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VCE Mathematical Methods ½
Polynomials Exam Skills [0.6]
Workshop

Error Logbook:

Mistake/Misconception #1		**Mistake/Misconception #2**	
Question #: 3 (Exam 1)	Page #:	Question #: 5 (E1)	Page #:
Notes: $(x+a)^3 = x^3+3ax^2+3a^2x+a^3$ $(x-a)^3 = x^3-3ax^2+3a^2x-a^3$		Notes: Check for values of k in quadratic factor to see if x is a factor of it!	
Mistake/Misconception #3		**Mistake/Misconception #4**	
Question #:	Page #:	Question #:	Page #:
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Section A: Recap

Degree of Polynomial Functions

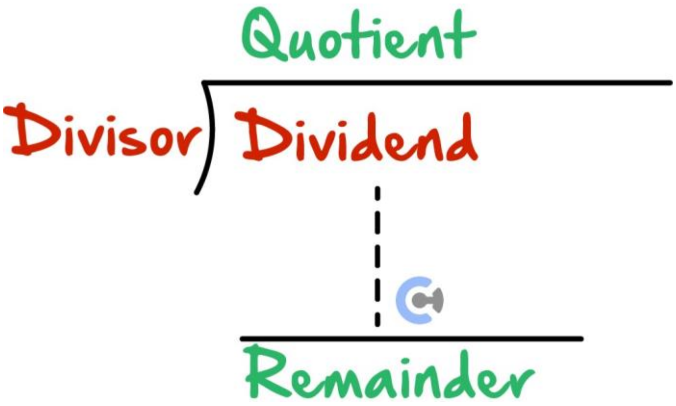
$$\textit{Degree} = \textit{Highest Power of the Polynomial}$$

Roots of Polynomial Functions

$$\textit{Roots} = x\textit{-intercept}$$

Polynomial Long Division

- Division of polynomials:



$$\frac{\textit{Dividend}}{\textit{Divisor}} = \textit{Quotient} + \frac{\textit{Remainder}}{\textit{Divisor}}$$

Space for Personal Notes

Remainder Theorem

Remainder of $\frac{p(x)}{x-a} = P(a)$

- **Definition**:
  - Finds the remainder of long division without the need of long division,

$$\boldsymbol{when\ P(x)\ is\ divided\ by\ (x-\alpha), the\ remainder\ is\ P(\alpha).}$$

- **Steps**
  1. Find x-values which make the divisor equal to 0.
  2. Substitute it into the dividend function.

Factor Theorem

- For every x-intercept, there is a factor:

$$\boldsymbol{If\ P(\alpha) = 0\ then, (x-\alpha)\ is\ a\ factor\ of\ P(x).}$$

Factorising Polynomials

- The steps are:
  - Find a single root by trial and error.
    - (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

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Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$\boldsymbol{Potential\ root = \pm\frac{Factors\ of\ constant\ term\ a_0}{Factors\ of\ leading\ coefficient\ a_n}}$$

- If the roots are rational numbers, the roots can only be $\pm\frac{\text{Factors of constant term } a_0}{\text{Factors of leading coefficient } a_n}$.

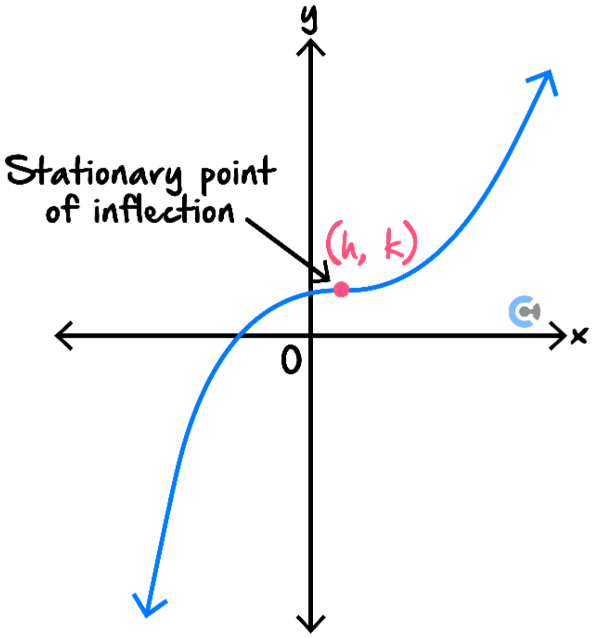
Sum and Difference of Cubes

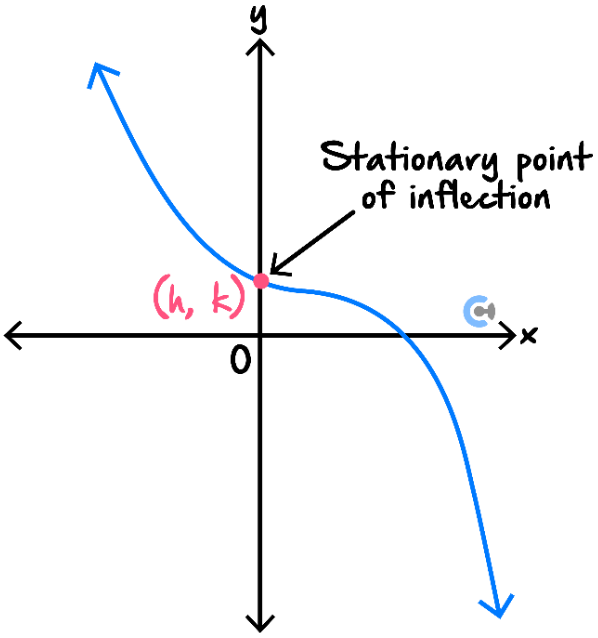
$$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$$

$$\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

Graphs of $a(x - h)^n + k$, where n is an Odd Positive Integer

- All graphs look like a “cubic”.

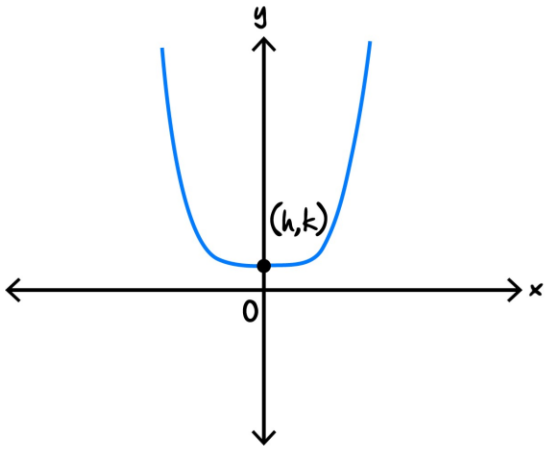


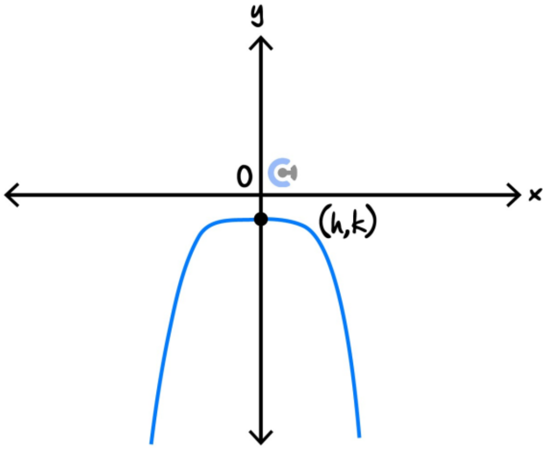


- The point (h, k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

Graphs of $a(x-h)^n+k$, where n is an Even Positive Integer

- All graphs look like a “quadratic”.

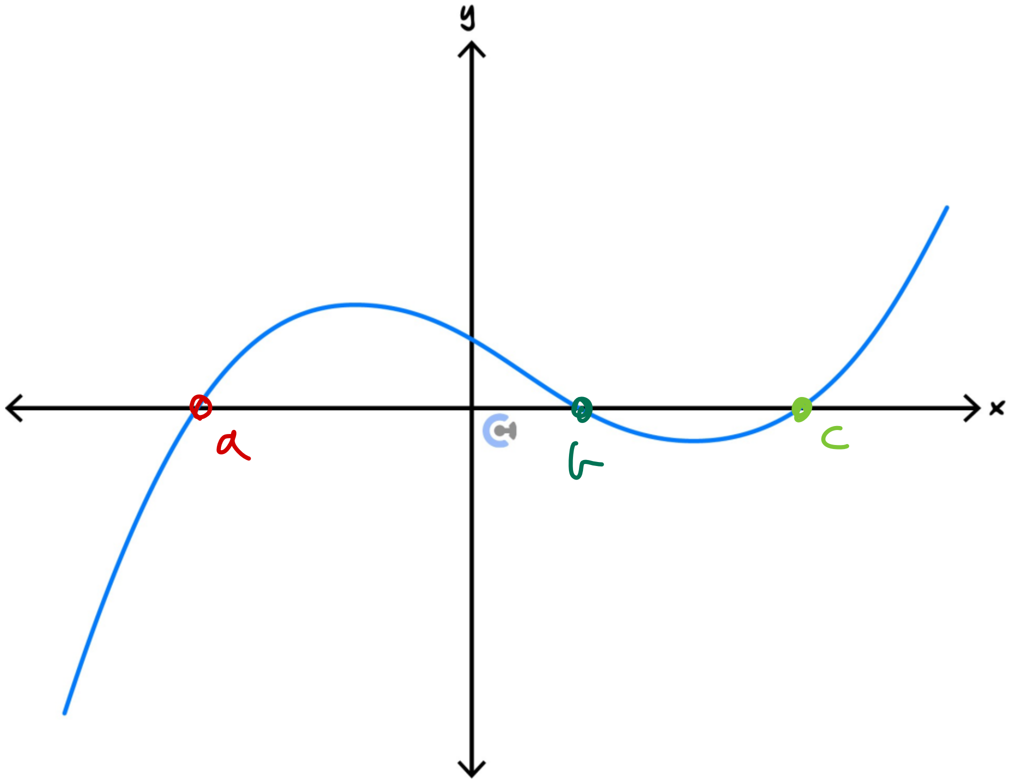




- The point (h,k) gives us the turning point.

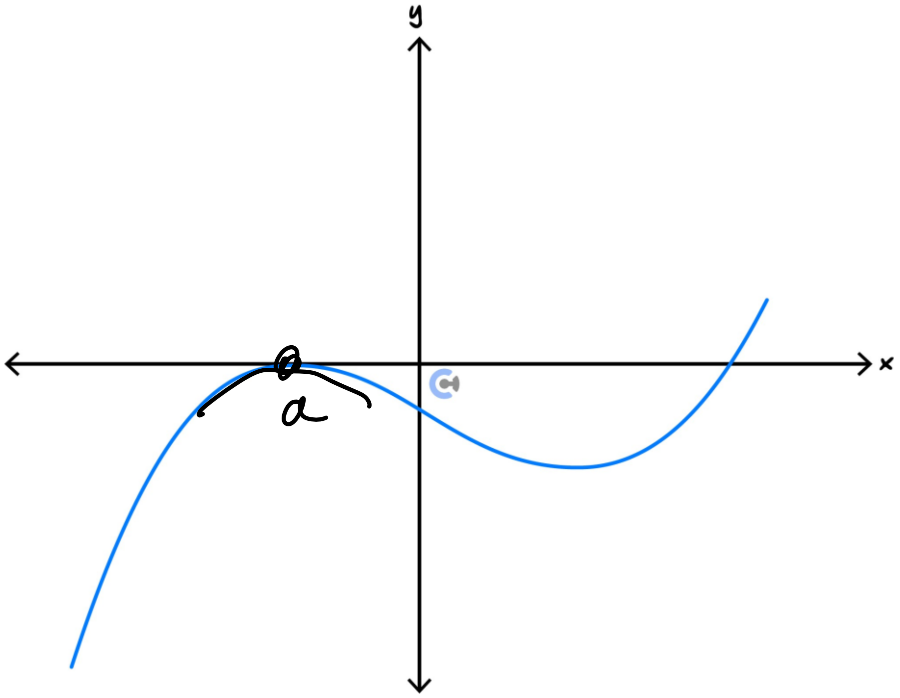
Graphs of Factorised Polynomials

- **All non-repeated linear factors correspond to x-intercepts of the graph.**



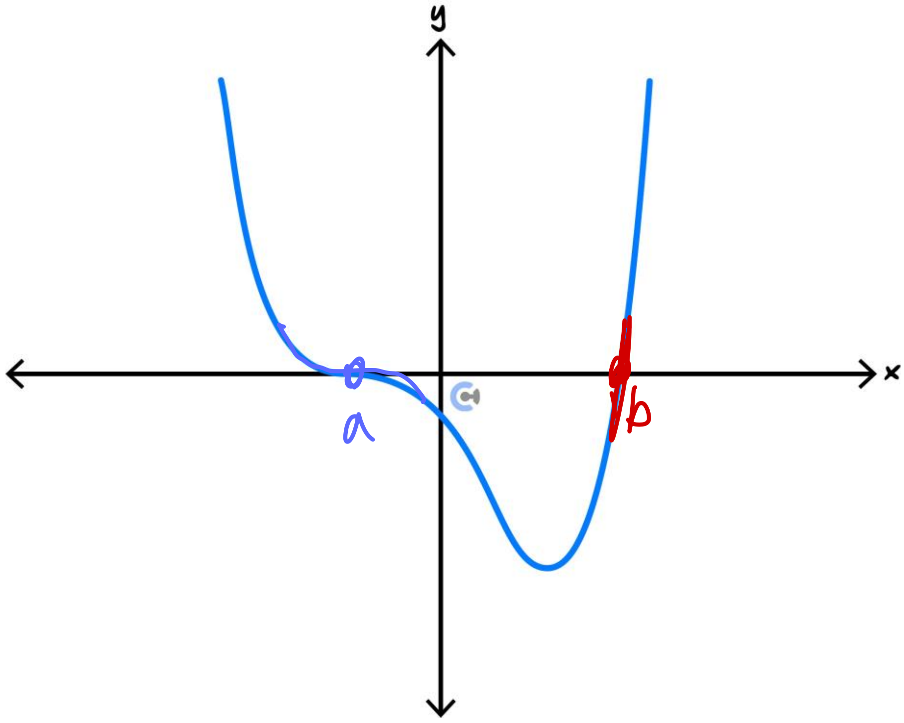
- E.g., $f(x)=(x-a)(x-b)(x-c)$ results in x-intercepts at $(a,0)$, $(b,0)$, and $(c,0)$.

- All squared linear factors correspond to x-intercepts and T.P. of the graph.



- E.g., $f(x) = \underline{(x-a)}^2(x-b)$ will have an x-intercept $(a, 0)$ which is also a local minimum/maximum.

- All cubed linear factors correspond to x-intercepts and SPI of the graph.



- E.g., $f(x) = \underline{(x-a)}^3\underline{(x-b)}$ has an x-intercept $(a, 0)$ which is also a stationary point of inflection.

Steps to Graphing Factorised Polynomials

- **Steps:**
  1. Plot x-intercepts.
  2. Determine whether the polynomial is positive or negative.
  3. Use the repeated factors to deduce the shape.
     - Non - Repeated: Only x-intercept.
     - Even Repeated: x-intercept and a turning point.
     - Odd Repeated: x-intercept and a stationary point of inflection.

Solving the Polynomial Inequality $f(x) > 0$

- Steps:
  1. Find the x-intercepts.
  2. Sketch the polynomial.
  3. Shade the places where the y-values are positive.

When does a cubic have n solutions?

- Steps:
  1. Factorise out the x term.
  2. Since the x term gives 1 solution, use discriminant to find when the quadratic has $n-1$ solutions.

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Definition

Bisection Method

- Step 1: Pick a random interval $[a, b]$ where $f(a) \times f(b) =$ Negative.
- Step 2: Find a midpoint to estimate the root.

$$\text{where } m = \frac{a+b}{2}$$

- Step 3: Create a new interval $[a, b]$ by making m either new a or new b.

$$\text{If } f(a) \times f(m) < 0$$

$$\text{New Interval: } [a, m]$$

$$\text{If } f(b) \times f(m) < 0$$

$$\text{New Interval: } [m, b]$$

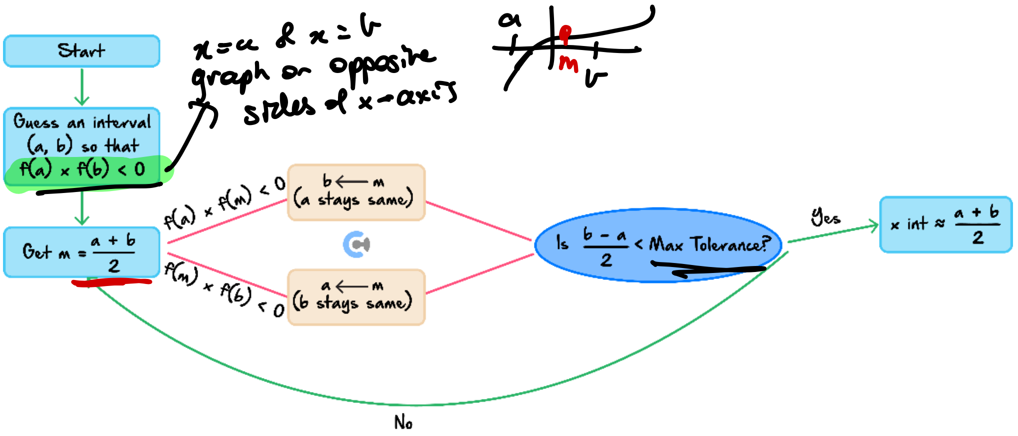
- Step 4: Repeat until the interval becomes short enough for good accuracy.
  - The smaller the interval $[a, b]$, more accurate our estimation gets.

$$\text{If } \frac{b-a}{2} < \text{Max Tolerance,}$$

$$\text{We stop.}$$

  - Maximum error is half of the width of the interval.

$$\text{Max Error} = \frac{b-a}{2}$$



Section B: Warmup

Question 1

a. Solve the inequality $x^3 + x + 6 > 4x^2$.

$x^3 - 4x^2 + x + 6 > 0$

Let $P(x) = x^3 - 4x^2 + x + 6$

$P(1) = 1 - 4 + 1 + 6 = 4 = 0$

$P(-1) = -1 - 4 - 1 + 6 = 0$

$\therefore$ $x + 1$ is a factor

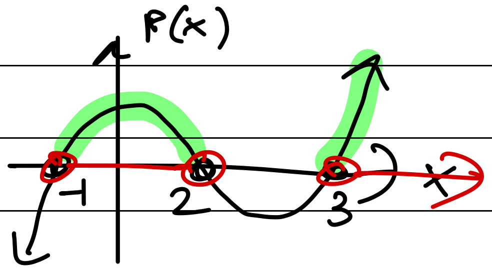
$$\begin{array}{r|l} & x^2 - 5x + 6 \\ \hline x+1 & x^3 - 4x^2 + x + 6 \\ & -(x^3 + x^2) \downarrow \\ & -5x^2 + x \\ & -(-5x^2 - 5x) \downarrow \\ & 6x + 6 \end{array}$$

$R = 0$

$P(x) = (x+1)(x^2 - 5x + 6)$

$= (x+1)(x-3)(x-2)$

$P(x) > 0$



$\therefore$ $-1 < x < 2$

OR $x > 3$

b. Find the values of k such that $x^3 + 2kx^2 + 3x = 0$ has only one real solution.

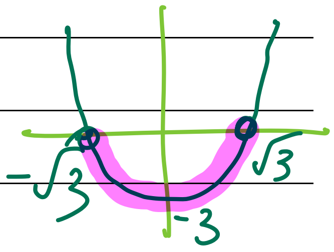
$x(\underbrace{x^2 + 2kx + 3}) = 0$

Need $\Delta < 0$ (No Solutions for Quad)

$(2k)^2 - 4(1)(3) < 0$

$4k^2 - 12 < 0$

$k^2 - 3 < 0$



Ans : $-\sqrt{3} < k < \sqrt{3}$

$*$ Error $= \frac{b-a}{2}$

Need error < 0.1

c. Apply the bisection method with initial interval $[1,2]$ and tolerance 0.1 to find an approximate solution to the equation $x^2 - 2 = 0$.

$f(x) = x^2 - 2$

a	b	$m = \frac{a+b}{2}$	$f(a)$	$f(b)$	$f(m)$	Error $*$
1	2	$3/2$	-1	2	$\frac{9}{4} - \frac{8}{4} = \frac{1}{4}$	0.5
1	$3/2$	$5/4$	-1	$\frac{1}{4}$	$\frac{25}{16} - \frac{32}{16} < 0$	0.25
$5/4$	$3/2$	$11/8$	< 0	$\frac{1}{4}$	$\frac{121}{64} - \frac{128}{64} < 0$	0.125
$\frac{11}{8}$	$3/2$					$\boxed{0.0625}$

!!

$\left(\frac{11}{8}, \frac{3}{2}\right) \to$ Final interval

Approx $= \dfrac{\frac{11}{8} + \frac{3}{2}}{2} = \frac{23}{16}$ //

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Section C: Exam 1 (23 Marks)

Question 2 (9 marks)

Let $f(x) = ax^3 - 5x^2 + bx + 9$. When $f(x)$ is divided by $x - 2$ the remainder is -7 and when $f(x)$ is divided by $x + 1$ the remainder is 8.

a. Show that $a = 2$ and $b = -6$. (2 marks)

$f(2) = -7 :\quad 8a - 20 + 2b + 9 = -7$

$4a + b = 2 \quad [1]$

$f(-1) = 8 :\quad -a - 5 - b + 9 = 8$

$a + b = -4 \quad [2]$

$[1] - [2] :\quad 3a = 6$

$a = 2$

$2 + b = -4$

$\therefore b = -6$

$\therefore a = 2 \;\&\; b = -6$ as required

b. Express $f(x)$ as the product of three linear factors. (3 marks)

$f(x) = 2x^3 - 5x^2 - 6x + 9$

$f(1) = 2 - 5 - 6 + 9 = 11 - 11 = 0$

$\therefore x - 1$ is a factor

$$\begin{array}{r} 2x^2 - 3x - 9 \\ x - 1 \;\big)\; 2x^3 - 5x^2 - 6x + 9 \\ -(2x^3 - 2x^2) \\ \hline -3x^2 - 6x \\ -(-3x^2 + 3x) \\ \hline -9x + 9 \\ -(-9x + 9) \\ \hline 0 \end{array}$$

$f(x) = (x-1)(2x^2 - 3x - 9)$

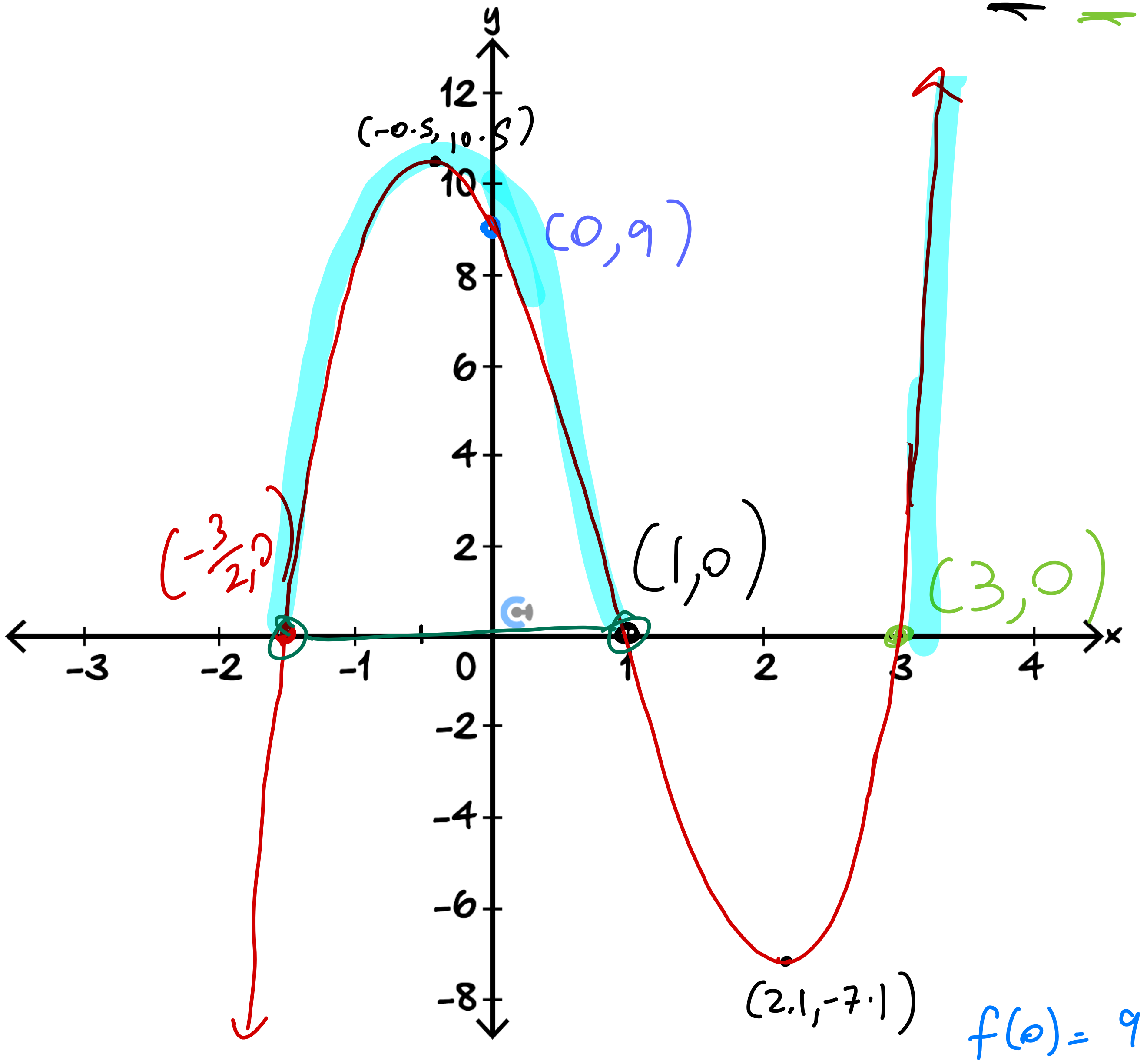
$= (x-1)(2x^2 - 6x + 3x - 9)$

$= (x-1)[2x(x-3) + 3(x-3)]$

$= (x-1)(2x+3)(x-3)$

c. Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts. Note that f has turning points at approximately $(-0.5, 10.5)$ and $(2.1, -7.1)$. (2 marks)

$f(x) = (2x+3)(x-1)(x-3)$



$f(0) = 9$

$(0, 9)$

d. Hence, solve the inequality $2x^3 - 5x^2 - 6x > -9$. (2 marks)

$2x^3 - 5x^2 - 6x + 9 > 0$

$\therefore f(x) > 0$

$\boxed{-\frac{3}{2} < x < 1 \text{ or } x > 3}$

$(x+b)^3 = x^3 + 3x^2b + 3b^2x + b^3$

Question 3 (5 marks)

Consider the function $f(x) = \frac{x^3}{4} - \frac{3x^2}{4} + \frac{3x}{4} + \frac{7}{4}$.

a. Write $f(x)$ in the form $a(x+b)^3 + c$ for real values a, b, and c. (2 marks)

$$f(x) = \frac{1}{4}\left(x^3 - 3x^2 + 3x + 7\right)$$

$$= \frac{1}{4}\left(\underbrace{x^3 - 3x^2 + 3x - 1}_{(x-1)^3} + 8\right)$$

$$= \frac{1}{4}\left[(x-1)^3 + 8\right]$$

$$= \frac{1}{4}(x-1)^3 + 2$$

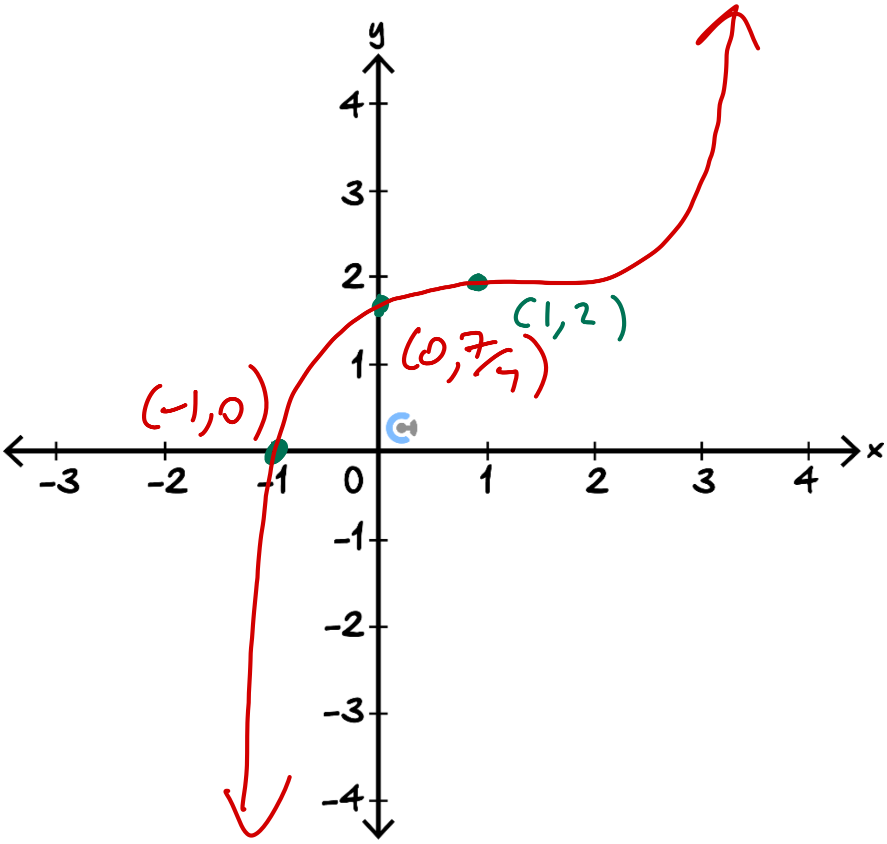
$a = \frac{1}{4}$

$b = -1$

$c = 2$

b. Sketch the graph of $y = f(x)$ on the axes below. Label any axes intercepts and stationary points of inflection with coordinates. (3 marks)

SPI : $(1, 2)$



Question 4 (4 marks)

The bisection method may be used to approximate $\sqrt{3}$ by finding a root to $x^2 - 3 = 0$.

a. Use the bisection method with initial interval [1,2] and tolerance 0.1 to find an approximate solution to $x^2 - 3 = 0$. Leave your answer in the form $\frac{a}{b}$, for positive integers a and b. (3 marks)

$f(x) = x^2 - 3$

a	b	$m = \frac{a+b}{2}$	$f(a)$	$f(b)$	$f(m)$	Error $= \frac{b-a}{2}$
1	2	$\frac{3}{2}$	-2	1	$\frac{9}{4} - \frac{12}{4} < 0$	0.5
$\frac{3}{2}$	2	$\frac{7}{4}$	< 0	1	$\frac{49}{16} - \frac{48}{16} > 0$	0.25
$\frac{3}{2}$	$\frac{7}{4}$	$\frac{13}{8}$	< 0	> 0	$\frac{169}{64} - \frac{192}{64} < 0$	0.125
$\frac{13}{8}$	$\frac{7}{4}$					0.0625

$\left(\frac{13}{8}, \frac{7}{4}\right)$

$\sqrt{3} \approx \dfrac{\frac{13}{8} + \frac{7}{4}}{2} = \frac{27}{16}$

b. Determine whether $\frac{7}{4}$ is more than or less than $\sqrt{3}$. (1 mark)

$\sqrt{3} \approx \left(\frac{13}{8}, \frac{7}{4}\right)$

$\frac{7}{4} > \sqrt{3}$

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Question 5 (5 marks)

Consider $f(x) = x^3 - 2kx^2 + 4kx + 4x$, where k is a real constant.

Find the values of k such that $f(x) = 0$ has:

a. One solution. (3 marks)

$$f(x) = x\left(\underbrace{x^2 - 2kx + (4k+4)}\right)$$

Quad factor has $\Delta = (-2k)^2 - 4(1)(4k+4)$

$= 4(k^2 - 4k - 4)$

1 solution, need $\Delta < 0$

$4(k^2 - 4k - 4) < 0$

$k^2 - 4k - 4 < 0$

$(k-2)^2 - 8 < 0$

$$\boxed{\therefore\ 2 - 2\sqrt{2} < k < 2 + 2\sqrt{2}}$$

b. Two solutions. (1 mark)

$\Delta = 0$ OR $x = 0$ is a solution of $x^2 - 2kx + 4k + 4$

$(k-2)^2 - 8 = 0$

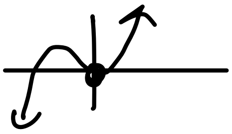
$4k + 4 = 0$

$k = 2 \pm 2\sqrt{2}$

$k = -1$

c. Three solutions. (1 mark)

$\Delta > 0$ AND $k \neq -1$



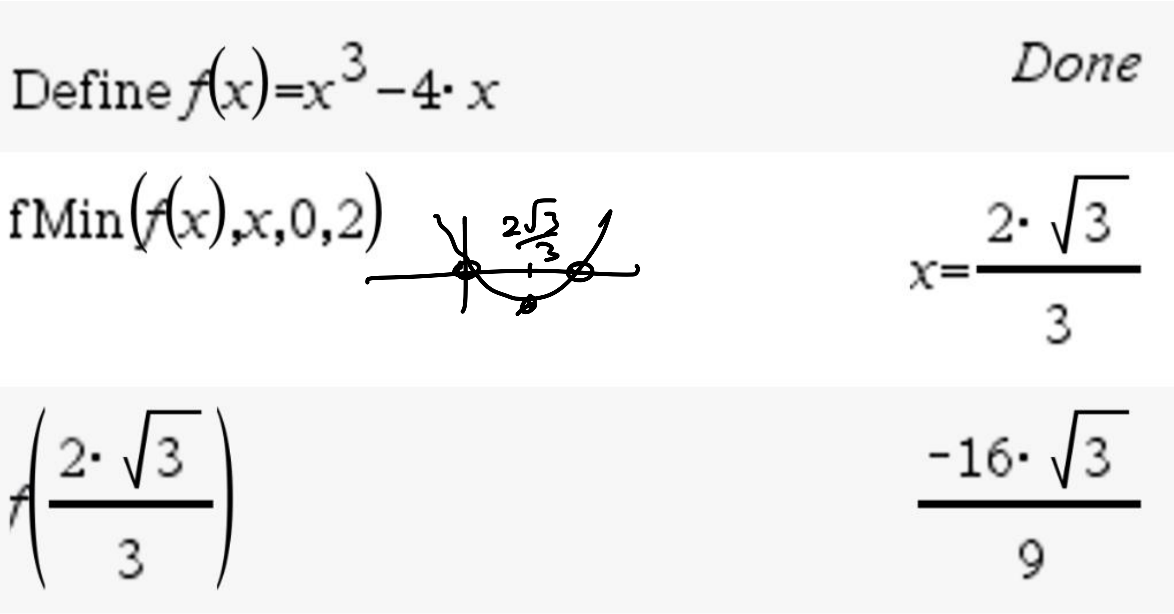
$k < 2 - 2\sqrt{2}$ or $k > 2 + 2\sqrt{2}$

but $k \neq -1$

Section D: Tech Active Exam Skills

Calculator Commands: Turning Point

- ALWAYS sketch the graph to find approximate bounds for where the turning point you want is located.
- To find a local maximum we maximise the function over a specific domain.
- To find a local minimum we minimise the function over a specific domain.
- **TI and Casio:** Use $fmin(expression, variable, lower\ (optional), upper\ (optional))$ or $fmax(expression, variable, lower\ (optional), upper\ (optional))$.
- **TI:** Menu→ $4 \rightarrow \frac{7}{8}$.



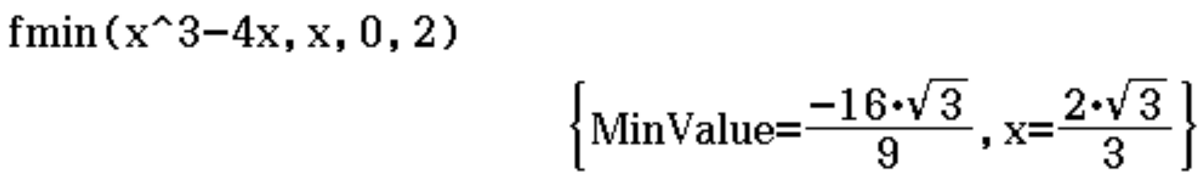
Define $f(x)=x^3-4\cdot x$ Done

fMin$(f(x),x,0,2)$ $x=\frac{2\cdot\sqrt{3}}{3}$

$\frac{2\sqrt{3}}{3}$

$f\left(\frac{2\cdot\sqrt{3}}{3}\right)$ $\frac{-16\cdot\sqrt{3}}{9}$

- **Casio:** Action→Calculation→ $fmin/fmax$



fmin(x^3−4x, x, 0, 2)

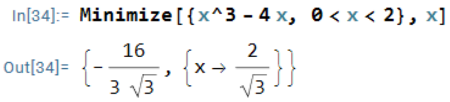
$\left\{\text{MinValue}=\frac{-16\cdot\sqrt{3}}{9}, x=\frac{2\cdot\sqrt{3}}{3}\right\}$

.

NOTE: TI only gives the x-value for the min/max so we then need to sub it back into our function. Casio gives us both!

<u>Calculator Commands</u>

- **Mathematica:** Minimise[ ] and Maximise[ ] commands.
- Minimise[$f[x], x$] will minimise $f[x]$ over its whole domain.
- To restrict the domain, we must use Minimise[$\{f[x], a \leq x \leq b\}, x$].

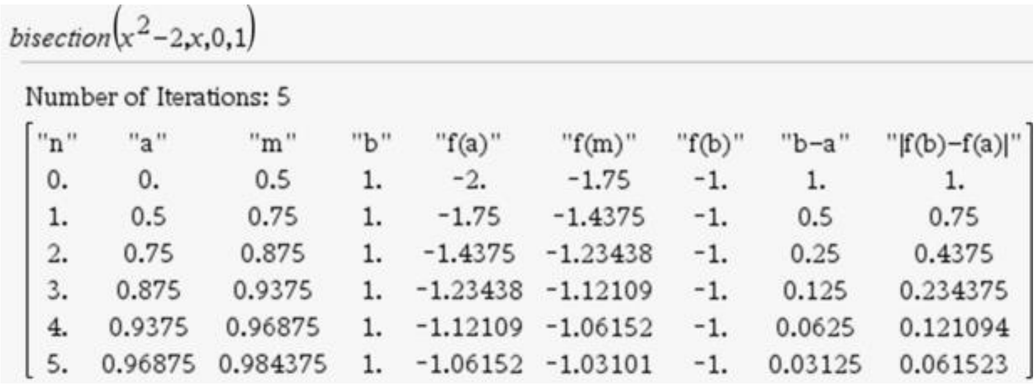


```
In[34]:= Minimize[{x^3 - 4 x, 0 < x < 2}, x]
```

Out[34]= $\left\{-\frac{16}{3\sqrt{3}}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$

<u>TI UDF:</u> Bisection Method

- **Overview:**
  - Apply the bisection method to a function to approximate x-intercepts.
- **Input:**
  - $\text{bisection}(<\text{function}>, <\text{variable}>, <\text{lower bound}>, <\text{upper bound}>)$
- **Other Notes:**
  - The program will ask for the threshold type to terminate the algorithm.
  - Select None [0] to provide a specific number of iterations.
  - Select x [1] to provide a threshold for $b - a$, after which the program will stop if $b - a$ becomes smaller than the threshold.
  - Select y [2] to provide a threshold for $|f(b) - f(a)|$, after which the program will stop if $|f(b) - f(a)|$ becomes smaller than the threshold.



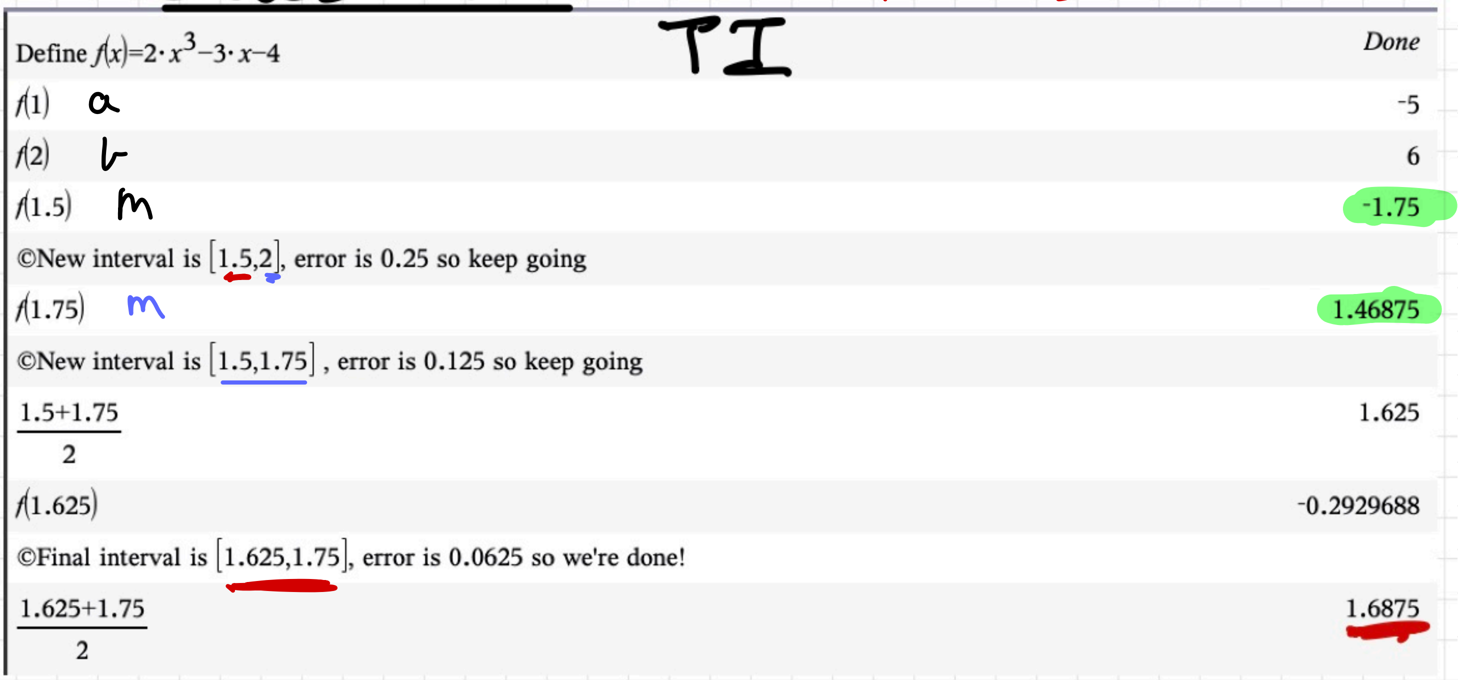
$bisection(x^2-2,x,0,1)$

```
Number of Iterations: 5
["n"   "a"      "m"       "b"  "f(a)"     "f(m)"    "f(b)" "b−a"    "|f(b)−f(a)|"]
 0.    0.       0.5       1.   -2.        -1.75     -1.    1.       1.
 1.    0.5      0.75      1.   -1.75      -1.4375   -1.    0.5      0.75
 2.    0.75     0.875     1.   -1.4375    -1.23438  -1.    0.25     0.4375
 3.    0.875    0.9375    1.   -1.23438   -1.12109  -1.    0.125    0.234375
 4.    0.9375   0.96875   1.   -1.12109   -1.06152  -1.    0.0625   0.121094
 5.    0.96875  0.984375  1.   -1.06152   -1.03101  -1.    0.03125  0.061523
```

Question 6

$f[x\_]:=2x^3-3x-4$

TI



Define $f(x)=2\cdot x^3-3\cdot x-4$ *Done*

$f(1)$ a -5

$f(2)$ b 6

$f(1.5)$ m -1.75

©New interval is $[1.5,2]$, error is 0.25 so keep going

$f(1.75)$ m 1.46875

©New interval is $[1.5,1.75]$, error is 0.125 so keep going

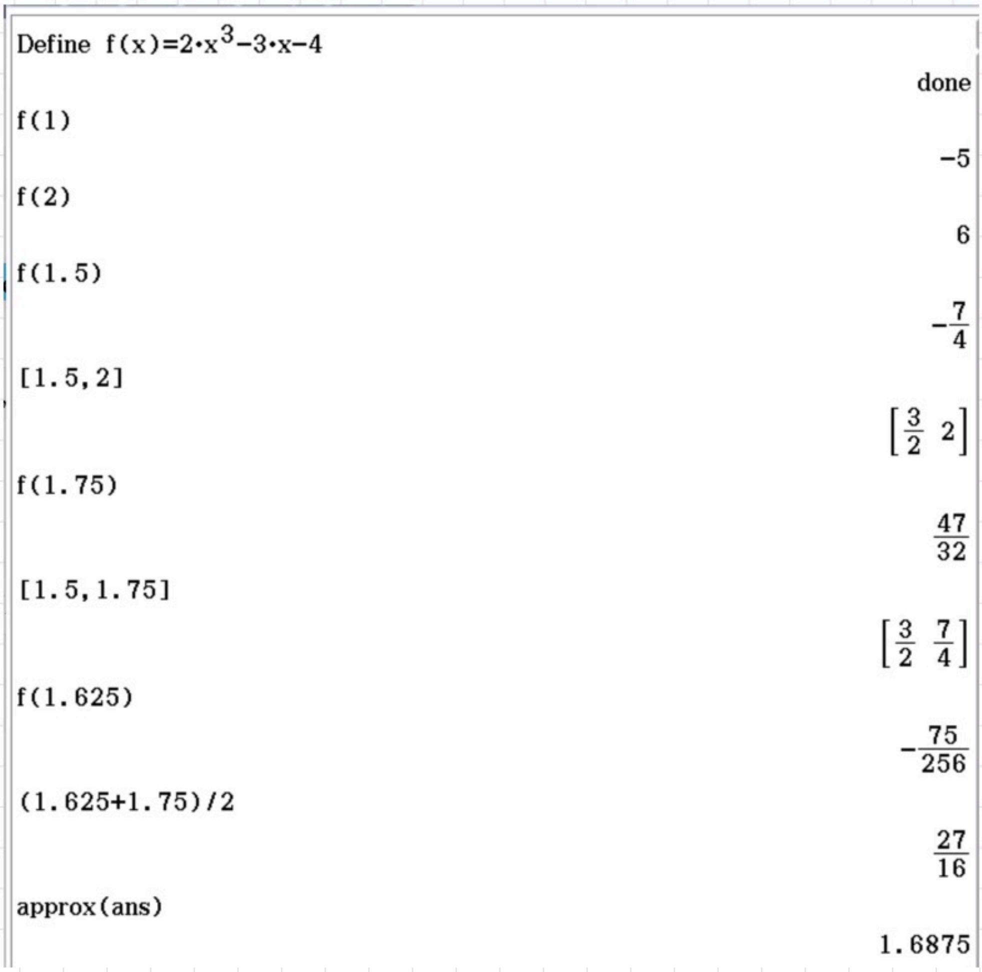
$\frac{1.5+1.75}{2}$ 1.625

$f(1.625)$ -0.2929688

©Final interval is $[1.625,1.75]$, error is 0.0625 so we're done!

$\frac{1.625+1.75}{2}$ 1.6875

Classpad



Define $f(x)=2\cdot x^3-3\cdot x-4$

done

f(1)

-5

f(2)

6

f(1.5)

$-\frac{7}{4}$

[1.5,2]

$\begin{bmatrix}\frac{3}{2} & 2\end{bmatrix}$

f(1.75)

$\frac{47}{32}$

[1.5,1.75]

$\begin{bmatrix}\frac{3}{2} & \frac{7}{4}\end{bmatrix}$

f(1.625)

$-\frac{75}{256}$

(1.625+1.75)/2

$\frac{27}{16}$

approx(ans)

1.6875

Section E: Exam 2 (25 Marks)

Do Section E!

Question 6 (1 mark)

The equation $2x^3 - 3x - 4 = 0$ has one real solution, which lies in the interval [1,2]. Approximate the solution using the bisection method with a maximum error of 0.1. What is the approximate solution?

A. $x \approx 1.655$

B. $x \approx 1.6250$

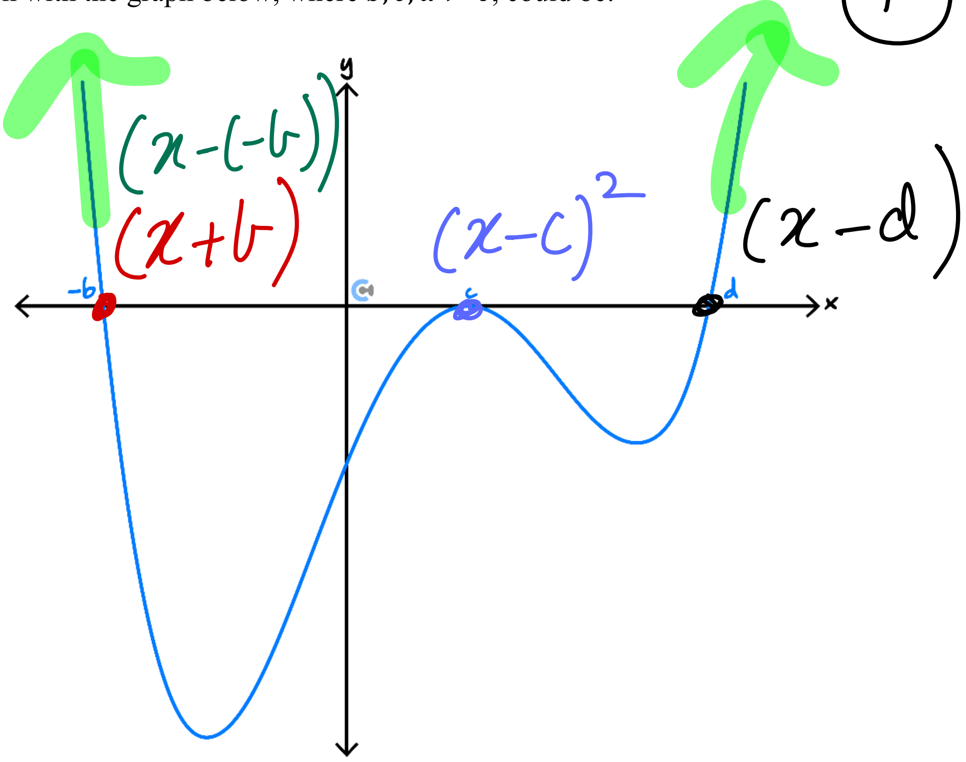
[✓ **C.** $x \approx 1.6875$]

D. $x \approx 1.6225$

Question 7 (1 mark)

The rule for a function with the graph below, where $b, c, d > 0$, could be:

+



A. $y = -2(x+b)(x-c)^2(x-d)$

[✓ **B.** $y = 3(x+b)(x-c)^2(x-d)$]

C. $y = -2(x-b)(x-c)^2(x-d)$

D. $y = 2(x-b)(x-c)^2(x-d)$

Question 8 (1 mark) def f(x)

The polynomial $\underbrace{x^3 + (a+2)x^2 + bx + 8}$ is perfectly divisible by $\underbrace{x+2}$ and has remainder of 2 when divided by $x - 3$. The values (a, b) are:

A. $(-5, -6)$

B. $\left(-\frac{21}{5}, -\frac{22}{5}\right)$

C. $\left(-\frac{3}{5}, -\frac{9}{5}\right)$

D. $\left(-\frac{7}{5}, \frac{3}{5}\right)$

$f(-2) = 0$

$f(3) = 2$

CAS : Solve $\left\{\begin{array}{l} f(-2)=0 \\ f(3)=2 \end{array}\right.$ $\{a, b\}$

M: $f[-2]==0$ && $f[3]==2$, $\{a, b\}$

Question 9 (1 mark)

All real values of x that satisfy the inequality $\underline{9x^2 - 2x^3 > 54 - 27x}$ are:

A. $x < -3$ or $\frac{3}{2} < x < 6$.

B. $x < -6$ or $\frac{3}{2} < x < 3$.

C. $x < -3$ or $x > -\frac{3}{2}$.

D. $-3 < x < \frac{3}{2}$ or $x > 6$.

Question 10 (1 mark)

The equation $x^3 - 3kx^2 + 5x = 0$ has exactly one solution when:

A. $k = \pm\frac{2\sqrt{5}}{3}$

B. $-\frac{2\sqrt{5}}{3} < k < \frac{2\sqrt{5}}{3}$

C. $k > \frac{2\sqrt{5}}{3}$

D. $k < -\frac{2\sqrt{5}}{3}$

$x(x^2 - 3kx + 5)$

Solve$\left((-3k)^2 - 4(5) < 0\right)$

Question 11 (9 marks)

A car is travelling along a straight road from A to C.
The car will travel along a section of road ABC.

- Section AB passes along a bridge over a canyon.
- Section BC passes through a tunnel in a mountain.

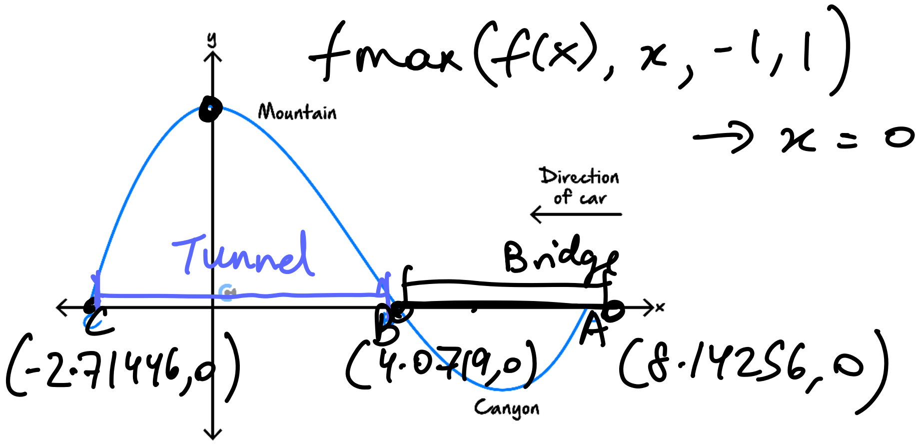
From A to C, the curve of the canyon and then mountain, directly below and above the road, is modelled by the graph of:

$$y = \frac{1}{250}(px^3 + qx^2 + r)$$

Where p, q, and r are real constants.
All measurements are in kilometres and a diagram of this situation is shown below.

fmax(f(x), x, -1, 1)

$\rightarrow x = 0$



a. The curve defined from A to C passes through the points $(1, 0.652)$, $(2, 0.48)$, and $(5, -0.18)$.

i. Use this information to write down three simultaneous equations in terms of p, q, and r. Write these equations with integer coefficients. (3 marks)

$f(1) = \frac{1}{250}(p+q+r) = 0.652$

$f(2) = \frac{1}{250}(8p+4q+r) = 0.48$

$f(5) = \frac{1}{250}(125p+25q+r) = -0.18$

$\times 250$

$$\therefore \begin{cases} p+q+r = 163 & [1] \\ 8p+4q+r = 120 & [2] \\ 125p+25q+r = -45 & [3] \end{cases}$$

ii. Hence, verify that $p = 2, q = -19$, and $r = 180$. (1 mark)

[1]: $2 - 19 + 180 = 163$ ✓

[2]: $8(2) + 4(-19) + 180 = 120$ [2] ✓

[3]: $125(2) + 25(-19) + 180 = -45$ ✓

∴ Solutions are verified

b. Find the exact height of the mountain, above the road, in metres. (1 mark)

$f(0) = 0.72$ km

$\boxed{\therefore \text{ height} = 720\text{m}}$

c. Find the length of the tunnel and the length of the bridge. Give your answers correct to the nearest metre. (3 marks)

Find coordinates of A, B & C (x-intercepts

Let $f(x) = \frac{1}{250}(2x^3 - 19x^2 + 180) = 0$

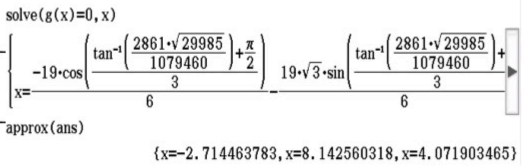
$\therefore x = -2.71446, x = 4.0719, x = 8.14256$

∴ Bridge length = 4.07066 km

$\boxed{= 4071 \text{ m}}$

Tunnel length = 6.78637 km

$\boxed{= 6786 \text{ m}}$

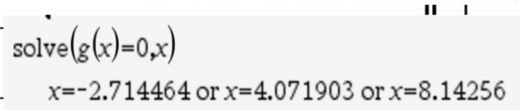


solve(g(x)=0, x)

$$\left\{x=\frac{-19\cdot\cos\left(\frac{\tan^{-1}\left(\frac{2861\cdot\sqrt{29985}}{1079460}\right)+\frac{\pi}{2}}{3}\right)}{6}-\frac{19\cdot\sqrt{3}\cdot\sin\left(\frac{\tan^{-1}\left(\frac{2861\cdot\sqrt{29985}}{1079460}\right)+}{3}\right)}{6}\right.$$

approx(ans)

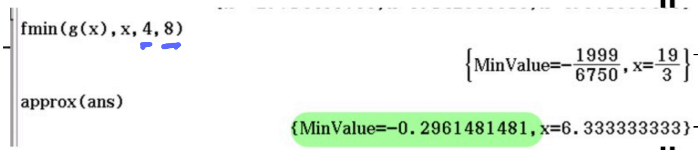
{x=-2.714463783, x=8.142560318, x=4.071903465}



solve(g(x)=0,x)

x=-2.714464 or x=4.071903 or x=8.14256

d. Find the maximum depth of the canyon below the road. Give your answer to the nearest metre. (1 mark)

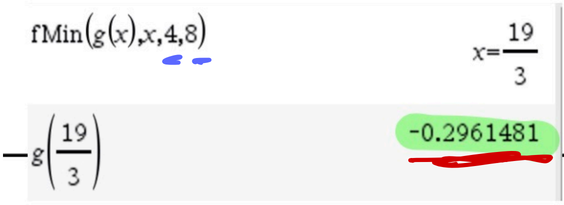


fmin(g(x), x, 4, 8)

$\left\{\text{MinValue}=-\frac{1999}{6750}, x=\frac{19}{3}\right\}$

approx(ans)

{MinValue=-0.2961481481, x=6.333333333}

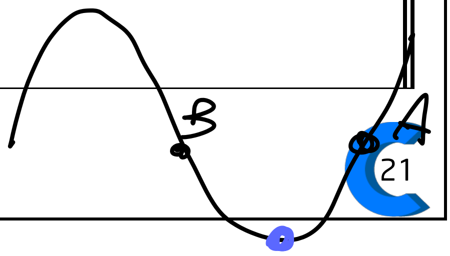


fMin(g(x),x,4,8)

$x=\frac{19}{3}$

$g\left(\frac{19}{3}\right)$ -0.2961481

$\boxed{296 \text{ m}}$



Question 12 (11 marks)

Consider the cubic polynomial $f(x) = x^3 + x^2 - 5x - 2$.

a. Explain why $f(x)$ must have a root between $x = 1$ and $x = 3$. (1 mark)

$f(1) = -5 < 0$

$f(3) = 19 > 0$

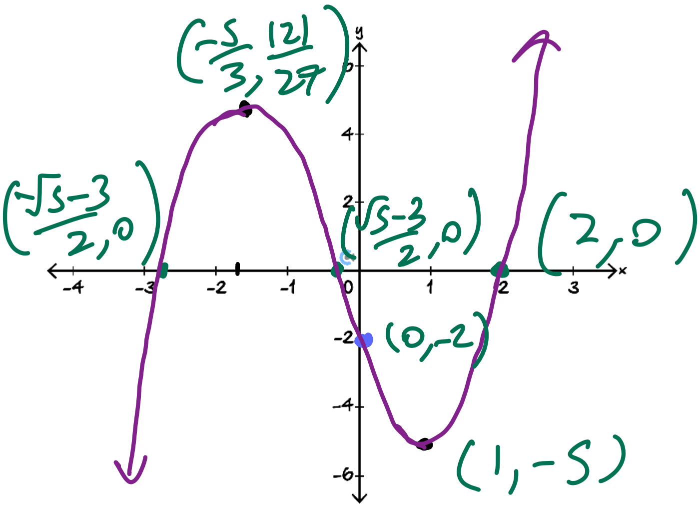
Opposite sides of x-axis

→ Root must be between $x = 1$ & $x = 3$

b. Write $f(x)$ in the form $f(x) = (x - a)Q(x)$ where $a > 0$ and $Q(x)$ is a quadratic function. (1 mark)

$f(x) = (x-2)(x^2+3x+1)$

c. It is known that the graph of $y = f(x)$ has turning points at x-values that are solutions to the equation $3x^2 + 2x - 5 = 0$. Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts and turning points with exact coordinates. (4 marks)



① x-ints: Solve $f(x) = 0 \rightarrow x = 2$, $x = \frac{-3+\sqrt{5}}{2}$, $x = \frac{-3-\sqrt{5}}{2}$

② y-int: $f(0) = -2$ $(0, -2)$

③ TPs

x coords

Solve $3x^2 + 2x - 5 = 0$

$x = 1$, $x = -5/3$

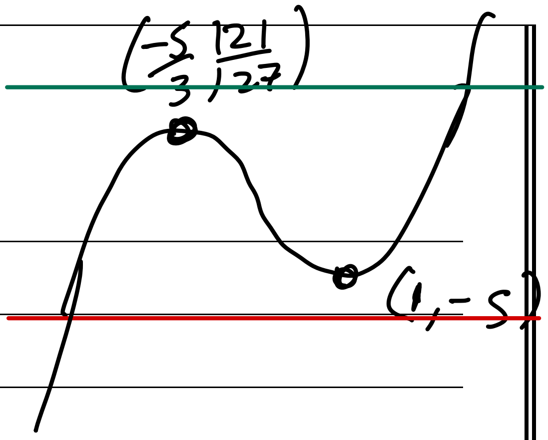
y-coords

$f(1) = -5$

$f(-\frac{5}{3}) = \frac{121}{27}$

$\left(-\frac{5}{3}, \frac{121}{27}\right)$ & $(1, -5)$

d. Find the values of b such that $g(x) = b$ has:



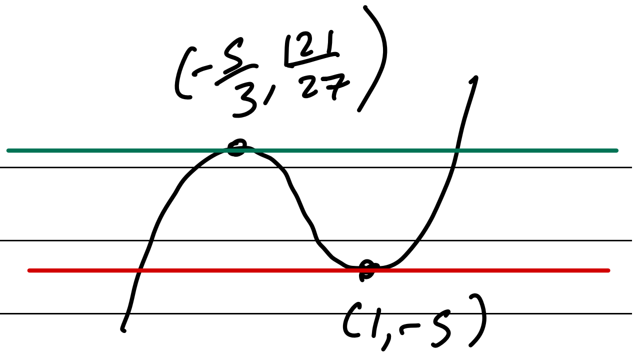
i. One solution. (2 marks)

$b < -5$

OR

$b > \frac{121}{27}$

ii. Two solutions. (1 mark)



$b = -5$ OR

$b = \frac{121}{27}$

e. Find the values of k for which the equation $x^3 + (k-2)x^2 + (1-2k)x - 2 = 0$ has three solutions. (2 marks)

Write as $(x-2)(x^2+kx+1) = 0$

$\Delta > 0$

Solve $k^2 - 4 > 0$

$\boxed{\therefore\ k < -2 \text{ or } k > 2}$

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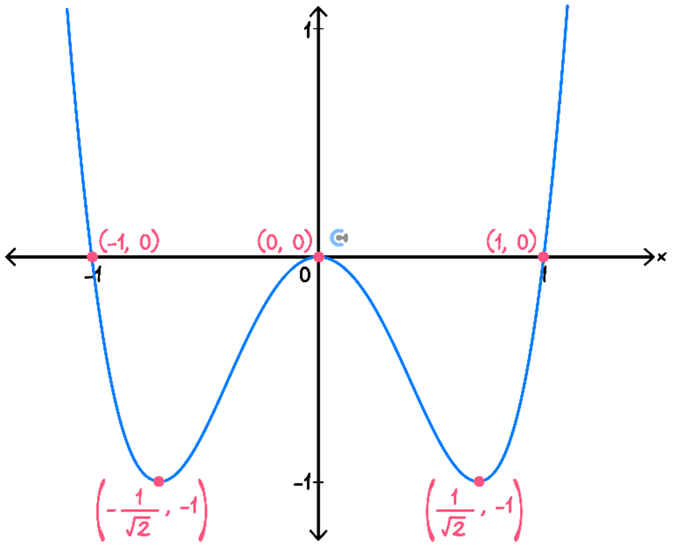
Section F: Extension Exam 1 (15 Marks)

Question 13 (4 marks)

The function $f(x)$ is a polynomial of degree 4. The graph of f is shown below.



a. Find the rule of $f(x)$. (2 marks)

Solution: $f(x) = 4x^2(x+1)(x-1)$

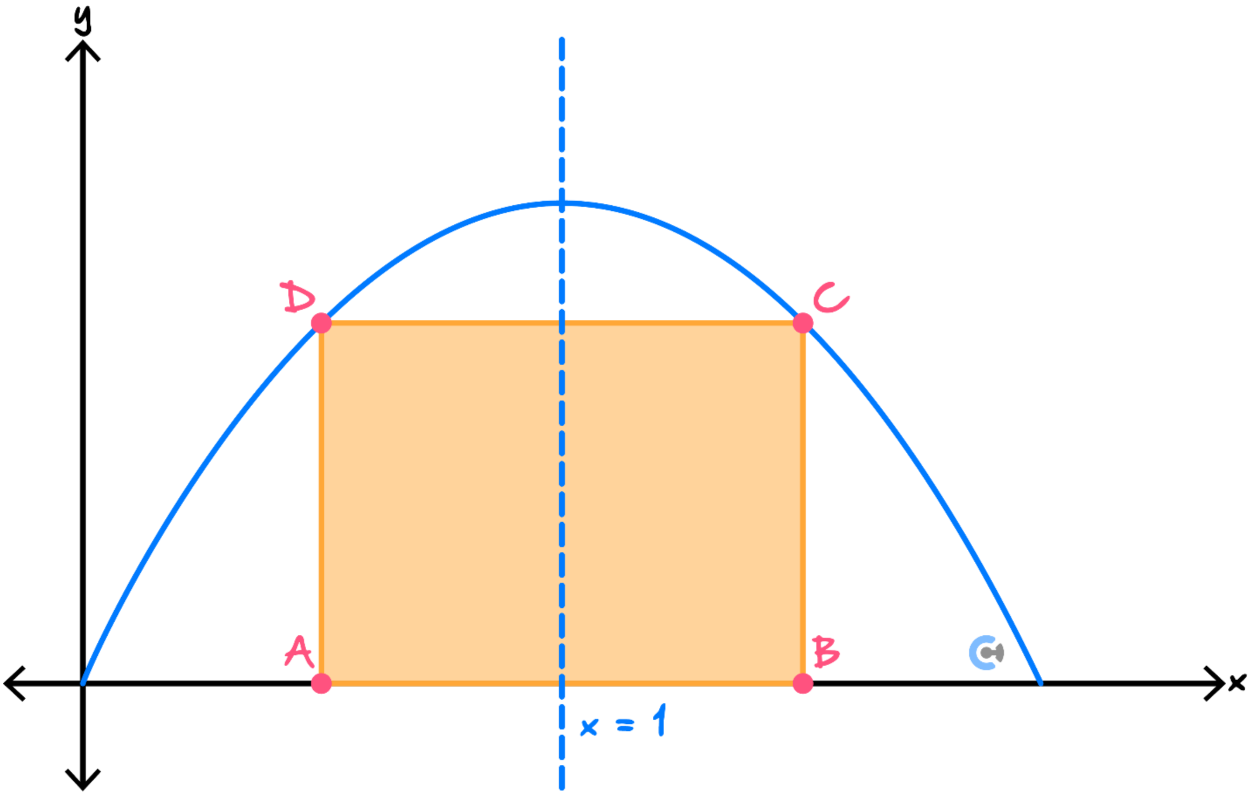
b. Find the values of k such that $f(x) + k = 0$, where k is a real number, has an even number of real solutions. (2 marks)

Solution: Only time there is an odd number of solutions is when $k = 0$. Therefore $k \neq 0$.

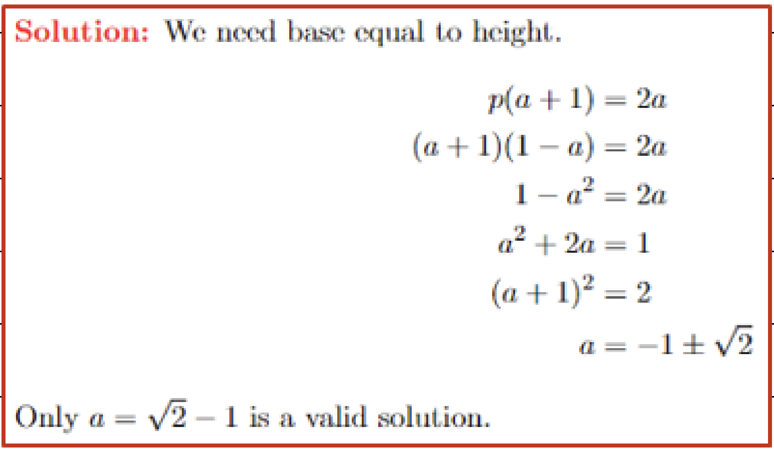
Question 14 (6 marks)

Consider the parabola $p(x) = x(2 - x)$, where $0 \leq x \leq 2$.

A rectangle $ABCD$ is inscribed between the graph of p and the x-axis. Its vertices are a distance of a units from the axis of symmetry, $x = 1$, as shown below.



a. Find the value of a when the rectangle is a square. (3 marks)



Solution: We need base equal to height.

$$\begin{aligned} p(a+1) &= 2a \\ (a+1)(1-a) &= 2a \\ 1 - a^2 &= 2a \\ a^2 + 2a &= 1 \\ (a+1)^2 &= 2 \\ a &= -1 \pm \sqrt{2} \end{aligned}$$

Only $a = \sqrt{2} - 1$ is a valid solution.

b. Find the rational value of a such that the rectangle $ABCD$ has an area of $\frac{3}{4}$ square units. (3 marks)

Solution: Area is given by $2a \times p(a+1) = 2a(1+a)(1-a) = 2a - 2a^3$.

We must solve $2a - 2a^3 = \frac{3}{4} \implies 8a - 8a^3 = 3$

$$8a^3 - 8a + 3 = 0$$

Use the rational root theorem to find that $a = \frac{1}{2}$ is a solution.

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Question 15 (5 marks)

Consider the function $g(x) = (x^2 - 4kx + 3)(x^2 - 2x + k)$, where k is a real number.
Find all possible values of k such that $g(x)$ has:

a. Four real roots. (3 marks)

Solution: We require the discriminant for both quadratic functions to be greater than zero.

$$\Delta_1 = 16k^2 - 12 > 0 \implies k < -\frac{\sqrt{3}}{2} \text{ or } k > \frac{\sqrt{3}}{2}$$

$$\Delta_2 = 4 - 4k > 0 \implies k < 1$$

Therefore $g(x)$ has four real roots for $k < -\frac{\sqrt{3}}{2}$ or $\frac{\sqrt{3}}{2} < k < 1$

b. Two real roots. (2 marks)

Solution: First quadratic no roots for $-\frac{\sqrt{3}}{2} < k < \frac{\sqrt{3}}{2}$ and second quadratic two roots for $k < 1$. This gives two roots for $-\frac{\sqrt{3}}{2} < k < \frac{\sqrt{3}}{2}$

Second quadratic no roots for $k > 1$ and first quadratic will give two roots for $k > \frac{\sqrt{3}}{2}$.
However, note that when $k = 1$,

$$g(x) = (x^2 - 4x + 3)(x^2 - 2x + 1) = (x - 1)(x - 3)(x - 1)^2 = (x - 1)^3(x - 3)$$

so two roots when $k = 1$.

Conclude that there are two real roots when $-\frac{\sqrt{3}}{2} < k < \frac{\sqrt{3}}{2}$ or $k \geq 1$

Section G: Extension Exam 2 (13 Marks)

Question 16 (1 mark)

Let $f(x) = x^3 + 3x^2 - 4x + 8$. The remainder when $f(x)$ is divided by $5x - 4$ is:

A. 104

B. 188

C. $\frac{904}{125}$

D. $\frac{617}{64}$

Question 17 (1 mark)

Consider the quartic $y = (x-2)^2(x^2 + 4kx + 6)$. It is known that the quartic has three distinct x-intercepts. The possible values of k are:

A. $k < -\sqrt{\frac{3}{2}}$ or $k > \sqrt{\frac{3}{2}}$

B. $-\sqrt{\frac{3}{2}} < k < \sqrt{\frac{3}{2}}$

C. $k = \pm\sqrt{\frac{3}{2}}$

D. $k < -\sqrt{3}$ or $k > \sqrt{3}$

Question 18 (1 mark)

A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has one distinct x-intercept.
All possible values of c are:

A. $c > 4$

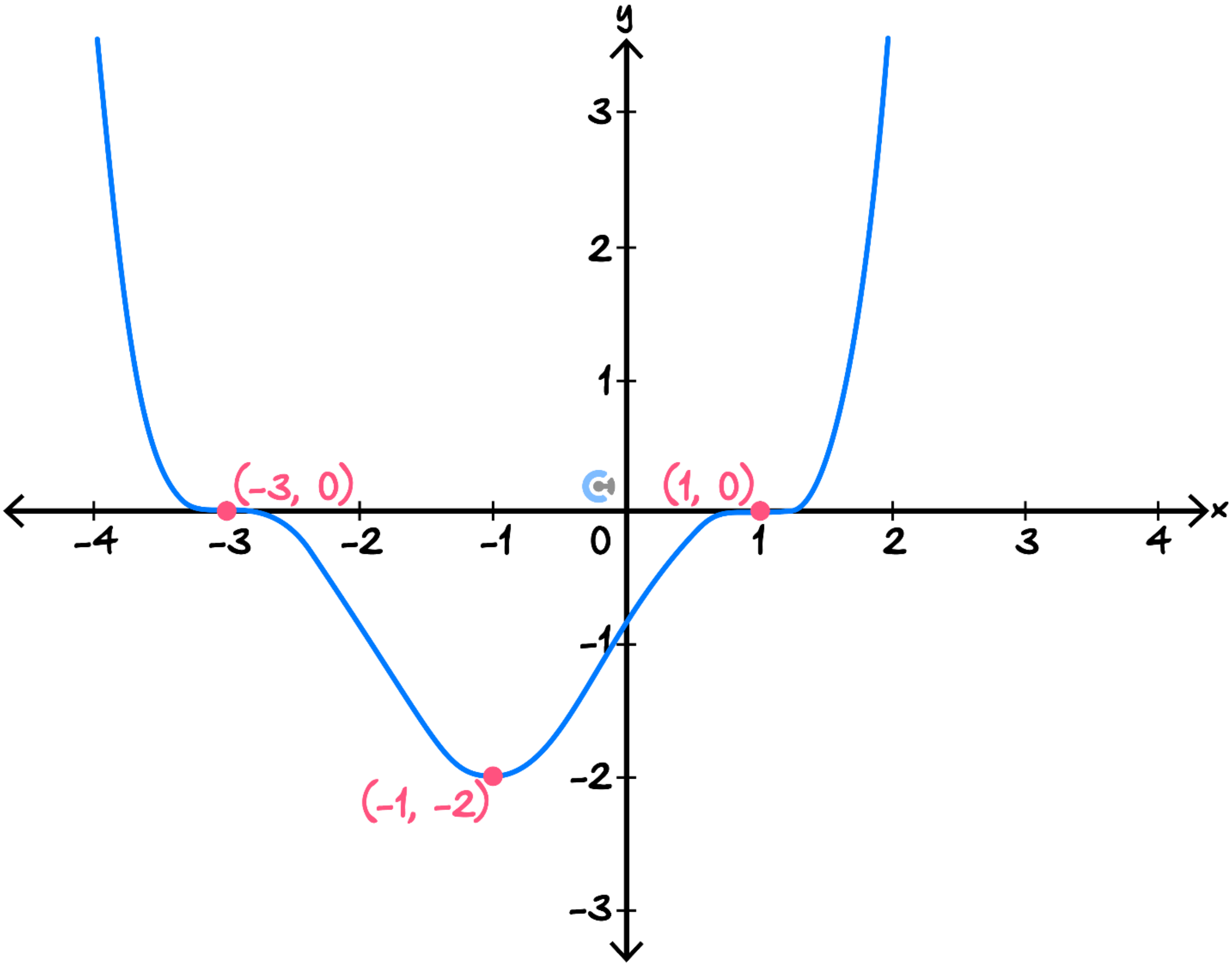
B. $0 < c < 4$

C. $c < 0$ or $c > 4$

D. $c > 4$

Question 19 (10 marks)

Consider the function of the form $f(x) = a(x-b)^3(x-c)^3$, where $b > c$, depicted on the graph below.



a. Find the values of a, b, and c. (2 marks)

Solution: From the inflection points we see that $b = 1$ and $c = -3$. Then using the point $(-1, -2)$ we find that $a = \frac{1}{32}$.

$$f(x) = \frac{1}{32}(x+3)^3(x-1)^3$$

b. Show that $x = -1$ is an axis of symmetry for the graph of f. (2 marks)

Solution:

$$\begin{aligned} f(-1+m) &= \frac{1}{32}(m+2)^3(m-2)^3 \\ &= \frac{1}{32}(-m-2)^3(2-m)^3 \\ &= f(-1-m) \end{aligned}$$

since this holds for any $m \in \mathbb{R}$, $x = -1$ is an axis of symmetry for f.

c. Find the value of $d > 0$ such that $f(x) + d = 0$ has one real solution. (1 mark)

Solution: $d = 2$

d. Consider the function $g(x) = (x + k + 3)^3(x + k - 1)^3$, where $k \in \mathbb{R}$.

i. Find the roots of g in terms of k. (1 mark)

Solution: $x = -k - 3, 1 - k$

ii. Hence, find the values of k so that $g(x)$ has only positive roots. (2 marks)

Solution: We require both $-k - 3 > 0$ and $1 - k > 0$. This yields $k < -3$.

iii. A function h is said to be even if $h(x) = h(-x)$ for all x.
Find the value of k such that $g(x)$ is an even function. (2 marks)

Solution: $k = -1$

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