the Polynomial Inequality $f(x) > 0$

- Steps:

1. Find the x-intercepts.
2. Sketch the polynomial.
3. Shade the places where the y-values are positive.

Definition

When does a cubic have n solutions?

- Steps:

1. Factorise out the x term.
2. Since the x term gives 1 solution, use discriminant to find when the quadratic has $n - 1$ solutions.

Space for Personal Notes

Definition

<u>Bisection Method</u>

- Step 1: Pick a random interval $[a, b]$ where $f(a) \times f(b) =$ Negative.
- Step 2: Find a midpoint to estimate the root.

$$\text{where } m = \frac{a+b}{2}$$

- Step 3: Create a new interval $[a, b]$ by making m either new a or new b.

$$\text{If } f(a) \times f(m) < 0$$

$$\text{New Interval: } [a, m]$$

$$\text{If } f(b) \times f(m) < 0$$

$$\text{New Interval: } [m, b]$$

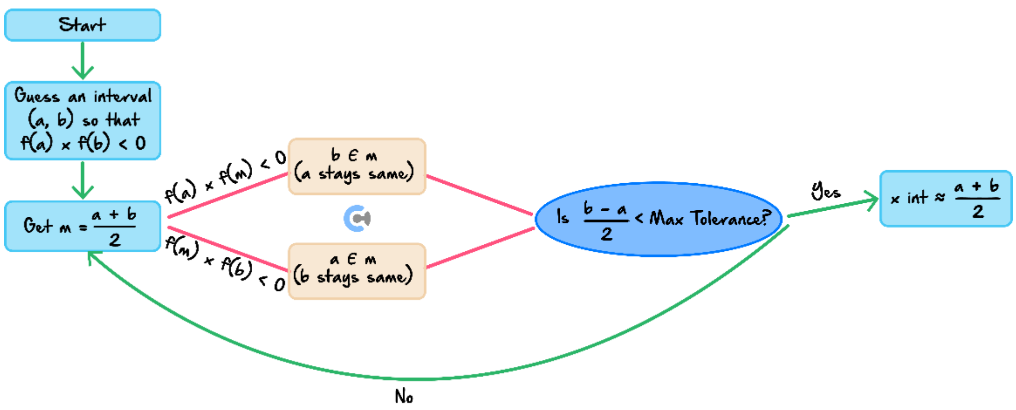
- Step 4: Repeat until the interval becomes short enough for good accuracy.
  - The smaller the interval $[a, b]$, more accurate our estimation gets.

$$\text{If } \frac{b-a}{2} < \text{Max Tolerance,}$$

$$\text{We stop.}$$

  - Maximum error is half of the width of the interval.

$$\text{Max Error} = \frac{b-a}{2}$$



Section B: Warmup

Question 1

a. Solve the inequality $x^3 + x + 6 > 4x^2$.

Solution: Write this as $x^3 - 4x^2 + x + 6 > 0$. This can be factored as

$$(x+1)(x-2)(x-3) > 0$$

and by considering the shape of the cubic the inequality holds for $-1 < x < 2$ or $x > 3$.

b. Find the values of k such that $x^3 + 2kx^2 + 3x = 0$ has only one real solution.

Solution: Factor as $x(x^2 + 2kx + 3)$. Only one solution $x = 0$ if quadratic has no solutions. So consider the discriminant.

$$\begin{aligned} 4k^2 - 12 &< 0 \\ k^2 &< 3 \end{aligned}$$

so only one solution for $-\sqrt{3} < k < \sqrt{3}$.

c. Apply the bisection method with initial interval [1,2] and tolerance 0.1 to find an approximate solution to the equation $x^2 - 2 = 0$.

Solution: Our intervals are: $[1,2] \to \left[1, \frac{3}{2}\right] \to \left[\frac{5}{4}, \frac{3}{2}\right] \to \left[\frac{11}{8}, \frac{3}{2}\right]$.

This last interval has width $\frac{1}{8} < 2 \times 0.1$. So our estimate is

$$\frac{11/8 + 12/8}{2} = \frac{23}{16}$$

Space for Personal Notes

Section C: Exam 1 (23 Marks)

Question 2 (9 marks)

Let $f(x) = ax^3 - 5x^2 + bx + 9$. When $f(x)$ is divided by $x - 2$ the remainder is -7 and when $f(x)$ is divided by $x + 1$ the remainder is 8.

a. Show that $a = 2$ and $b = -6$. (2 marks)

$\because R = f(2) = -7 \longrightarrow \therefore 8a - 20 + 2b + 9 = -7$

$8a + 2b = 4$

$4a + b = 2 \ldots (1)$

$\because R = f(-1) = 8$

$-a - 5 - b + 9 = 8$

$a + b = -4 \ldots (2)$

$(1) - (2)$

$4a + b = 2$

$-(a + b = -4)$

$\therefore 3a = 6$

$\therefore a = 2$

$2 + b = -4$

$\therefore b = -6$

b. Express $f(x)$ as the product of three linear factors. (3 marks)

$f(x) = 2x^3 - 5x^2 - 6x + 9$

1. Finding a Single Root (T4E):

Sub $x = 1$:

$f(1) = 2 - 5 - 6 + 9 = 0$

$\therefore (x-1)$ is a factor

2. Synthetic Division:

1	2	-5	-6	9
	↓	2	-3	-9
	2	-3	-9	0
	x^2	x	c	R

3. Factorise:

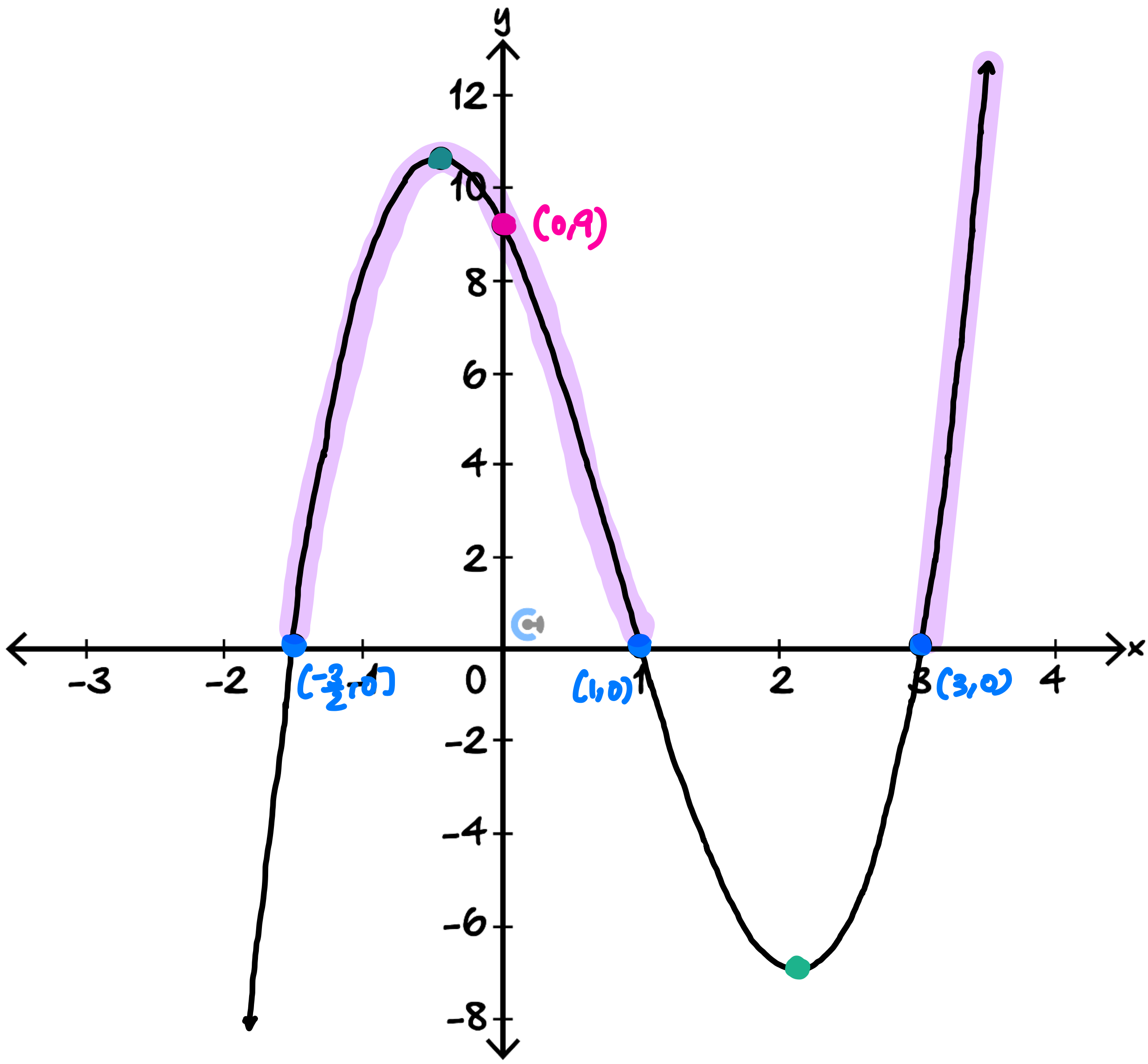
$= (x-1)(2x^2 - 3x - 9)$

$= (x-1)(2x+3)(x-3)$ //

$\therefore x = 1, -\frac{3}{2}, 3$

$\therefore y = 9$

c. Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts. Note that f has turning points at approximately $(-0.5, 10.5)$ and $(2.1, -7.1)$. (2 marks)



d. Hence, solve the inequality $2x^3 - 5x^2 - 6x > -9$. (2 marks)

$2x^3 - 5x^2 - 6x + 9 > 0$

$\therefore x \in \left(-\frac{3}{2}, 1\right) \cup (3, \infty)$

Question 3 (5 marks)

$(x+b)^2(x+b) = x^3+3b^2x+3bx^2+b^3$

Consider the function $f(x) = \frac{x^3}{4} - \frac{3x^2}{4} + \frac{3x}{4} + \frac{7}{4}$.

a. Write $f(x)$ in the form $a(x+b)^3 + c$ for real values a, b, and c. (2 marks)

$f(x) = \frac{1}{4}(x^3-3x^2+3x+7)$

$= \frac{1}{4}\left((x-1)^3+8\right)$ $\quad (x-1)^3 = x^3-3x^2+3x-1$

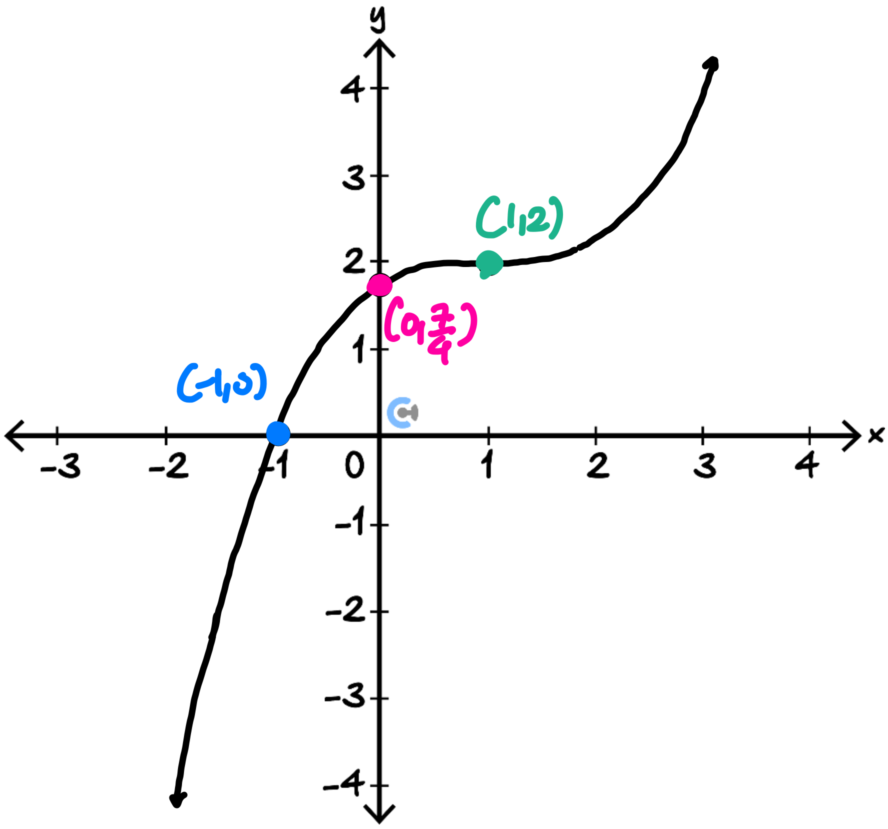
$f(x) = \frac{1}{4}(x-1)^3+2$

y-int: $y = \frac{1}{4}(-1)^3+2$
$= -\frac{1}{4}+2 = \frac{7}{4}$

SPOI : $(1,2)$

x-int: $0 = \frac{1}{4}(x-1)^3+2$
$-8 = (x-1)^3$
$x-1 = -2 \Rightarrow x = -1$

b. Sketch the graph of $y = f(x)$ on the axes below. Label any axes intercepts and stationary points of inflection with coordinates. (3 marks)

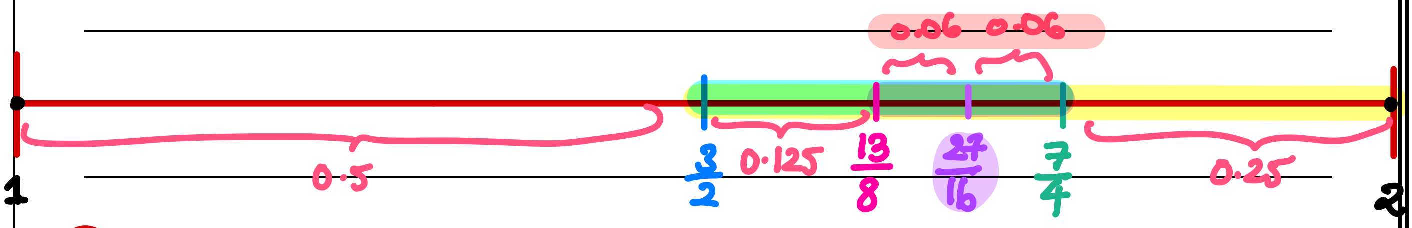


Question 4 (4 marks) (Tech Active)

The bisection method may be used to approximate $\sqrt{3}$ by finding a root to $x^2 - 3 = 0$.

Max Error ≤ 0.1

a. Use the bisection method with initial interval $[1,2]$ and tolerance 0.1 to find an approximate solution to $x^2 - 3 = 0$. Leave your answer in the form $\frac{a}{b}$, for positive integers a and b. (3 marks)



① Midpoint

② Max Error < Tolerance? $\Rightarrow$ End!

③ New Interval

$\therefore$ x-int $\approx \frac{27}{16}$

b. Determine whether $\frac{7}{4}$ is more than or less than $\sqrt{3}$. (1 mark)

x-int: $\sqrt{3} \approx \frac{27}{16} < \frac{7}{4}$

$\therefore \frac{7}{4} > \sqrt{3}$

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Question 5 (5 marks)

Consider $f(x) = x^3 - 2kx^2 + 4kx + 4x$, where k is a real constant.

Find the values of k such that $f(x) = 0$ has:

a. One solution. (3 marks)

$f(x) = 0$:

$x^3 - 2kx^2 + 4kx + 4x = 0$

$x(\underbrace{x^2 - 2kx + 4k + 4}) = 0$

$(1) + (0) = 1$

$\therefore \Delta < 0$

$(-2k)^2 - 4(1)(4k+4) < 0$

$4k^2 - 16k - 16 < 0$

$k^2 - 4k - 4 < 0$

$(k-2)^2 - 8 < 0$

$(k-2)^2 < 8$

$-2\sqrt{2} < (k-2) < 2\sqrt{2}$

$\therefore 2 - 2\sqrt{2} < k < 2 + 2\sqrt{2}$

b. Two solutions. (1 mark)

$\Delta = 0 \Rightarrow (k-2)^2 = 8$

$k - 2 = \pm 2\sqrt{2}$

$\therefore k = 2 \pm 2\sqrt{2}$ & $k = -1$

$x(x^2 - 2kx + 4k + 4)$

$x = 0$

$(4k+4)$

$\therefore k = -1$

$(x^2 + 2x)$

c. Three solutions. (1 mark)

$\Delta > 0 \Rightarrow (k-2)^2 > 8$

$(k-2) > 2\sqrt{2}$ or $(k-2) < -2\sqrt{2}$

$x(x+2)$

$x(x^2 + 2x)$

$x = 0$ $\quad x = 0, -2$

$\therefore k > 2 + 2\sqrt{2}$ or $k < 2 - 2\sqrt{2}$ & $k \neq -1$

Section D: Tech Active Exam Skills

Calculator Commands: Turning Point

- ALWAYS sketch the graph to find approximate bounds for where the turning point you want is located.
- To find a local maximum we maximise the function over a specific domain.
- To find a local minimum we minimise the function over a specific domain.
- **TI and Casio:** Use $fmin(expression, variable, lower\ (optional), upper\ (optional))$ or $fmax(expression, variable, lower\ (optional), upper\ (optional))$.
- **TI:** Menu→ $4 \rightarrow \frac{7}{8}$.

Define $f(x)=x^3-4 \cdot x$ *Done*

$\text{fMin}(f(x),x,0,2)$ $x=\frac{2\cdot\sqrt{3}}{3}$

$f\left(\frac{2\cdot\sqrt{3}}{3}\right)$ $\frac{-16\cdot\sqrt{3}}{9}$

- **Casio:** Action→Calculation→ $fmin/fmax$

fmin(x^3−4x, x, 0, 2)

$\left\{\text{MinValue}=\frac{-16\cdot\sqrt{3}}{9}, x=\frac{2\cdot\sqrt{3}}{3}\right\}$

.

NOTE: TI only gives the x-value for the min/max so we then need to sub it back into our function. Casio gives us both!

Calculator Commands

- **Mathematica:** Minimise[ ] and Maximise[ ] commands.
- Minimise[$f[x], x$] will minimise $f[x]$ over its whole domain.
- To restrict the domain, we must use Minimise[$\{f[x], a \leq x \leq b\}, x$].

In[34]:= **Minimize**[{x^3 - 4 x, 0 < x < 2}, x]

Out[34]= $\left\{-\frac{16}{3\sqrt{3}}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$

TI UDF: Bisection Method

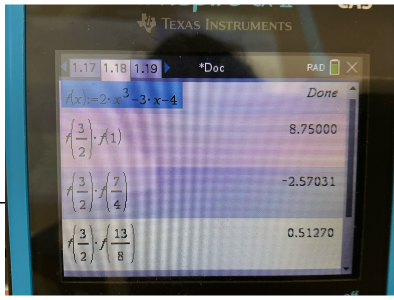
- **Overview:**
  - Apply the bisection method to a function to approximate x-intercepts.
- **Input:**
  - bisection(< function >, < variable >, < lower bound >, < upper bound >)
- **Other Notes:**
  - The program will ask for the threshold type to terminate the algorithm.
  - Select None [0] to provide a specific number of iterations.
  - Select x [1] to provide a threshold for $b - a$, after which the program will stop if $b - a$ becomes smaller than the threshold.
  - Select y [2] to provide a threshold for $|f(b) - f(a)|$, after which the program will stop if $|f(b) - f(a)|$ becomes smaller than the threshold.

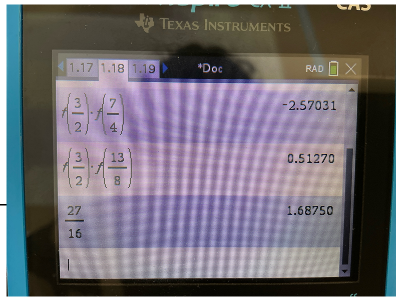
bisection$(x^2-2, x, 0, 1)$

Number of Iterations: 5

"n"	"a"	"m"	"b"	"f(a)"	"f(m)"	"f(b)"	"b−a"	"|f(b)−f(a)|"
0.	0.	0.5	1.	-2.	-1.75	-1.	1.	1.
1.	0.5	0.75	1.	-1.75	-1.4375	-1.	0.5	0.75
2.	0.75	0.875	1.	-1.4375	-1.23438	-1.	0.25	0.4375
3.	0.875	0.9375	1.	-1.23438	-1.12109	-1.	0.125	0.234375
4.	0.9375	0.96875	1.	-1.12109	-1.06152	-1.	0.0625	0.121094
5.	0.96875	0.984375	1.	-1.06152	-1.03101	-1.	0.03125	0.061523

Section E: Exam 2 (25 Marks)





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Question 6 (1 mark)

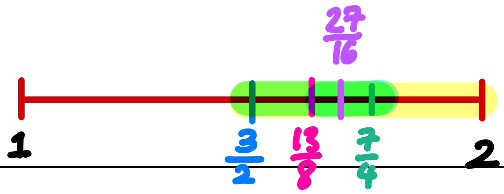
The equation $2x^3 - 3x - 4 = 0$ has one real solution, which lies in the interval $[1,2]$. Approximate the solution using the bisection method with a maximum error of 0.1. What is the approximate solution?

A. $x \approx 1.655$

B. $x \approx 1.6250$

C. $x \approx 1.6875$

D. $x \approx 1.6225$

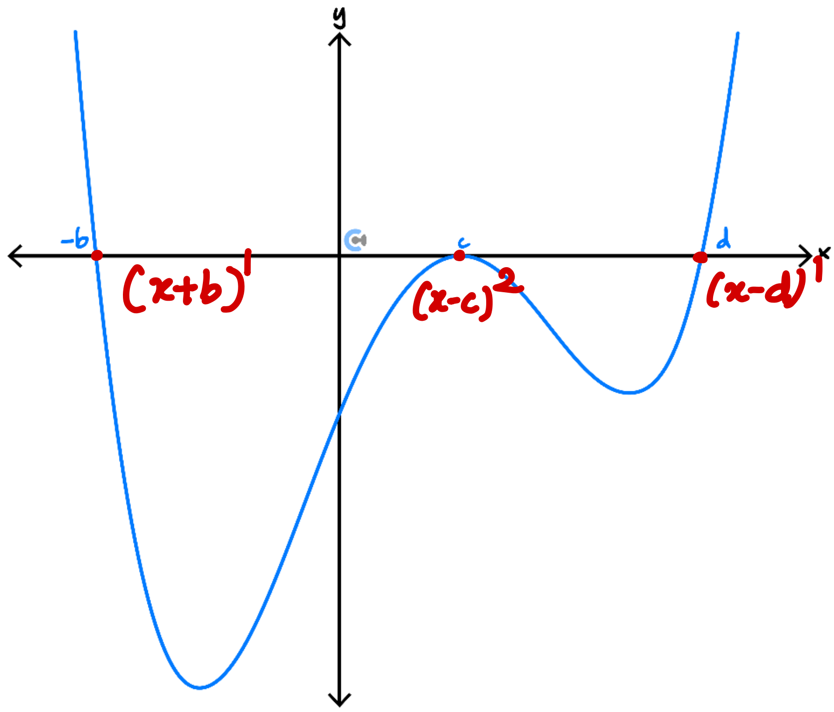


Estimate	Max Error
$\frac{3}{2}$	0.5
$\frac{7}{4}$	0.25
$\frac{13}{8}$	0.125
$\frac{27}{16}$	0.0625

Question 7 (1 mark)

The rule for a function with the graph below, where $b, c, d > 0$, could be:

+ve parabola



A. $y = -2(x+b)(x-c)^2(x-d)$

B. $y = 3(x+b)(x-c)^2(x-d)$

C. $y = -2(x-b)(x-c)^2(x-d)$

D. $y = 2(x-b)(x-c)^2(x-d)$

Question 8 (1 mark)

The polynomial $x^3 + (a+2)x^2 + bx + 8$ is perfectly divisible by $x + 2$ and has remainder of 2 when divided by $x - 3$. The values (a, b) are:

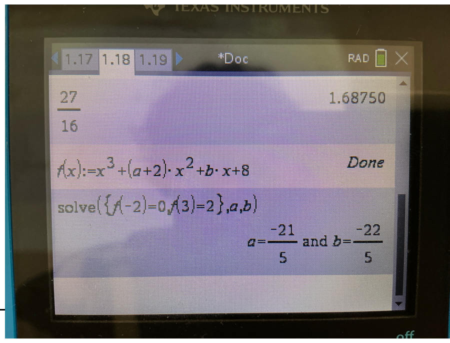
$f(-2) = 0$ $f(3) = 2$

A. $(-5, -6)$

B. $\left(-\frac{21}{5}, -\frac{22}{5}\right)$

C. $\left(-\frac{3}{5}, -\frac{9}{5}\right)$

D. $\left(-\frac{7}{5}, \frac{3}{5}\right)$



Question 9 (1 mark)

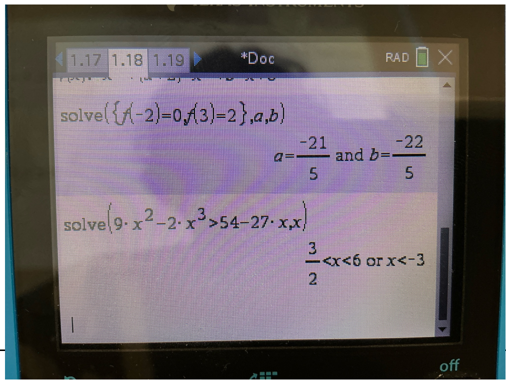
All real values of x that satisfy the inequality $9x^2 - 2x^3 > 54 - 27x$ are:

A. $x < -3$ or $\frac{3}{2} < x < 6$.

B. $x < -6$ or $\frac{3}{2} < x < 3$.

C. $x < -3$ or $x > -\frac{3}{2}$.

D. $-3 < x < \frac{3}{2}$ or $x > 6$.



Question 10 (1 mark)

The equation $x^3 - 3kx^2 + 5x = 0$ has exactly one solution when:

A. $k = \pm\frac{2\sqrt{5}}{3}$

B. $-\frac{2\sqrt{5}}{3} < k < \frac{2\sqrt{5}}{3}$

C. $k > \frac{2\sqrt{5}}{3}$

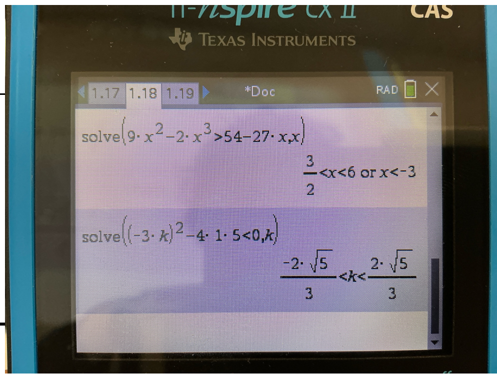
D. $k < -\frac{2\sqrt{5}}{3}$

$x(x^2 - 3kx + 5) = 0$

$(1) + (0) = (1)$

$\Delta < 0$

$(-3k)^2 - 4(1)(5) < 0$



Question 11 (9 marks)

A car is travelling along a straight road from A to B.
The car will travel along a section of road $ABCD$.

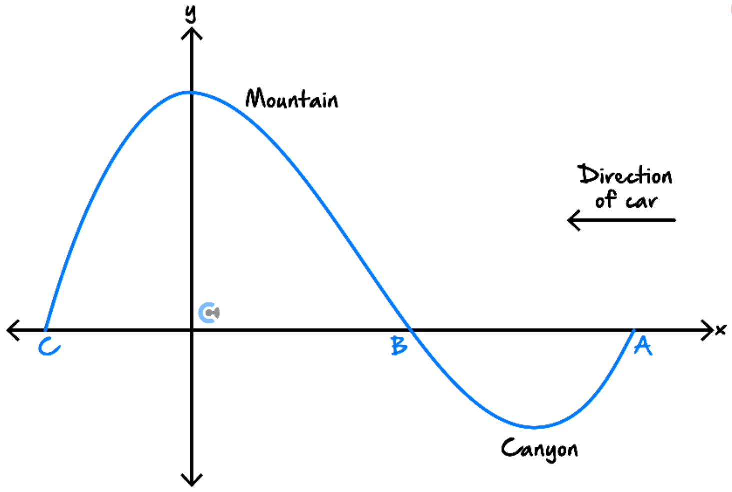
- Section AB passes along a bridge over a canyon.
- Section BC passes through a tunnel in a hill.

From A to C, the curve of the canyon and then hill, directly below and above the road, is modelled by the graph of:

$$y = \frac{1}{250}(px^3 + qx^2 + r)$$

Where p, q, and r are real constants.
All measurements are in kilometres and a diagram of this situation is shown below.

★ Peak of mountain occurs when graph touches the y axis



a. The curve defined from A to C passes through the points $(1, 0.652)$, $(2, 0.48)$, and $(5, -0.18)$.

i. Use this information to write down three simultaneous equations in terms of p, q, and r. Write these equations with integer coefficients. (3 marks)

$\frac{1}{250}(p+q+r) = 0.652$

$\hookrightarrow 4(p+q+r) = 652$

$p+q+r = 163 \quad \ldots (1)$

$\frac{1}{250}(8p+4q+r) = 0.48$

$\hookrightarrow 4(8p+4q+r) = 480$

$8p+4q+r = 120 \quad \ldots (2)$

$\frac{1}{250}(125p+25q+r) = -0.18$

$\hookrightarrow 4(125p+25q+r) = -180$

$125p+25q+r = -45 \quad \ldots (3)$

ii. Hence, verify that $p = 2, q = -19$, and $r = 180$. (1 mark)

$2 - 19 + 180 = 163$ ✓

$8(2) + 4(-19) + 180 = 120 \Rightarrow 16 - 76 + 180 = 120$ ✓

$125(2) + 25(-19) + 180 = 250 - 475 + 180 = -45$ ✓

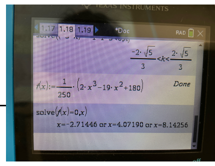
$\therefore p = 2,\ q = -19,\ r = 180$ //

b. Find the exact height of the mountain in metres. (1 mark)

y-int: $\therefore y = \frac{180}{250} = \frac{18}{25} = 0.72\text{km} = 720\text{m}$ //

c. Find the length of the tunnel and the length of the bridge. Give your answers correct to the nearest metre. (3 marks)

Solve $f(x) = 0$:



$\therefore C: (-2.71446, 0)$, $B: (4.07190, 0)$, $A: (8.14256, 0)$

Tunnel length $= AB = 8.14256 - 4.07190 = 4.07066\text{km}$

$= 4070.66\text{m}$ $\approx 4071\text{m}$ //

Bridge Length $= BC = 4.07190 + 2.71446$

$= 6.78636\text{ km}$

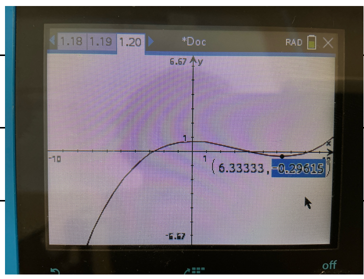
$= 6786.36\text{m} \approx 6786\text{m}$ //

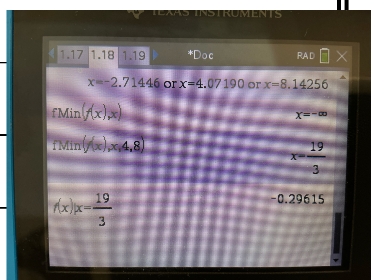
d. Find the maximum depth of the canyon below the road. Give your answer to the nearest metre. (1 mark)

Max Depth $= 0.29615\text{km}$

$= 296.15\text{m}$

$\approx 296.15\text{m}$





Question 12 (11 marks)

Consider the cubic polynomial $f(x) = x^3 + x^2 - 5x - 2$.

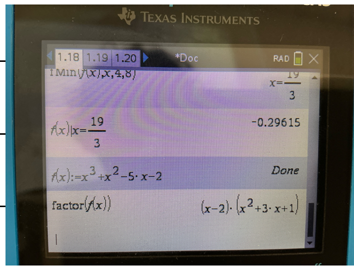
a. Explain why $f(x)$ must have a root between $x = 1$ and $x = 3$. (1 mark)

$f(1) = -5$
$f(3) = 19$

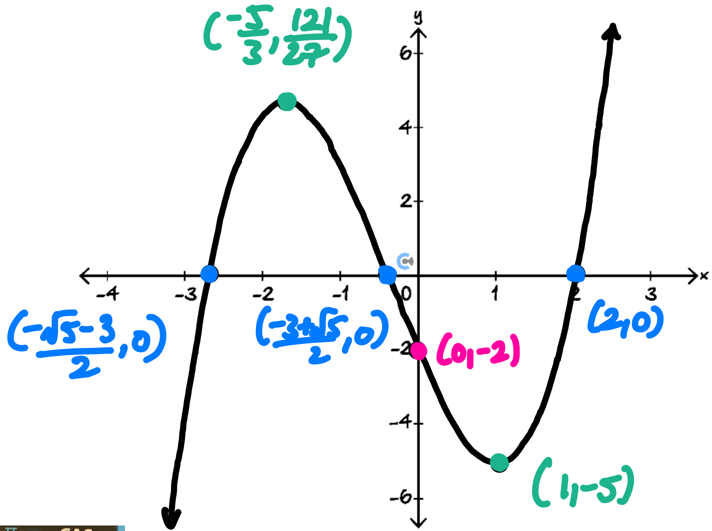
$\therefore$ Since $x=1$ & $x=3$ are on opposite sides of the x-axis $\Rightarrow$ $\therefore$ Must be an x-intercept inbetween

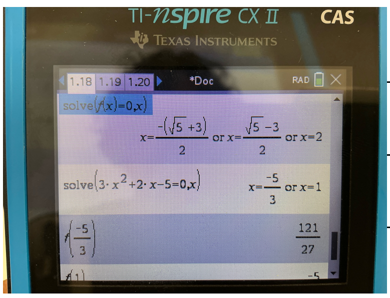
b. Write $f(x)$ in the form $f(x) = (x - a)Q(x)$ where $a > 0$ and $Q(x)$ is a quadratic function. (1 mark)

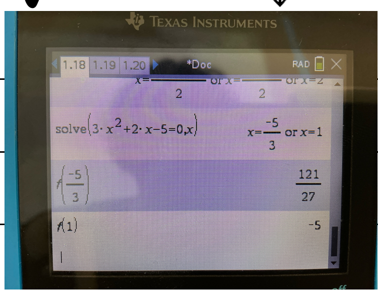
$f(x) = (x-2)(x^2+3x+1)$ //



c. It is known that the graph of $y = f(x)$ has turning points at x-values that are solutions to the equation $3x^2 + 2x - 5 = 0$. Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts and turning points with exact coordinates. (4 marks)







y-int: $y = -2$

d. Find the values of b such that $f(x) = b$ has:

intersection

i. One solution. (2 marks)

$y = f(x)$ & $y = b$

$b < -5$

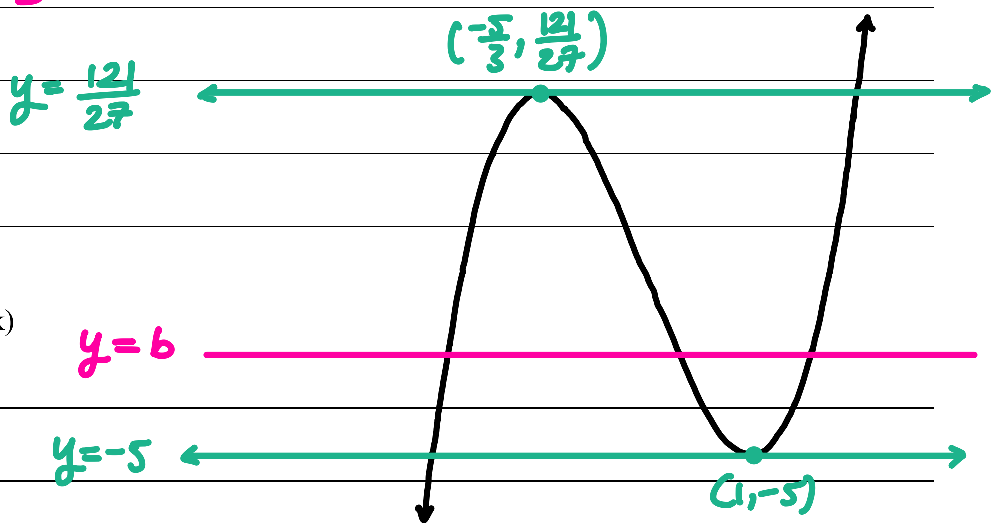
or

$b > \frac{121}{27}$

ii. Two solutions. (1 mark)

$\therefore b = -5$ or

$b = \frac{121}{27}$



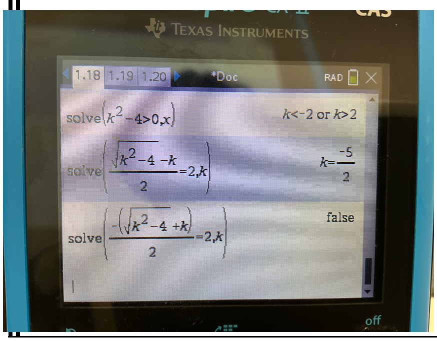
e. Find the values of k for which the equation $x^3 + (k-2)x^2 + (1-2k)x - 2 = 0$ has three solutions. (2 marks)

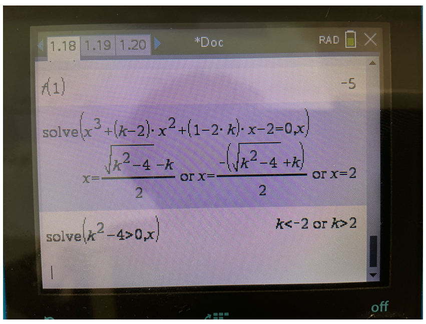
$\therefore x = \frac{-k \pm \sqrt{k^2 - 4}}{2}$ or $x = 2$

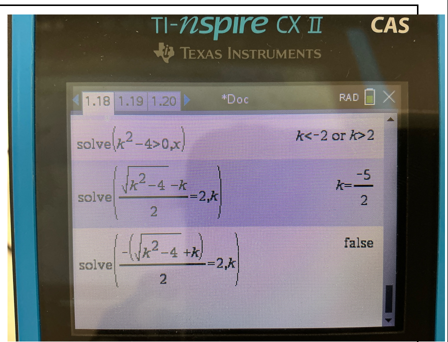
$\therefore$ For 3 solns : $\Delta > 0$

$k^2 - 4 > 0 \Rightarrow k < -2$ or $k > 2$

$k < -2$ & $k \neq \frac{-5}{2}$ or $k > 2$



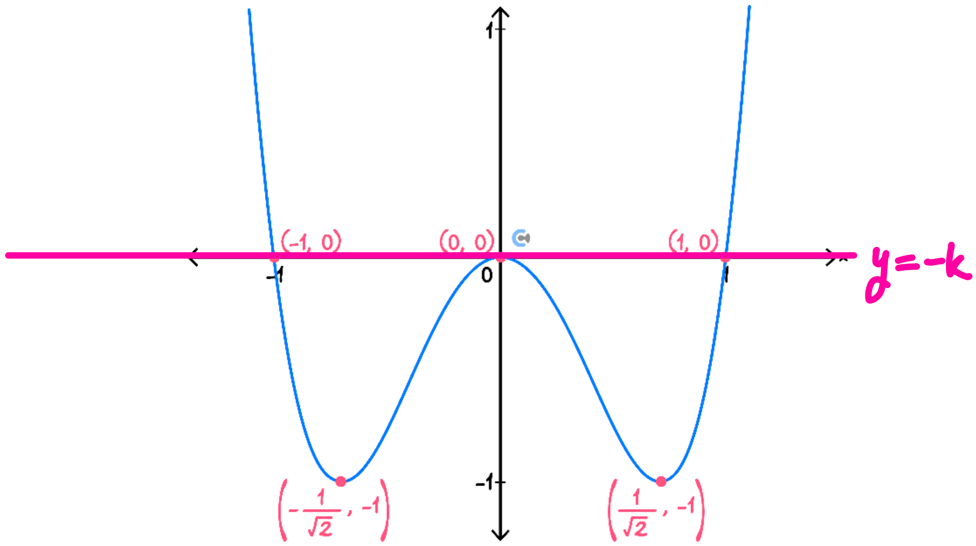




Section F: Extension Exam 1 (15 Marks)

Question 13 (4 marks)

The function $f(x)$ is a polynomial of degree 4. The graph of f is shown below.



a. Find the rule of $f(x)$. (2 marks)

$\because f(x) = a(x+1)(x-0)^2(x-1)$

Sub $\left(\frac{1}{\sqrt{2}}, -1\right)$:

$-1 = a\left(\frac{1}{\sqrt{2}}+1\right)\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}-1\right)$

$= a\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)$

$-1 = -\frac{1}{4}a \Rightarrow \therefore a = 4$

$f(x) = 4x^2(x+1)(x-1)$

$= 4x^2(x^2-1)$

$= 4x^4 - 4x^2$

b. Find the values of k such that $f(x) + k = 0$, where k is a real number, has an even number of real solutions. (2 marks)

$\therefore \underbrace{f(x)}_{y=f(x)} = \underbrace{-k}_{y=-k}$

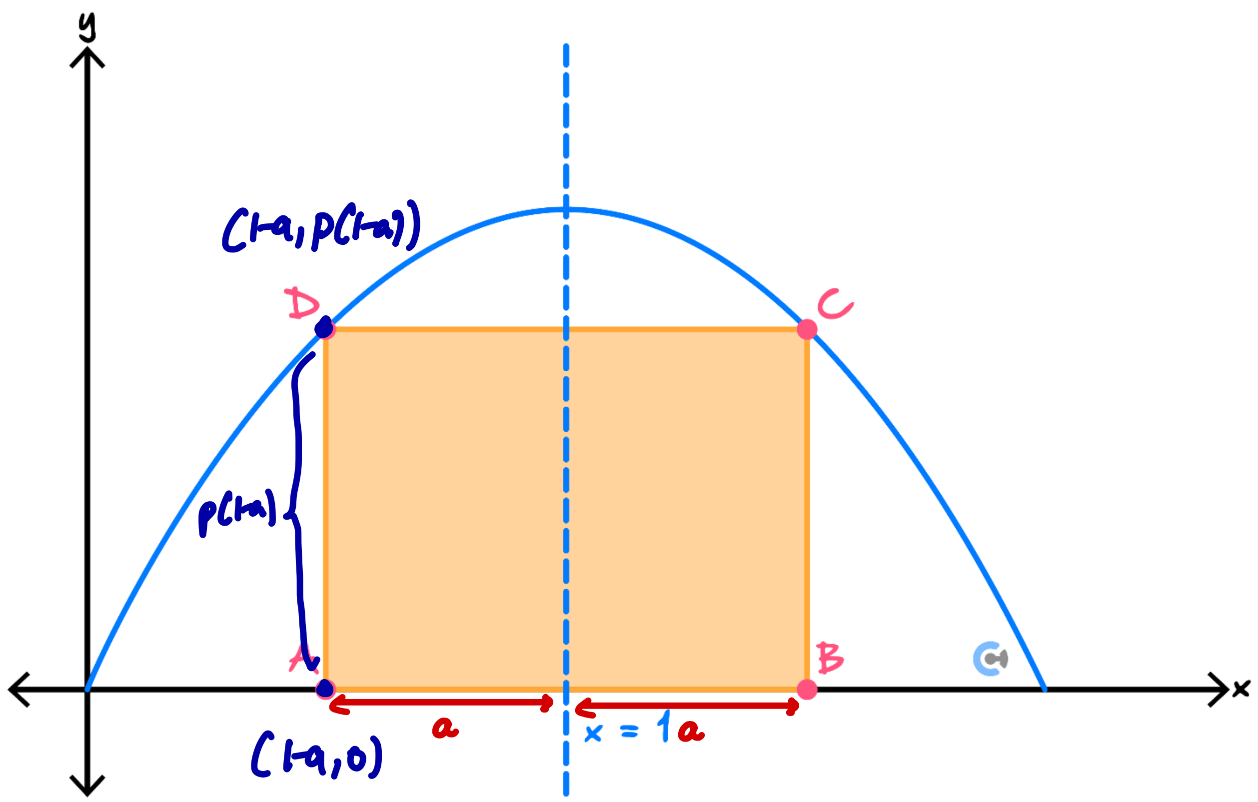
$\therefore -k \neq 0$

$\therefore k \neq 0$ or $k \in \mathbb{R}\setminus\{0\}$

Question 14 (6 marks)

Consider the parabola $p(x) = x(2 - x)$, where $0 \leq x \leq 2$.

A rectangle $ABCD$ is inscribed between the graph of p and the x-axis. Its vertices are a distance of a units from the axis of symmetry, $x = 1$, as shown below.



a. Find the value of a when the rectangle is a square. (3 marks)

$\because\ p(1-a) = 2a \Rightarrow (1-a)(2-(1-a)) = 2a$

$(1-a)(1+a) = 2a$

$1-a^2 = 2a$

$a^2+2a-1 = 0$

$(a+1)^2-2 = 0$

$(a+1)^2 = 2$

$a+1 = \pm\sqrt{2}$

$\therefore\ a = -1+\sqrt{2}$ or $a = -1-\sqrt{2}$ reject as $a>0$

~~**b.** Find the rational value of a such that the rectangle $ABCD$ has an area of $\frac{3}{4}$ square units. (3 marks)~~

$\because A = 2a \cdot p(1-a) = \frac{3}{4}$

$\therefore 2a\left(1-a^2\right) = \frac{3}{4}$

$2a - 2a^3 = \frac{3}{4}$

$\therefore 2a^3 - 2a + \frac{3}{4} = 0$

$8a^3 - 8a + 3 = 0$

Using RRT:

$RR = \pm \frac{\{1,3\}}{\{1,2,4,8\}} = \pm\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}\right\}$

$= 8\left(\frac{1}{2}\right)^3 - 8\left(\frac{1}{2}\right) + 3$

$= 1 - 4 + 3 = 0$

$\therefore a = \frac{1}{2}$ is the rational solution

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Question 15 (5 marks)

Consider the function $g(x) = (x^2 - 4kx + 3)(x^2 - 2x + k)$, where k is a real number.
Find all possible values of k such that $g(x)$ has:

a. Four real roots. (3 marks)

$f(x) = \underbrace{(x^2-4kx+3)}_{\Delta>0}\ \&\ \underbrace{(x^2-2x+k)}_{\Delta>0}$

$\therefore (-4k)^2 - 4(1)(3) > 0$ & $\therefore (-2)^2 - 4(1)(k) > 0$

$16k^2 - 12 > 0$ & $4 - 4k > 0$

$k^2 > \frac{3}{4}$ & $4k < 4$

$k > \frac{\sqrt{3}}{2}$ or $k < \frac{-\sqrt{3}}{2}$ & $k < 1$ $\Rightarrow$ $\therefore \frac{\sqrt{3}}{2} < k < 1$ or $k < \frac{-\sqrt{3}}{2}$

b. Two real roots. (2 marks)

Case 1: $\Delta > 0$ (2 solns) & $\Delta < 0$ (0 solns)

$f(x) = \underbrace{(x^2-4kx+3)}_{\Delta>0}\ \text{AND}\ \underbrace{(x^2-2x+k)}_{\Delta<0}$

$(-4k)^2 - 4(1)(3) > 0$ & $(-2)^2 - 4(1)(k) < 0$

$16k^2 - 12 > 0$; $4 - 4k < 0$

$k^2 > \frac{3}{4}$; $4k > 4$

$\therefore k > \frac{\sqrt{3}}{2}$ or $k < \frac{-\sqrt{3}}{2}$; $\therefore k > 1$

$\therefore k > 1$

Case 2: $\Delta < 0$ (0 solns) & $\Delta > 0$ (2 solns)

$f(x) = \underbrace{(x^2-4kx+3)}_{\Delta<0}\ \text{AND}\ \underbrace{(x^2-2x+k)}_{\Delta>0}$

$(-4k)^2 - 4(1)(3) < 0$; $(-2)^2 - 4(1)(k) > 0$

$16k^2 - 12 < 0$; $4 - 4k > 0$

$k^2 < \frac{3}{4}$; $4k < 4$

$\frac{-\sqrt{3}}{2} < k < \frac{\sqrt{3}}{2}$; $k < 1$

$\therefore \frac{-\sqrt{3}}{2} < k < \frac{\sqrt{3}}{2}$

$\because k = 1 \Rightarrow g(x) = (x^2 - 4x + 3)(x^2 - 2x + 1)$

$= (x-3)(x-1)(x-1)^2$

$= (x-3)(x-1)^3$ $\Rightarrow \therefore k = 1$ is also a solution

$\therefore k \in [1, \infty) \cup \left(\frac{-\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$

Section G: Extension Exam 2 (13 Marks)

Question 16 (1 mark)

Let $f(x) = x^3 + 3x^2 - 4x + 8$. The remainder when $f(x)$ is divided by $5x - 4$ is:

A. 104

B. 188

C. $\frac{904}{125}$

D. $\frac{617}{64}$

$R = f\left(\frac{4}{5}\right) = \frac{904}{125}$ //

Question 17 (1 mark)

Consider the quartic $y = (x-2)^2(x^2 + 4kx + 6)$. It is known that the quartic has three distinct x-intercepts. The possible values of k are:

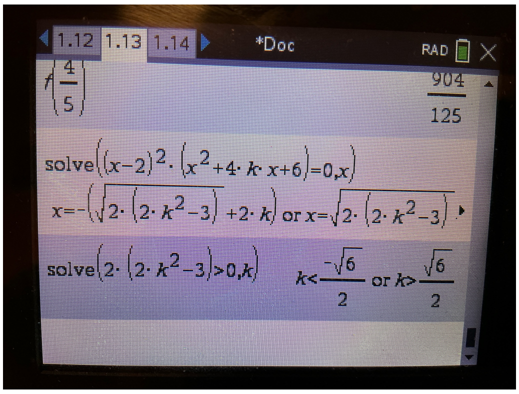
A. $k < -\sqrt{\frac{3}{2}}$ or $k > \sqrt{\frac{3}{2}}$

B. $-\sqrt{\frac{3}{2}} < k < \sqrt{\frac{3}{2}}$

C. $k = \pm\sqrt{\frac{3}{2}}$

D. $k < -\sqrt{3}$ or $k > \sqrt{3}$

$\frac{\sqrt{6}}{2} = \frac{\cancel{\sqrt{2}} \cdot \sqrt{3}}{\cancel{\sqrt{2}} \cdot \sqrt{2}} = \sqrt{\frac{3}{2}}$



Question 18 (1 mark)

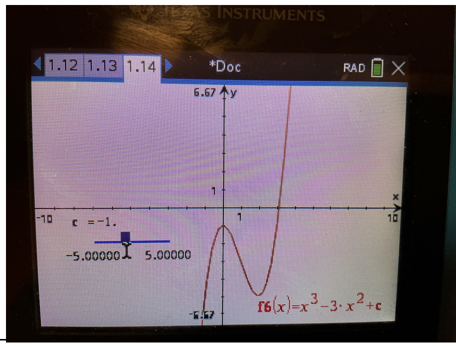
A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has one distinct x-intercept. All possible values of c are:

A. $c > 4$

B. $0 < c < 4$

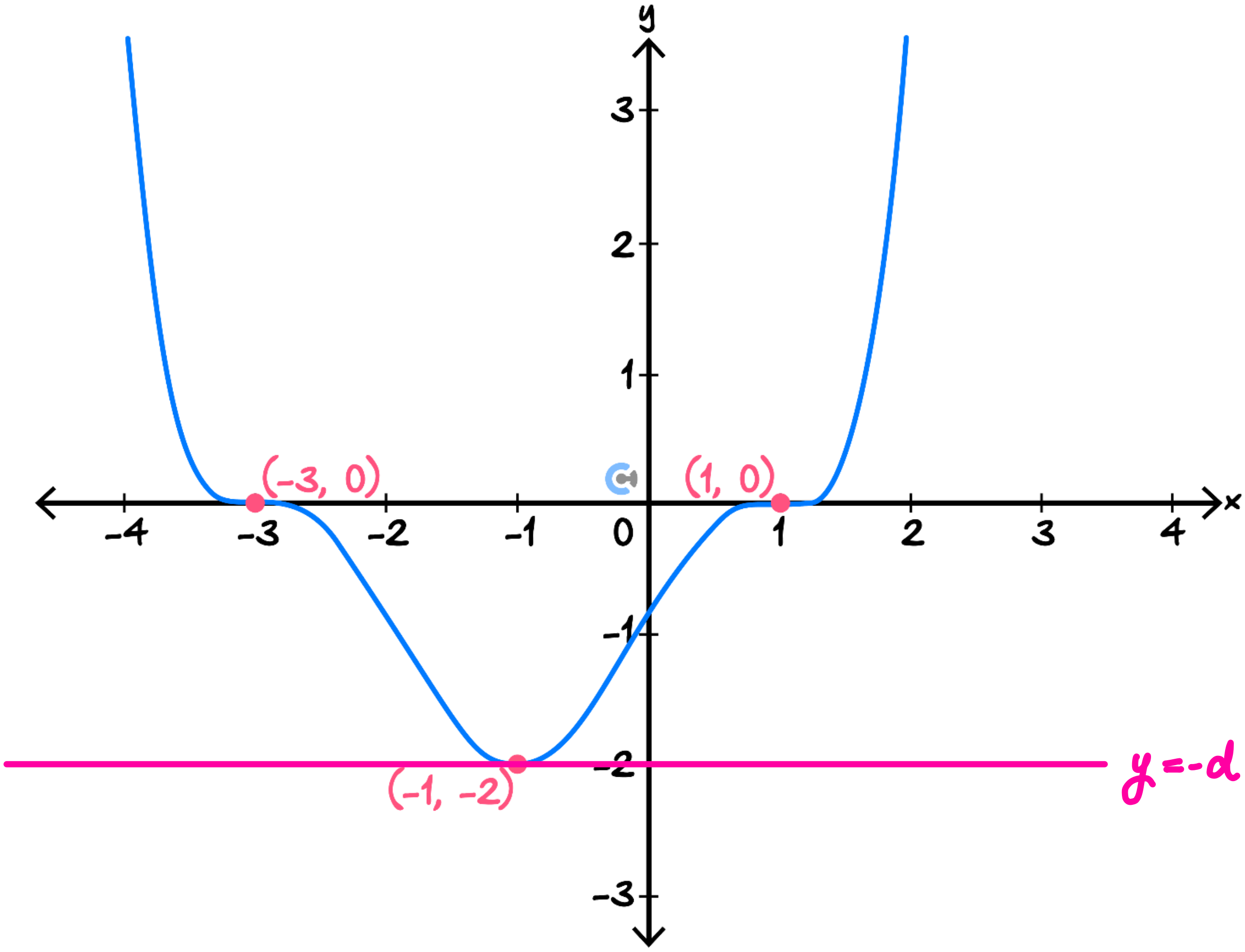
C. $c < 0$ or $c > 4$

D. $c > 4$



Question 19 (10 marks)

Consider the function of the form $f(x) = a(x-b)^3(x-c)^3$, where $b > c$, depicted on the graph below.



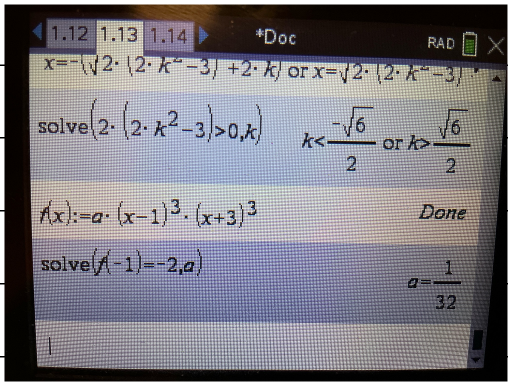
a. Find the values of a, b, and c. (2 marks)

$\because b = 1, \; c = -3$

Sub$(-1,-2)$:

$-2 = a(-1-1)^3(-1+3)^3$

$\therefore a = \frac{1}{32}$



$\therefore a = \frac{1}{32}, \; b = 1, \; c = -3$

~~**b.** Show that $x = -1$ is an axis of symmetry for the graph of f. (2 marks)~~

$$f(-1-x) = \frac{1}{32}(-1-x+3)^3(-1-x-1)^3$$
$$= \frac{1}{32}(2-x)^3(-2-x)^3$$
$$= \frac{-1}{32}(2-x)^3(x+2)^3$$
$$= \frac{1}{32}(x-2)^3(x+2)^3$$

$$f(-1+x) = \frac{1}{32}(-1+x+3)^3(-1+x-1)^3$$
$$= \frac{1}{32}(2+x)^3(x-2)^3$$

$\therefore$ Since $f(-1+x) = f(-1-x)$, $x=-1$ is the symmetry line

c. Find the value of $d > 0$ such that $f(x) + d = 0$ has one real solution. (1 mark)

$$\underbrace{f(x)}_{y=f(x)} = \underbrace{-d}_{y=-d}$$

$\because -d = -2$

$\therefore d = 2$

d. Consider the function $g(x) = (x + k + 3)^3(x + k - 1)^3$, where $k \in \mathbb{R}$.

i. Find the roots of g in terms of k. (1 mark)

$\therefore x = -k-3$ or $x = 1-k$

ii. Hence, find the values of k so that $g(x)$ has only positive roots. (2 marks)

$\therefore -k-3>0$ & $1-k>0$

$k<-3$ & $k<1$

$\therefore k<-3$

iii. A function h is said to be even if $h(x) = h(-x)$ for all x.
Find the value of k such that $g(x)$ is an even function. (2 marks)

$$(x+k+3)^3(x+k-1)^3 = (-x+k+3)^3(-x+k-1)^3$$
$$= (x-k-3)^3(x-k+1)^3$$

Case 1: $(x+k+3)^3(x+k-1)^3 = (x-k-3)^3(x-k+1)^3$

$x+k+3 = x-k-3$ & $x+k-1 = x-k+1$

$2k=-6$ $2k=2$

$k=-3$ & $k=1$ No!

Case 2: $(x+k+3)^3(x+k-1)^3 = (x-k-3)^3(x-k+1)^3$

$x+k+3 = x-k+1$ & $x+k-1 = x-k-3$

$2k=-2$ $2k=-2$

$k=-1$ & $k=-1$ YES!

$\therefore k=-1$

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