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VCE Mathematical Methods ½
Polynomials [0.5]
Workshop Solutions

Section A: Recap

Degree of Polynomial Functions

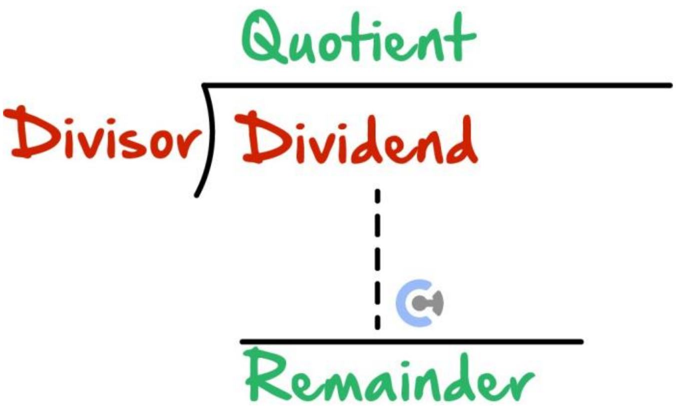
$$\textit{Degree} = \textit{Highest Power of the Polynomial}$$

Roots of Polynomial Functions

$$\textit{Roots} = x - \textit{intercept}(s)$$

Polynomial Long Division

- Division of polynomials



$$\frac{\textit{Dividend}}{\textit{Divisor}} = \textit{Quotient} + \frac{\textit{Remainder}}{\textit{Divisor}}$$

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Remainder Theorem

- **Definition**:
  - Finds the remainder of long division without the need of long division,

$$\textit{when } P(x) \textit{ is divided by } (x-\alpha), \textit{the remainder is } P(\alpha).$$

- **Steps**
  1. Find x-values which make the divisor equal to 0.
  2. Substitute it into the dividend function.

Factor Theorem

- For every x-intercept, there is a factor:

$$\textit{If } P(\alpha) = 0 \textit{ then}, (x-\alpha) \textit{ is a factor of } P(x).$$

Factorising Polynomials

- The steps are:
  - Find a single root by trial and error.
    - (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

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Definition

Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$\textit{Potential root} = \pm\frac{\textit{Factors of constant term } a_0}{\textit{Factors of leading coefficient } a_n}$$

- If the roots are rational numbers, the roots can only be $\pm\frac{\textit{Factors of constant term } a_0}{\textit{Factors of leading coefficient } a_n}$.

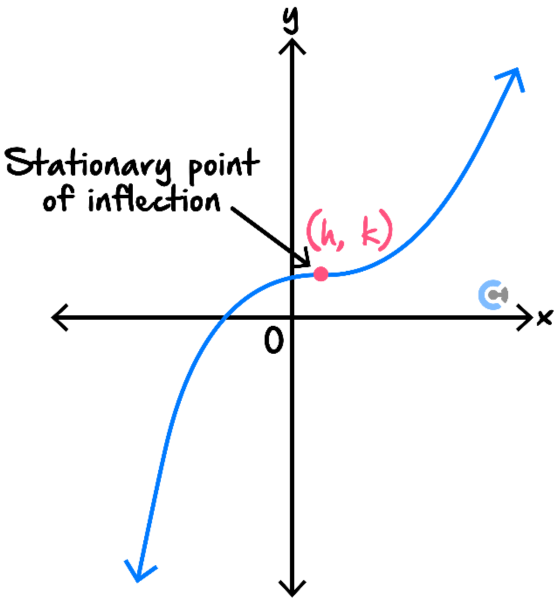
Sum and Difference of Cubes

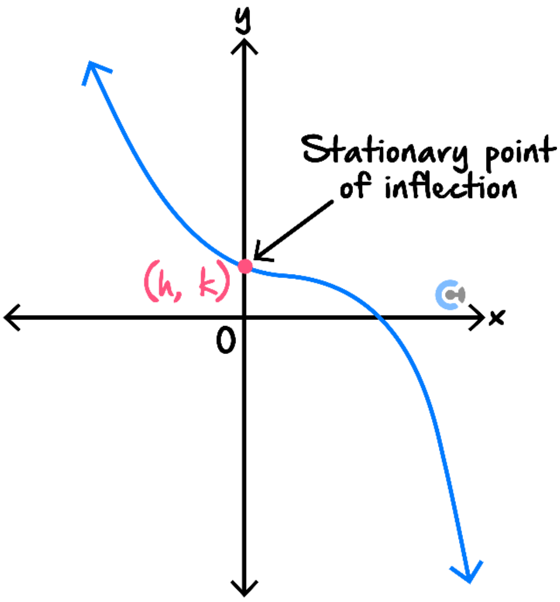
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Graphs of $a(x - h)^n + k$, where n is an Odd Positive Integer

- All graphs look like a "cubic".

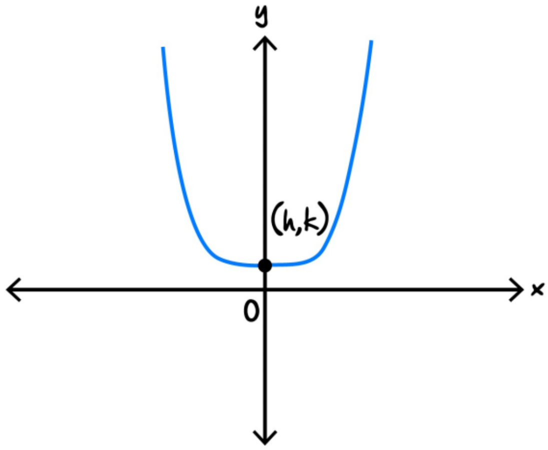


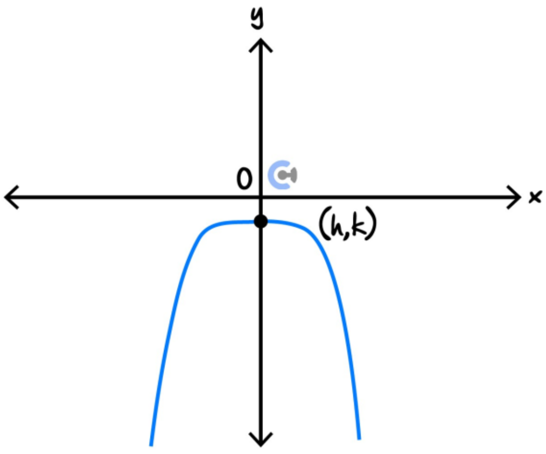


- The point (h, k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

Graphs of $a(x-h)^n + k$, Where n is an Even Positive Integer

- All graphs look like a "quadratic".

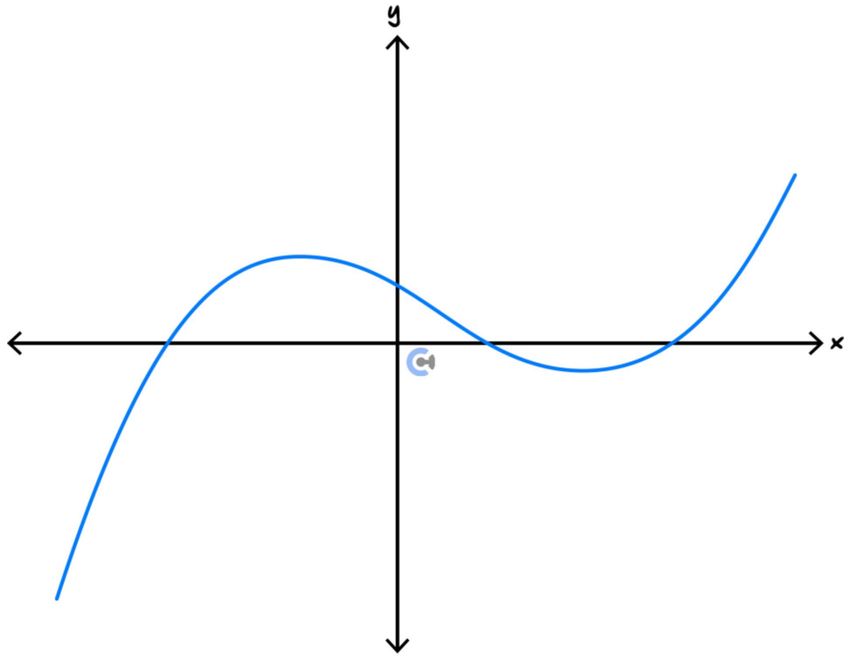




- The point (h, k) gives us the turning point.

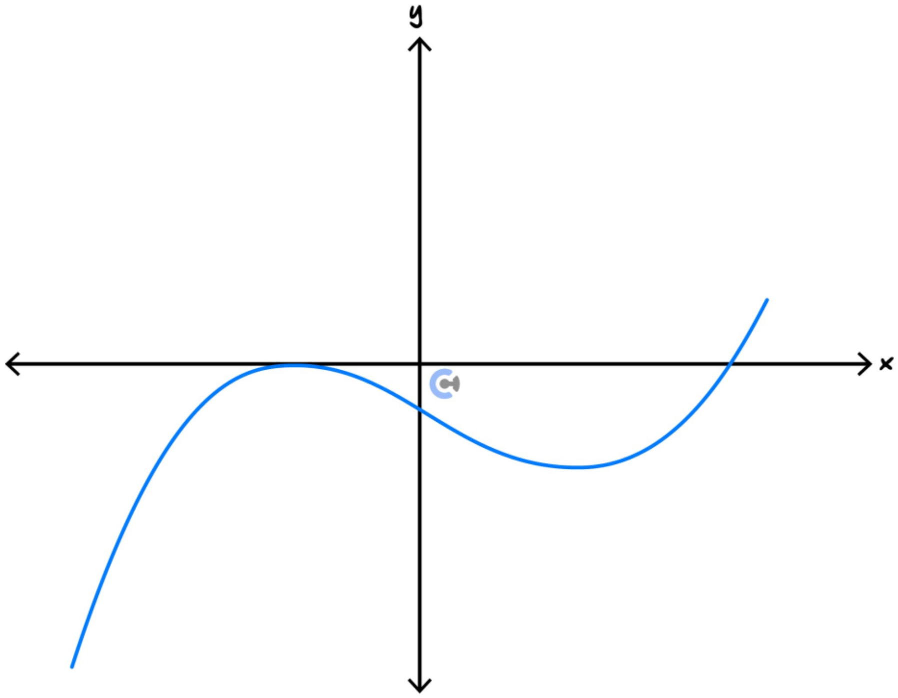
Graphs of Factorised Polynomials

- **All non-repeated linear factors correspond to x-intercepts of the graph.**



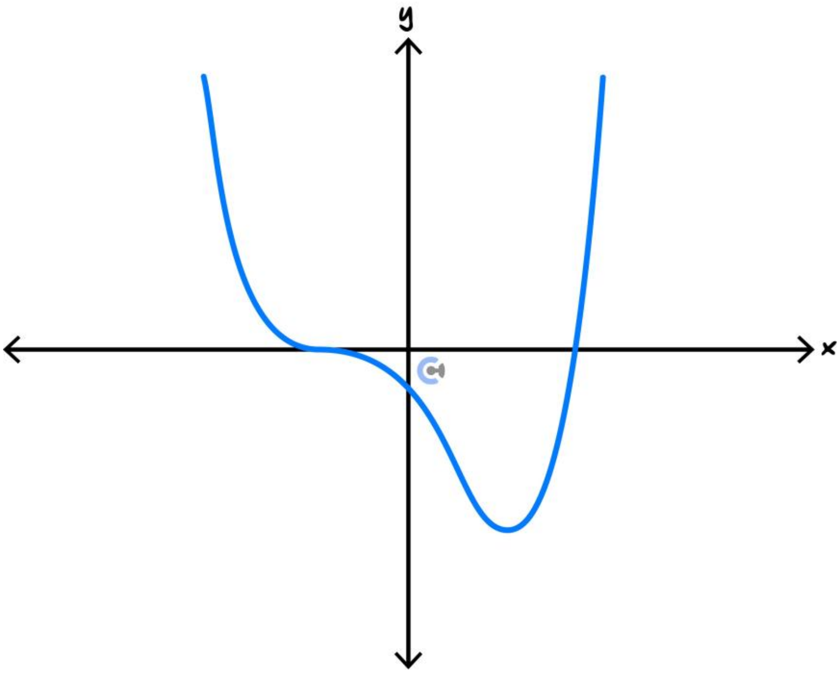
- E.g., $f(x) = (x-a)(x-b)(x-c)$ results in x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$.

- All squared linear factors correspond to x-intercepts and T.P. of the graph.



- E.g., $f(x) = (x-a)^2(x-b)$ will have an x-intercept $(a, 0)$ which is also a local minimum/maximum.

- All cubed linear factors correspond to x-intercepts and SPI of the graph.



- E.g., $f(x) = (x-a)^3(x-b)$ has an x-intercept $(a, 0)$ which is also a stationary point of inflection.

Definition

Steps to Graphing Factorised Polynomials

- **Steps:**

  1. Plot x-intercepts.
  2. Determine whether the polynomial is positive or negative.
  3. Use the repeated factors to deduce the shape.
     - Non - Repeated: Only x-intercept.
     - Even Repeated: x-intercept and a turning point.
     - Odd Repeated: x-intercept and a stationary point of inflection.

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Section B: Warmup

Question 1

a. Use polynomial long division to write $f(x) = \frac{x^3+2x^2+3}{x-2}$ in the form $f(x) = Q(x) + \frac{a}{x-2}$, for quadratic function Q and integer a.

$$f(x) = x^2 + 4x + 8 + \frac{19}{x-2}$$

b. Find the remainder of the division $\frac{f(x)}{g(x)}$ where $f(x) = x^3 + 3x^2 + 2$ and $g(x) = x + 1$.

$$f(-1) = 4$$

c. Find all the roots of $f(x) = x^3 + 2x^2 - x - 2$.

$f(x) = (x-1)(x+1)(x+2)$ so roots are $x = -2, \pm 1$

d. Factorise the expression $8x^3 - 27$.

$8x^3 - 27 = (2x-3)(4x^2+6x+9)$

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Section C: Exam 1 (18 Marks)

Question 2 (5 marks)

We know that $f(x) = \frac{12}{2x+3}$ for all positive values of x.

a. Simplify $\frac{x^2+4x-5}{12} \times f(x)$. Give your answer in the form of $\frac{ax+5}{b} - \frac{c}{2x+3}$ where a and b are positive integers and c is a rational number. (3 marks)

$$\frac{2x+5}{4} - \frac{35}{4(2x+3)}$$

Consider $g(x) = \frac{5-81x^2}{4}$ for all values of x.

b. Solve $g(x) = -13$. (2 marks)

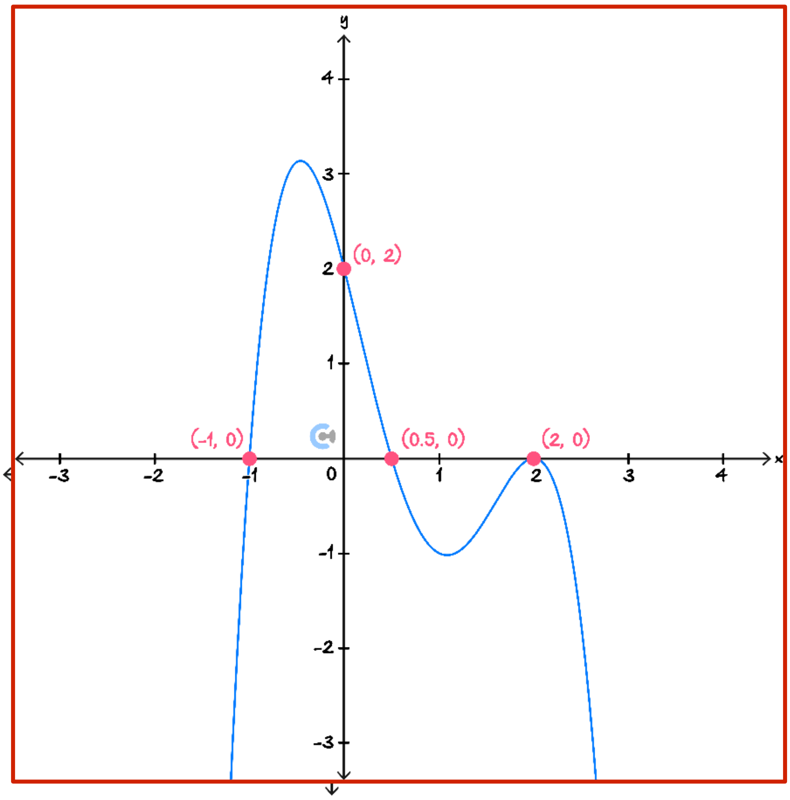
$$x = \pm\frac{\sqrt{57}}{9}$$

Question 3 (3 marks)

Consider the function:

$$f(x) = (2-x)^2(x+1)\left(\frac{1}{2}-x\right)$$

It is known that f has turning points at approximately $(-0.5, 3.1)$ and $(1.1, -1)$
Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts with coordinates.



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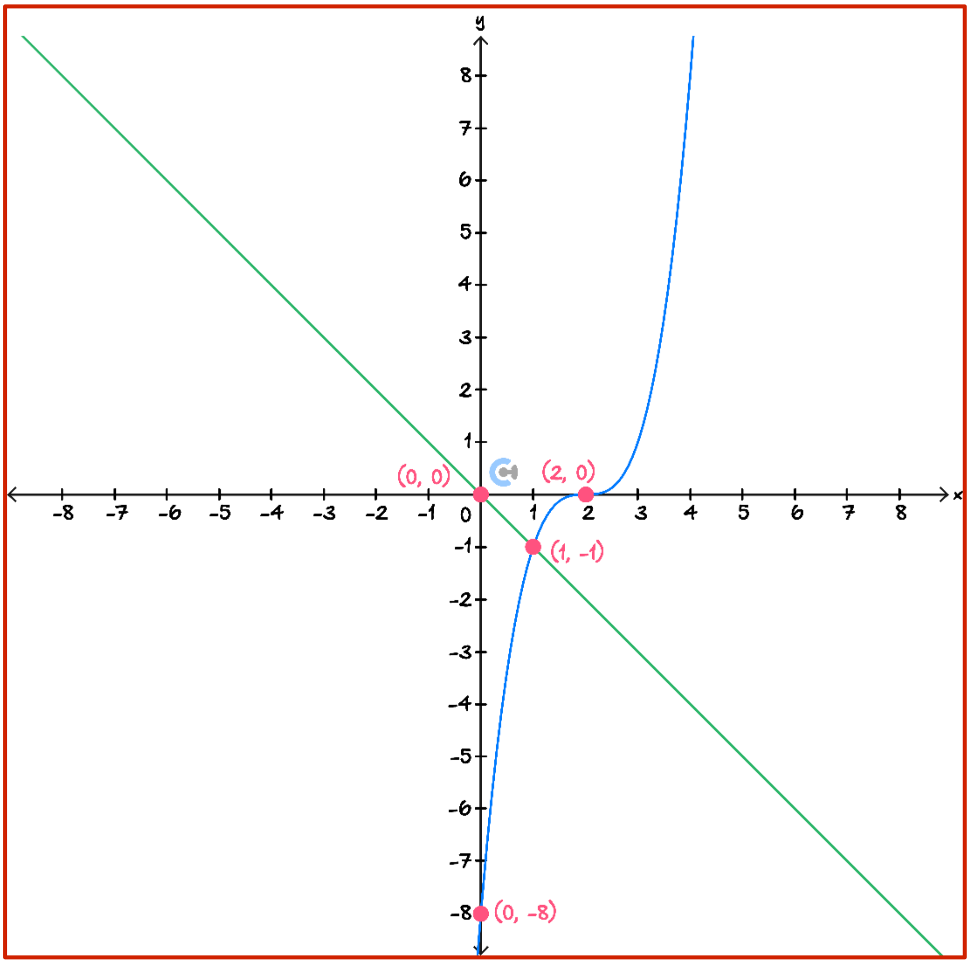
Question 4 (4 marks)

Consider the functions $f(x) = (x-2)^3$ and $g(x) = -x$.

a. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the axes below. Label all axes intercepts. (3 marks)



b. Hence, solve the equation $(x-2)^3 = -x$ for x. (1 mark)

$x = 1$

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Question 5 (6 marks)

a. Consider the function $f(x) = 3x^3 + ax^2 + bx - 12$. If $x - 2$ is a factor of $f(x)$ and the remainder of $f(x) \div x - 1 = -18$, find the value(s) of a and b. (3 marks)

$$a = 3, b = -12.$$

b. Hence, simplify the following using polynomial long division: $\frac{3x^3+3x^2-12x-12}{x^2+4x+1}$. (3 marks)

$$\frac{21x - 3}{x^2 + 4x + 1} + 3x - 9$$

Section D: Tech Active Exam Skills

Calculator Commands: Turning Point

- ALWAYS sketch the graph to find approximate bounds for where the turning point you want is located.
- To find a local maximum we maximise the function over a specific domain.
- To find a local minimum we minimise the function over a specific domain.
- **TI and Casio:** Use fmin(expression, variable, lower (optional), upper (optional)) or fmax(expression, variable, lower (optional), upper (optional)).
- **TI:** Menu→ $4 \rightarrow \frac{7}{8}$.

Define $f(x)=x^3-4\cdot x$	*Done*
$\text{fMin}(f(x),x,0,2)$	$x=\frac{2\cdot\sqrt{3}}{3}$
$f\left(\frac{2\cdot\sqrt{3}}{3}\right)$	$\frac{-16\cdot\sqrt{3}}{9}$

- **Casio:** Action→Calculation→ fmin/fmax

$$\text{fmin}(x\hat{}3-4x, x, 0, 2)$$

$$\left\{\text{MinValue}=\frac{-16\cdot\sqrt{3}}{9}, x=\frac{2\cdot\sqrt{3}}{3}\right\}$$

.

NOTE: TI only gives the x-value for the min/max so we then need to sub it back into our function. Casio gives us both!

<u>Calculator Commands</u>

- **Mathematica:** Minimize[ ] and Maximize[ ] commands.
- Minimize[$f[x], x$] will minimize $f[x]$ over its whole domain.
- To restrict the domain, we must use Minimize[$\{f[x], a \leq x \leq b\}, x$].

```
In[34]:= Minimize[{x^3 - 4 x, 0 < x < 2}, x]
```

Out[34]= $\left\{-\frac{16}{3\sqrt{3}}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$

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Section E: Exam 2 (26 Marks)

Question 6 (1 mark)

The stationary point of inflection of the cubic, $y = (x-2)^3 + 4$ occurs at:

A. $(-2, 4)$

B. $(2, 4)$

C. $(-2, -4)$

D. $(4, 2)$

Question 7 (1 mark)

The data $(3, 3)$, $(4, 0)$, $(5, 3)$, $(6, 48)$ and $(7, 243)$ can be modelled by the equation $y = a(x-b)^4$. The values of a and b respectively are:

A. 3 and 4.

B. -3 and 4.

C. 4 and 3.

D. -3 and -4.

Question 8 (1 mark)

The values of x that satisfy $x^3 - 6x^2 - 27x + 140 = 0$ are:

A. $x = -5,4,5$

B. $x = -5,-4,6$

C. $x = -5,4,7$

D. $x = 2,3,7$

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Question 9 (1 mark)

Using the Rational Root Theorem, count the number of possible roots that $y = 3x^3 + 7x^2 - 5x + 11$ has.

A. 4

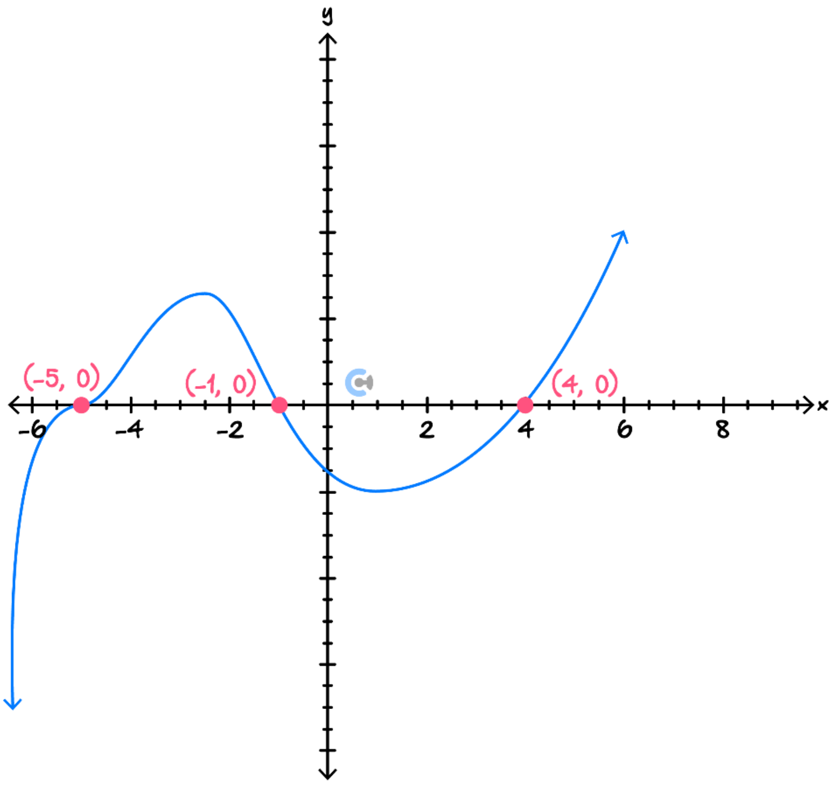
B. 6

C. 7

D. 8

Question 10 (1 mark)

The equation that best represents the graph below is:



A. $y = (x-4)^3(x+5)(x-1)$

B. $y = (x+5)^3(x+1)(x-4)$

C. $y = (x+1)^3(x+5)^2(x-4)$

D. $y = (x+4)(x+5)(x-1)$

Question 11 (1 mark)

Find the remainder of the division, $f(x) \div g(x)$, where $f(x) = 7x^3 - 4x^2 + 5x - 11$ and $g(x) = 3x - 2$.

A. 0

B. 13

C. $\frac{233}{17}$

D. -199

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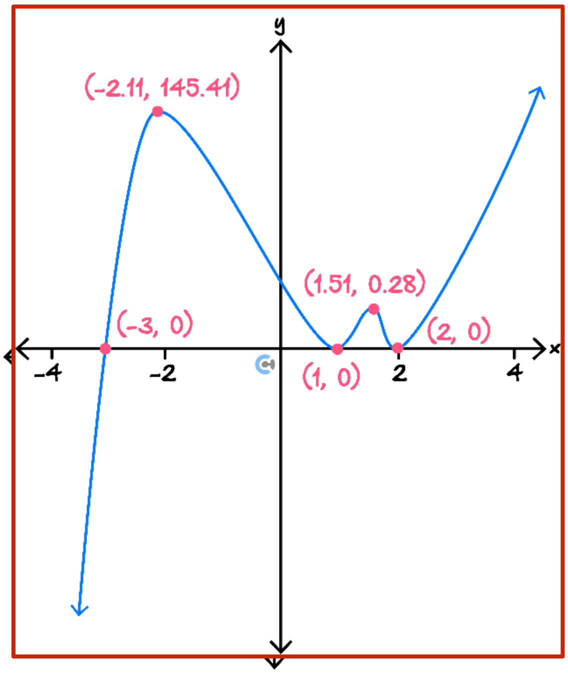
Question 12 (10 marks)

Consider the quintic polynomial $f(x) = (x+3)(x-1)^2(x-2)^2$. The function has turning points at $(-2.11, 145.41), (a, 0), (1.51, 0.28), (b, 0)$, where $a > b$.

a. State the values of a and b. (1 mark)

$a = 2$ and $b = 1$

b. Sketch the graph on the axes below labelling all x-intercepts and turning points. Ignore the y-axis scale. (3 marks)



c. Find all solutions to $f(x) = 0.28$ for x correct to 2 decimal places. (2 marks)

$x = -3.00, 0.78, 1.49, 1.54, 2.19$

d. Find the values of k (to 2 decimal places) when:

i. $f(x) = k$ has 1 solution. (1 mark)

$k < 0 \text{ or } k > 145.41$

ii. $f(x) = k$ has 5 solutions. (2 marks)

$0 < k < 0.28$

iii. $f(x) = k$ has 2 solutions. (1 mark)

$k = 145.41$

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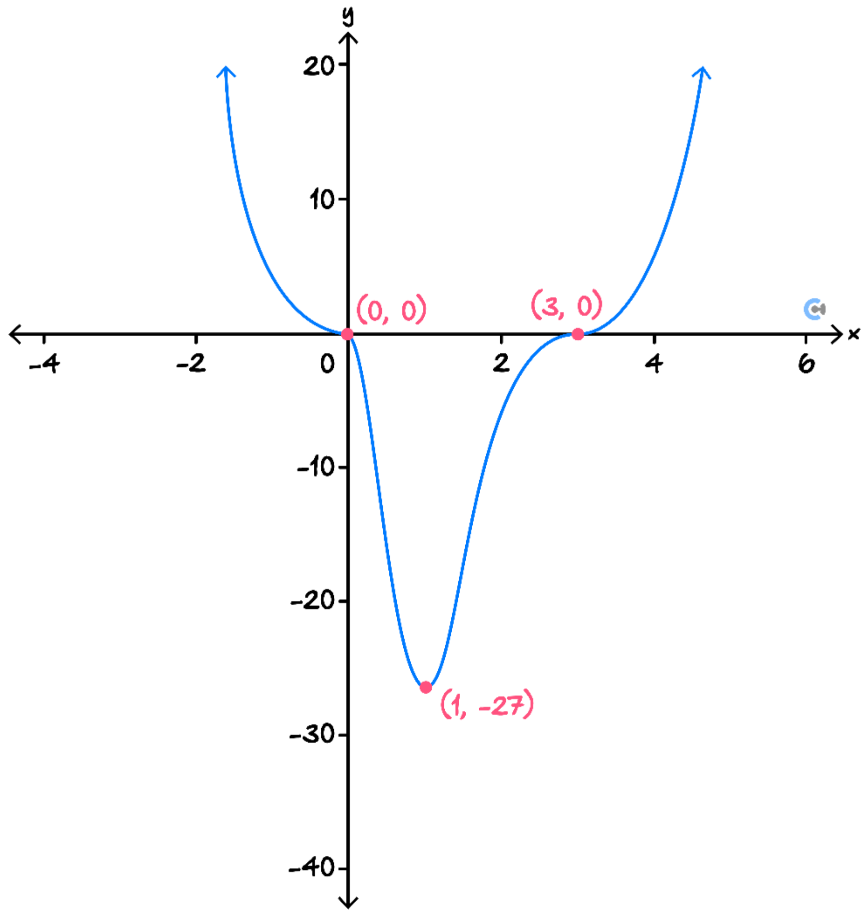
Question 13 (9 marks)

Consider the function of the form $f(x) = a(x-b)^3(x-c)^3$ where $b > c$, shown on the axes below.



a. Find the values of a, b and c. (3 marks)

$a = \frac{27}{8}, b = 3, c = 0.$

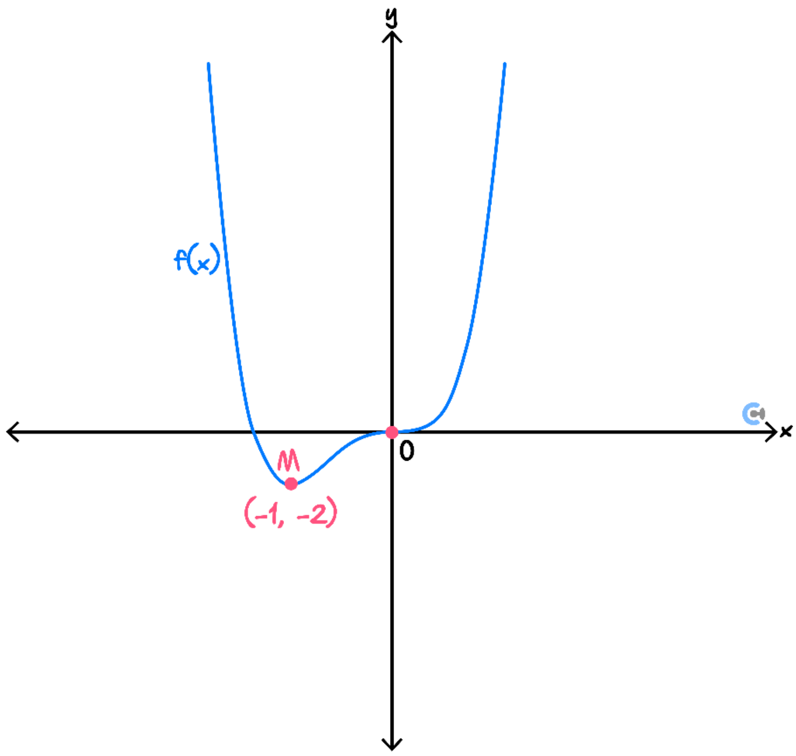
b. Hence, simplify $\frac{f(x)}{(x-3)^3}$, leaving your answer in the form of $\frac{px^q}{m}$, where p, q and m are positive integers. Show all working. (2 marks)

$$\frac{27}{8}x^3$$

c. It is known that the division, $f(x) \div (x-k)$, leaves a remainder of 0. Find the possible values of k. (2 marks)

$$k = 0, 3$$

Consider the quartic $f: R \to R, f(x) = 5x^4 + 6x^3 - x^2$. Part of the graph of $f(x)$ is shown below.



d. State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts. (1 mark)

$b > 2$

Let $p: R \to R, p(x) = 5x^4 + 3(a+2)x^3 - x^2 - 2ax + 3a^2, a \in R$.

e. State the value of a for which $p(x) = f(x)$ has infinite solutions. (1 mark)

$a = 0$

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Section F: Extension Exam 1 (14 Marks)

Question 14 (8 marks)

Consider the function $f(x) = x^5 + x^4 - 5x^3 - 5x^2 + 4x + 4$.

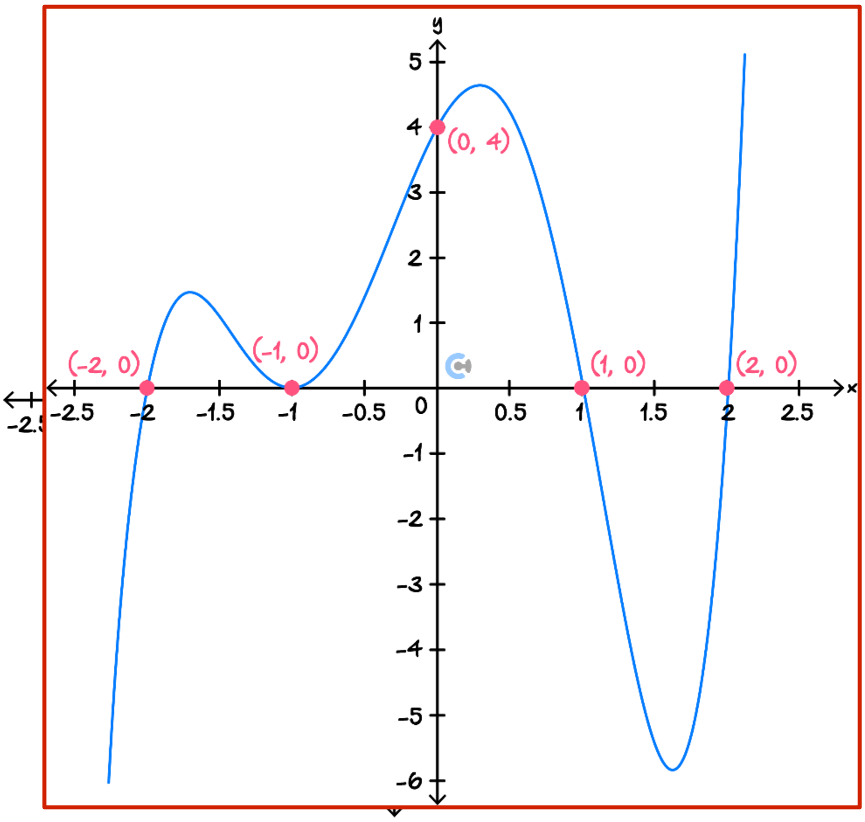
a. Show that $(x^2 - 4)$ is a factor of f. (2 marks)

Note that $f(2) = 32 + 16 - 40 - 20 + 8 + 4 = 0$ and therefore $(x - 2)$ is a factor of f.
Also $f(-2) = -32 + 16 + 40 - 20 - 8 + 4 = 0$ and therefore $(x + 2)$ is a factor of f.
Therefore $(x + 2)(x - 2) = x^2 - 4$ is a factor of f.

b. Find all roots of f. (3 marks)

We note that $f(x) = (x^2 - 4)(x^3 + x^2 - x - 1)$.
Let us find the roots of $P(x) = x^3 + x^2 - x - 1$.
Note that $P(1) = 0$ therefore $x - 1$ is a factor.
Now $P(x) = (x - 1)(x^2 + 2x + 1) = (x - 1)(x + 1)^2$.
Therefore $f(x) = (x + 2)(x - 2)(x - 1)(x + 1)^2$ and so $f(x)$ has roots $x = -2, -1, 1, 2$

c. It is known that f has two turning points when $x > 0$.
Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts and ignore the y-scale. (3 marks)



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Question 15 (6 marks)

A cubic function $f(x) = ax^3 + bx^2 + cx + d$ passes through the points $(-2, -22)$, $(0,4)$ $(1,2)$ and $(2,6)$.

a. Show that $a = 2, b = -3, c = -1$ and $d = 4$. (3 marks)

We note that $f(0) = 2$ and thus $d = 4$. Now we have that

$$f(-2) = -22 \implies -8a + 4b - 2c + 4 = -22 \quad (1)$$
$$f(1) = 2 \implies a + b + c + 4 = 2 \quad (2)$$
$$f(2) = 6 \implies 8a + 4b + 2c + 4 = 6 \quad (3)$$

(1) + (2) yields $8b + 8 = -16 \implies b = -3$.
Sub this into (2) and (3) to get the equations

$$a + c = 1 \quad (4)$$
$$8a + 2c = 14 \quad (5)$$

Sub in $c = 1 - a$ into (5)

$$8a + 2 - 2a = 14 \implies 6a = 12 \implies a = 2$$

and finally $c = 1 - a \implies c = -1$

b. Write the function $g(x) = \frac{(x+1)f(x)}{x-1}$ in the form $g(x) = C(x) + \frac{A}{x-1}$, for a cubic function and an integer A. (3 marks)

We have that $(x+1)f(x) = 2x^4 - x^3 - 4x^2 + 3x + 4$ and we then perform polynomial long division to get

$$g(x) = 2x^3 + x^2 - 3x + \frac{4}{x-1}$$

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Section G: Extension Exam 2 (15 Marks)

Question 16 (1 mark)

How many **rational** roots does the polynomial $4x^3 - x^2 - 12x + 3$ have?

A. 0

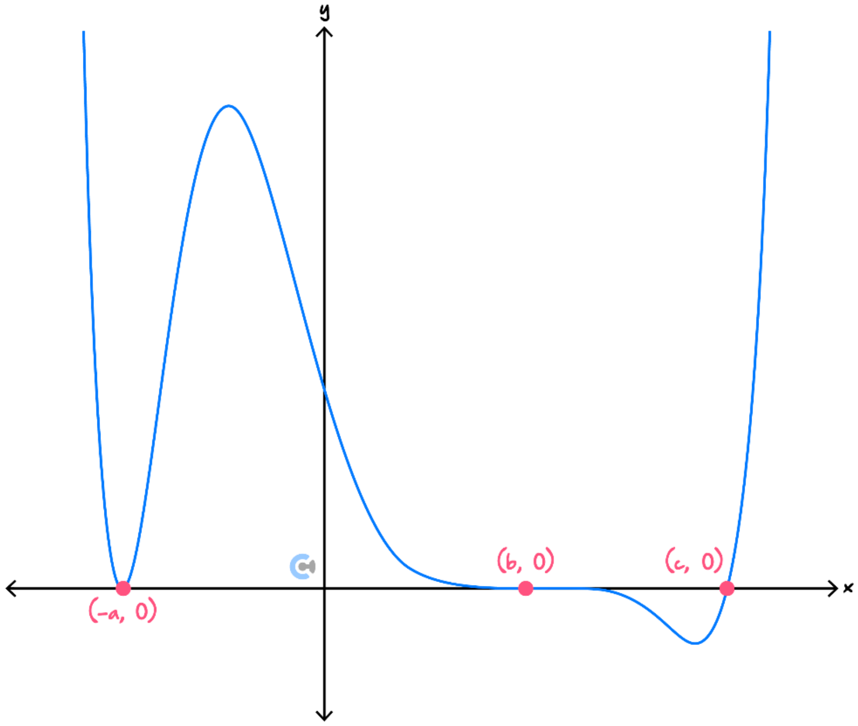
B. 1

C. 2

D. 3

Question 17 (1 mark)

If $a, b, c > 0$, which of the following could describe the graph below?



A. $y = (x - a)^2(x - b)^3(x - c)$

B. $y = (x + a)^2(x - b)^5(x - c)$

C. $y = (x - a)^2(x - b)^3(x - c)^2$

D. $y = (x + a)^2(x - b)(x - c)^3$

Question 18 (1 mark)

A set of three numbers that could be solutions to the equation $x^3 + ax^2 - 17x + 60 = 0$ is:

A. $\{1,4,7\}$

B. $\{-4,3,5\}$

C. $\{-5,2,3\}$

D. $\{-4,-5,3\}$

Question 19 (1 mark)

If $x + a$ is a factor of $x^3 + (1 - a)x^2 - 8x + 21$, where $a > 0$, then the value of a is:

A. 1

B. 2

C. 3

D. 4

Question 20 (1 mark)

The graph of $y = x^4 - 4kx^2 + 4$, where k is a real number, has four x-intercepts when:

A. $k < 1$

B. $k > 1$

C. $-1 \leq k \leq 1$

D. $k \leq 1$

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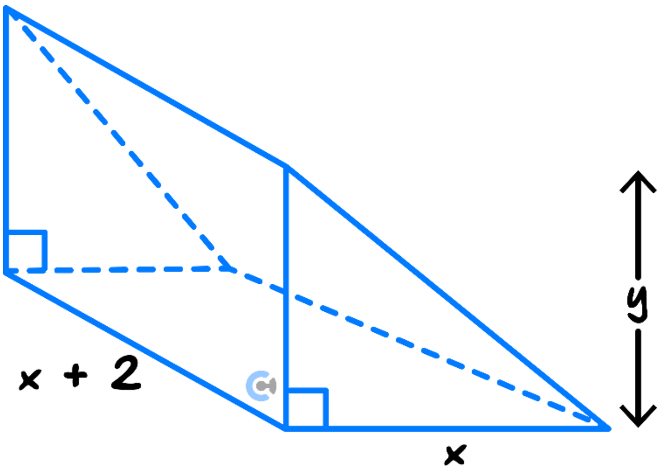
Question 21 (5 marks)

The triangular box below consists of sides $x\ cm$ and $x + 2\ cm$, in length and width, and a height of $y\ cm$.

It is known that the width, length and height of the box added together equals $30\ cm$.



a. Find an expression for V, the volume of the box, in terms of x. (2 marks)

We have that $x + x + 2 + y = 30 \implies y = 28 - 2x$.
Then $V = \frac{1}{2}(28 - 2x)x(x + 2) = x(x + 2)(14 - x)$

b. State the possible values that x can take. (2 marks)

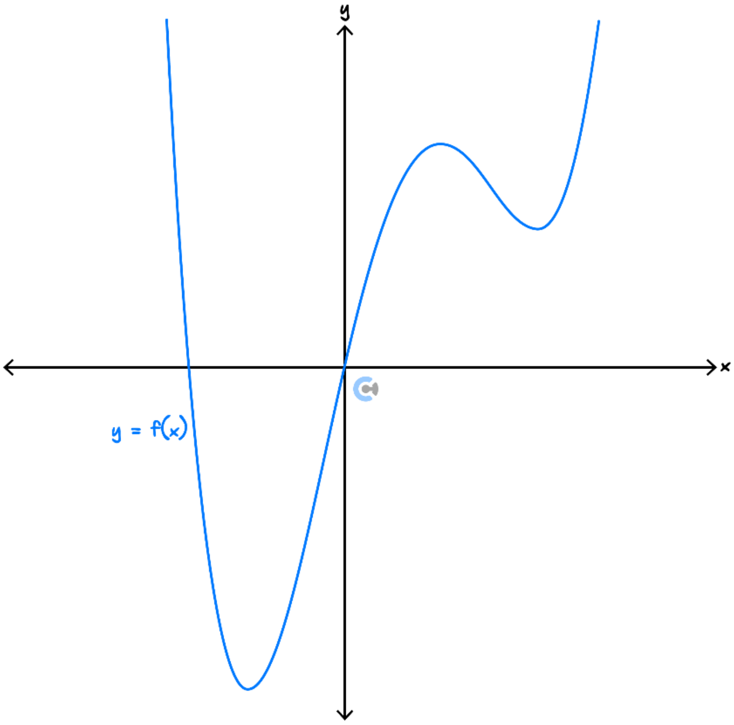
We must have $x > 0$ and $V > 0$.
Therefore $0 < x < 14$.

c. Find the maximum value of the box correct to the nearest cubic centimetre. (1 mark)

$495\ cm^3$

Question 22 (5 marks)

Consider the function $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x$. Part of the graph of $y = f(x)$ shown below.



a. Find the distance between the x-intercepts. Give your answer correct to two decimal places. (1 mark)

1.62.

b. It is known that f has turning points at x-values that are roots to the function $g(x) = 12x^3 - 24x^2 - 12x + 24$.
Find the coordinates of all turning points of f. (2 marks)

$(-1, -19), (1, 13)$ and $(2, 8)$

c. Hence, find the values of k such that $f(x) = k$ has two real solutions. (2 marks)

$-19 < k < 8$ or $k > 13$

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