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VCE Mathematical Methods ½
Polynomials [0.5]
Workshop

Section A: Recap

Degree of Polynomial Functions

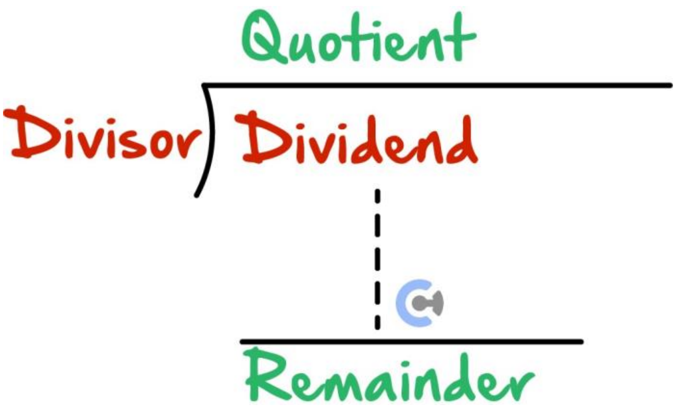
$$\textit{Degree} = \textit{Highest Power of the Polynomial}$$

Roots of Polynomial Functions

$$\textit{Roots} = x - \textit{intercept}$$

Polynomial Long Division

- Division of polynomials



$$\frac{\textit{Dividend}}{\textit{Divisor}} = \textit{Quotient} + \frac{\textit{Remainder}}{\textit{Divisor}}$$

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Remainder Theorem

- **Definition**:
  - Finds the remainder of long division without the need of long division,

$$\boldsymbol{when\ P(x)\ is\ divided\ by\ (x-\alpha), the\ remainder\ is\ P(\alpha).}$$

- **Steps**

1. Find x-values which make the divisor equal to 0.
2. Substitute it into the dividend function.

Factor Theorem

- For every x-intercept, there is a factor:

$$\boldsymbol{If\ P(\alpha) = 0\ then, (x-\alpha)\ is\ a\ factor\ of\ P(x).}$$

Factorising Polynomials

- The steps are:
  - Find a single root by trial and error.
    - (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

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Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$\text{Potential root} = \pm\frac{\text{Factors of constant term } a_0}{\text{Factors of leading coefficient } a_n}$$

- If the roots are rational numbers, the roots can only be $\pm\frac{\text{Factors of constant term } a_0}{\text{Factors of leading coefficient } a_n}$.

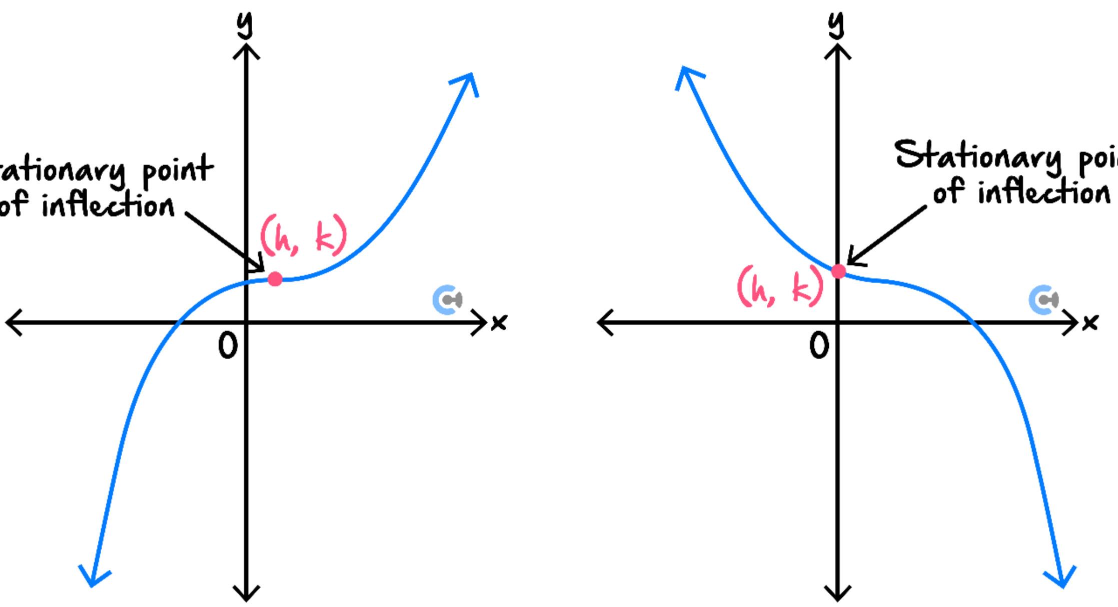
Sum and Difference of Cubes

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Graphs of $a(x-h)^n + k$, where n is an Odd Positive Integer

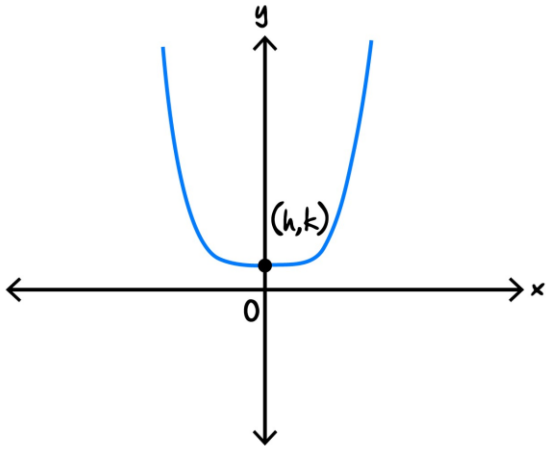
- All graphs look like a "cubic".

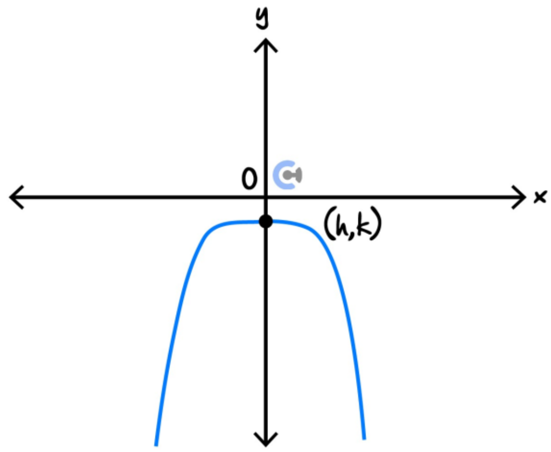


- The point (h, k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

Graphs of $a(x-h)^n + k$, Where n is an Even Positive Integer

- All graphs look like a "quadratic".

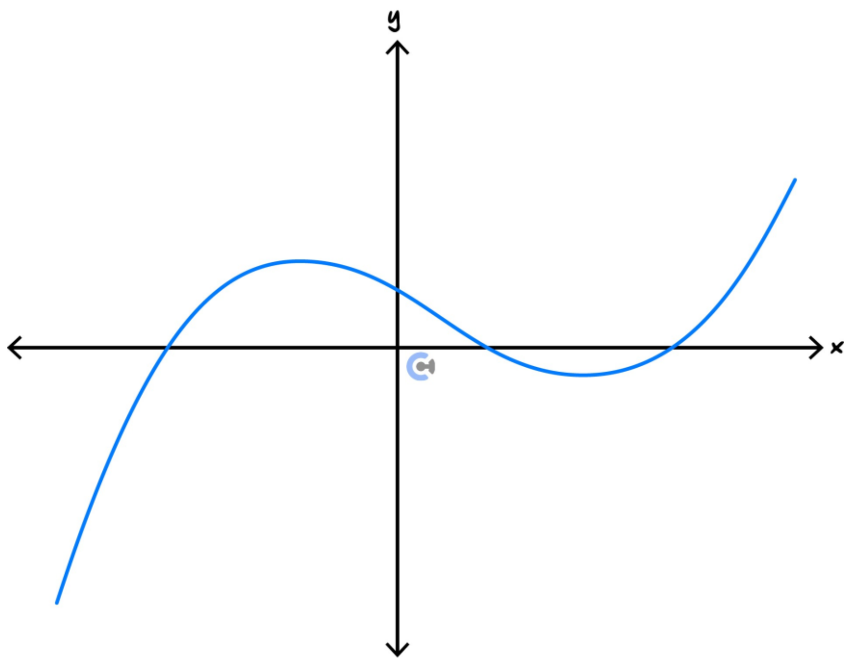




- The point (h, k) gives us the turning point.

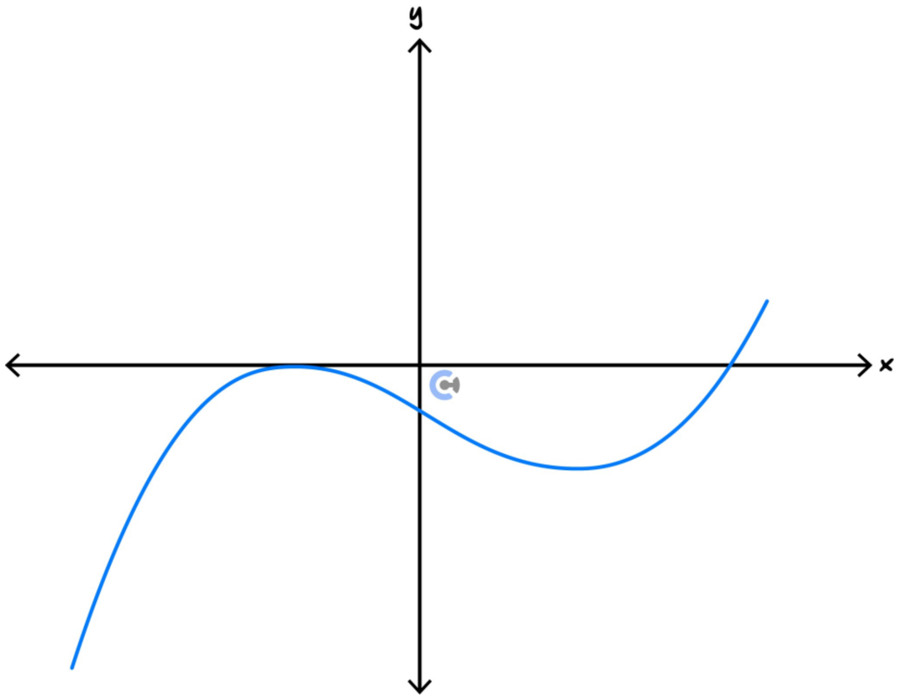
Graphs of Factorised Polynomials

- **All non-repeated linear factors correspond to x-intercepts of the graph.**



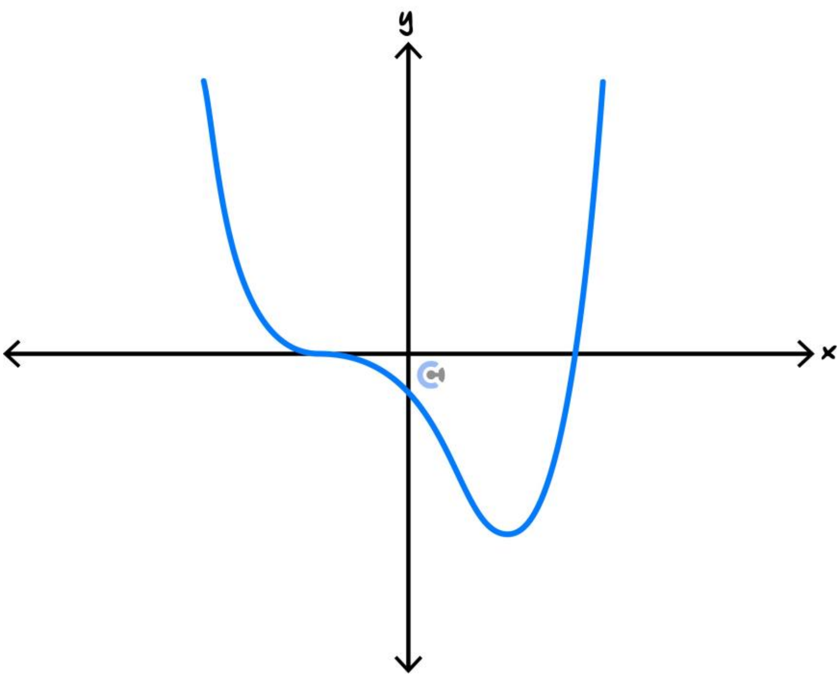
- E.g., $f(x) = (x-a)(x-b)(x-c)$ results in x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$.

- **All squared linear factors correspond to x-intercepts and T.P. of the graph.**



- E.g., $f(x) = (x-a)^2(x-b)$ will have an x-intercept $(a, 0)$ which is also a local minimum/maximum.

- **All cubed linear factors correspond to x-intercepts and SPI of the graph.**



- E.g., $f(x) = (x-a)^3(x-b)$ has an x-intercept $(a, 0)$ which is also a stationary point of inflection.

Definition

Steps to Graphing Factorised Polynomials

- **Steps:**

  1. Plot x-intercepts.
  2. Determine whether the polynomial is positive or negative.
  3. Use the repeated factors to deduce the shape.

     - Non - Repeated: Only x-intercept.
     - Even Repeated: x-intercept and a turning point.
     - Odd Repeated: x-intercept and a stationary point of inflection.

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Section B: Warmup

Question 1

a. Use polynomial long division to write $f(x) = \frac{x^3+2x^2+3}{x-2}$ in the form $f(x) = Q(x) + \frac{a}{x-2}$, for quadratic function Q and integer a.

b. Find the remainder of the division $\frac{f(x)}{g(x)}$ where $f(x) = x^3 + 3x^2 + 2$ and $g(x) = x + 1$.

c. Find all the roots of $f(x) = x^3 + 2x^2 - x - 2$.

d. Factorise the expression $8x^3 - 27$.

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Section C: Exam 1 (18 Marks)

Question 2 (5 marks)

We know that $f(x) = \frac{12}{2x+3}$ for all positive values of x.

a. Simplify $\frac{x^2+4x-5}{12} \times f(x)$. Give your answer in the form of $\frac{ax+5}{b} - \frac{c}{2x+3}$ where a and b are positive integers and c is a rational number. (3 marks)

$$\frac{x^2+4x-5}{12} \times f(x) = \frac{x^2+4x-5}{\cancel{12}} \times \frac{\cancel{12}}{2x+3}$$

$$= \frac{x^2+4x-5}{2x+3}$$

$$\begin{array}{r|l} & \frac{1}{2}x + \frac{5}{4} \\ \hline (2x+3) & x^2+4x-5 \\ & -(x^2+\frac{3}{2}x) \\ \hline & \frac{5}{2}x - 5 \\ & -(\frac{5}{2}x+\frac{15}{4}) \\ \hline & -\frac{35}{4} \end{array}$$

$$= \frac{1}{2}x + \frac{5}{4} - \frac{\frac{35}{4}}{2x+3}$$

$$= \frac{2x+5}{4} - \frac{\frac{35}{4}}{2x+3} //$$

Consider $g(x) = \frac{5-81x^2}{4}$ for all values of x.

b. Solve $g(x) = -13$. (2 marks)

$$\frac{5-81x^2}{4} = -13$$

$$5-81x^2 = -52$$

$$81x^2 = 57$$

$$x^2 = \frac{57}{81}$$

$$x = \pm\sqrt{\frac{57}{81}} = \pm\sqrt{\frac{19}{27}} = \pm\frac{\sqrt{57}}{9} //$$

$$= \pm\frac{\sqrt{19}}{3\sqrt{3}} = \pm\frac{\sqrt{57}}{9} //$$

Question 3 (3 marks)

$x = 2, -1, \frac{1}{2}$

$y = (2)^2(1)(\frac{1}{2}) = 2$

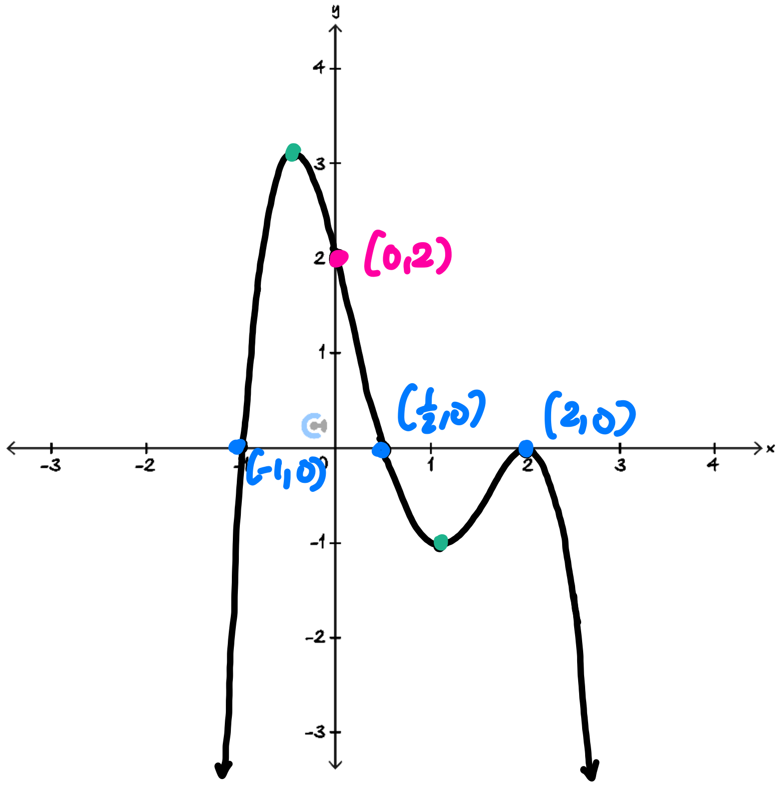
Consider the function:

$$f(x) = (2 - x)^2(x + 1)\left(\frac{1}{2} - x\right)$$

Shape $\Rightarrow (-x)^2(x)(-x)$
$= -x^4$

It is known that f has turning points at approximately $(-0.5, 3.1)$ and $(1.1, -1)$.
Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts with coordinates.

$-$ve parabola



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Y-int

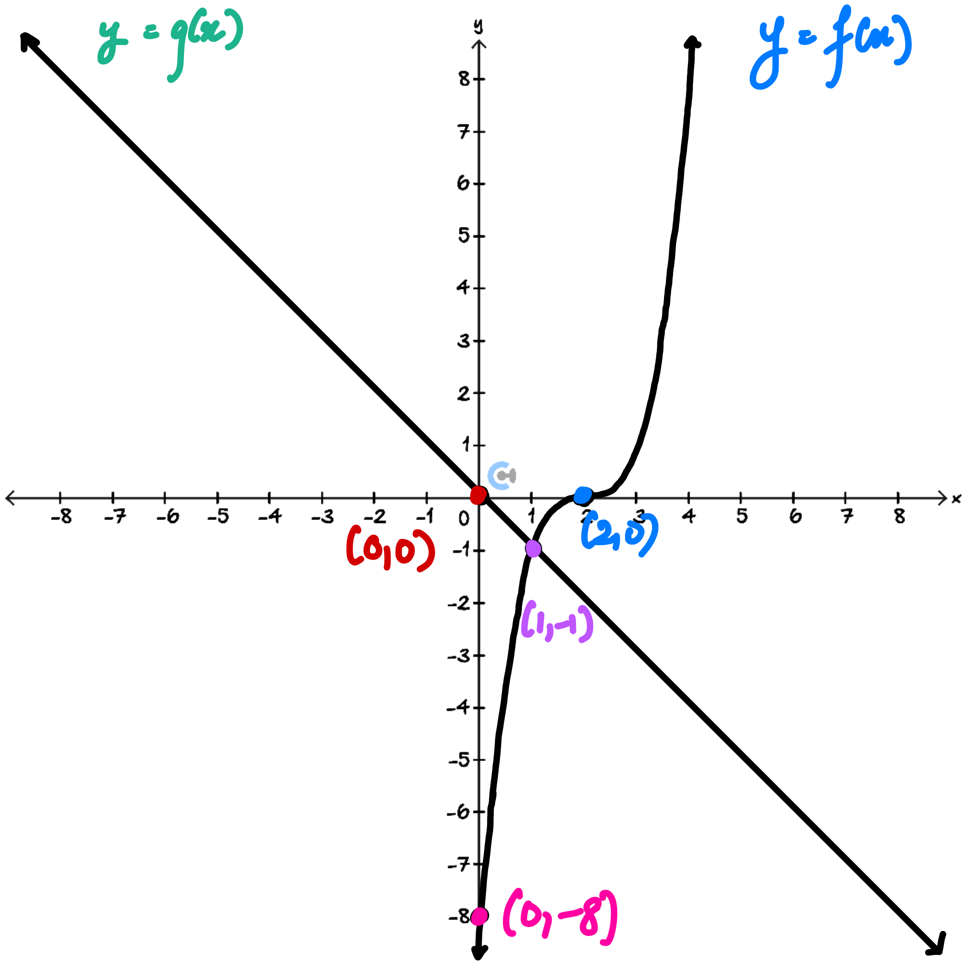
SPOI : (2,0)

$y = (-2)^3 = -8$

Question 4 (4 marks)

Consider the functions $f(x) = (x-2)^3$ and $g(x) = -x$.

a. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the axes below. Label all axes intercepts. (3 marks)



b. Hence, solve the equation $(x-2)^3 = -x$ for x. (1 mark)

$\therefore x = 1$

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Question 5 (6 marks)

a. Consider the function $f(x) = 3x^3 + ax^2 + bx - 12$. If $x - 2$ is a factor of $f(x)$ and the remainder of $f(x) \div x - 1 = -18$, find the value(s) of a and b. (3 marks)

$R = f(2) = 0 \Rightarrow 24 + 4a + 2b - 12 = 0$

$4a + 2b = -12$

$\boxed{2a + b = -6} \ldots ①$

$R = f(1) = -18 \Rightarrow 3 + a + b - 12 = -18$

$\boxed{a + b = -9} \ldots ②$

$2a + b = -6$

$-(a + b = -9)$ ① − ②

$\therefore a = 3 \Rightarrow 3 + b = -9$

$\therefore b = -12$

b. Hence, simplify the following using polynomial long division: $\frac{3x^3+3x^2-12x-12}{x^2+4x+1}$. (3 marks)

$3x - 9$

$(x^2+4x+1) \overline{)\, 3x^3 + 3x^2 - 12x - 12}$

$-(3x^3 + 12x^2 + 3x)$

$-9x^2 - 15x - 12$

$-(-9x^2 - 36x - 9)$

$21x - 3$

$= 3x - 9 + \frac{21x - 3}{x^2 + 4x + 1}$ //

Section D: Tech Active Exam Skills

Calculator Commands: Turning Point

- ALWAYS sketch the graph to find approximate bounds for where the turning point you want is located.
- To find a local maximum we maximise the function over a specific domain.
- To find a local minimum we minimise the function over a specific domain.
- **TI and Casio:** Use fmin(expression, variable, lower (optional), upper (optional)) or fmax(expression, variable, lower (optional), upper (optional)).
- **TI:** Menu→ $4 \to \frac{7}{8}$.

$$\text{Define } f(x)=x^3-4\cdot x \qquad \textit{Done}$$

$$\text{fMin}(f(x),x,0,2) \qquad x=\frac{2\cdot\sqrt{3}}{3}$$

$$f\left(\frac{2\cdot\sqrt{3}}{3}\right) \qquad \frac{-16\cdot\sqrt{3}}{9}$$

- **Casio:** Action→Calculation→ fmin/fmax

```
fmin(x^3-4x, x, 0, 2)
```

$$\left\{\text{MinValue}=\frac{-16\cdot\sqrt{3}}{9},\ x=\frac{2\cdot\sqrt{3}}{3}\right\}$$

.

NOTE: TI only gives the x-value for the min/max so we then need to sub it back into our function. Casio gives us both!

Calculator Commands

- **Mathematica:** Minimize[ ] and Maximize[ ] commands.
- Minimize[$f[x], x$] will minimize $f[x]$ over its whole domain.
- To restrict the domain, we must use Minimize[$\{f[x], a \leq x \leq b\}, x$].

```
In[34]:= Minimize[{x^3 - 4 x, 0 < x < 2}, x]
```

Out[34]= $\left\{-\frac{16}{3\sqrt{3}}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$

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Section E: Exam 2 (26 Marks)

Question 6 (1 mark)

The stationary point of inflection of the cubic, $y = (x-2)^3 + 4$ occurs at:

A. $(-2, 4)$

B. $(2, 4)$

C. $(-2, -4)$

D. $(4, 2)$

↳ SPOI : (2,4)

Question 7 (1 mark)

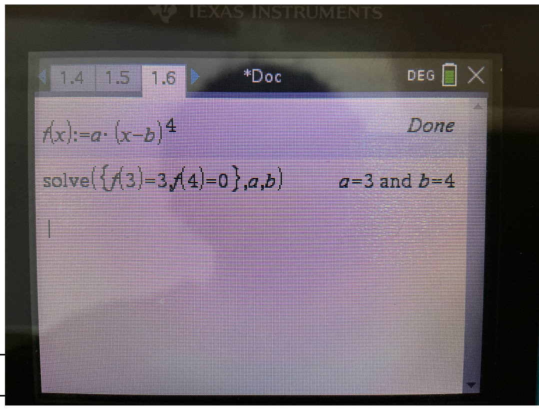
The data $(3, 3)$, $(4, 0)$, $(5, 3)$, $(6, 48)$ and $(7, 243)$ can be modelled by the equation $y = a(x-b)^4$. The values of a and b respectively are:

A. 3 and 4.

B. -3 and 4.

C. 4 and 3.

D. -3 and -4.



Question 8 (1 mark)

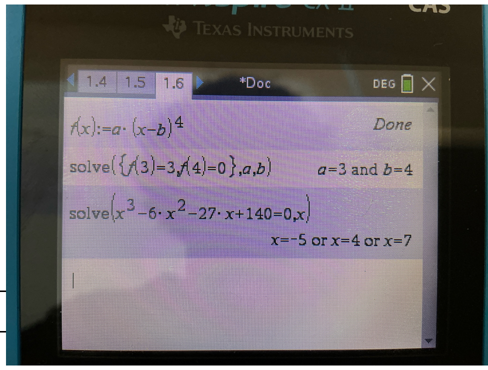
The values of x that satisfy $x^3 - 6x^2 - 27x + 140 = 0$ are:

A. $x = -5, 4, 5$

B. $x = -5, -4, 6$

C. $x = -5, 4, 7$

D. $x = 2, 3, 7$



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Question 9 (1 mark)

Using the Rational Root Theorem, count the number of possible roots that $y = 3x^3 + 7x^2 - 5x + 11$ has.

A. 4

B. 6

C. 7

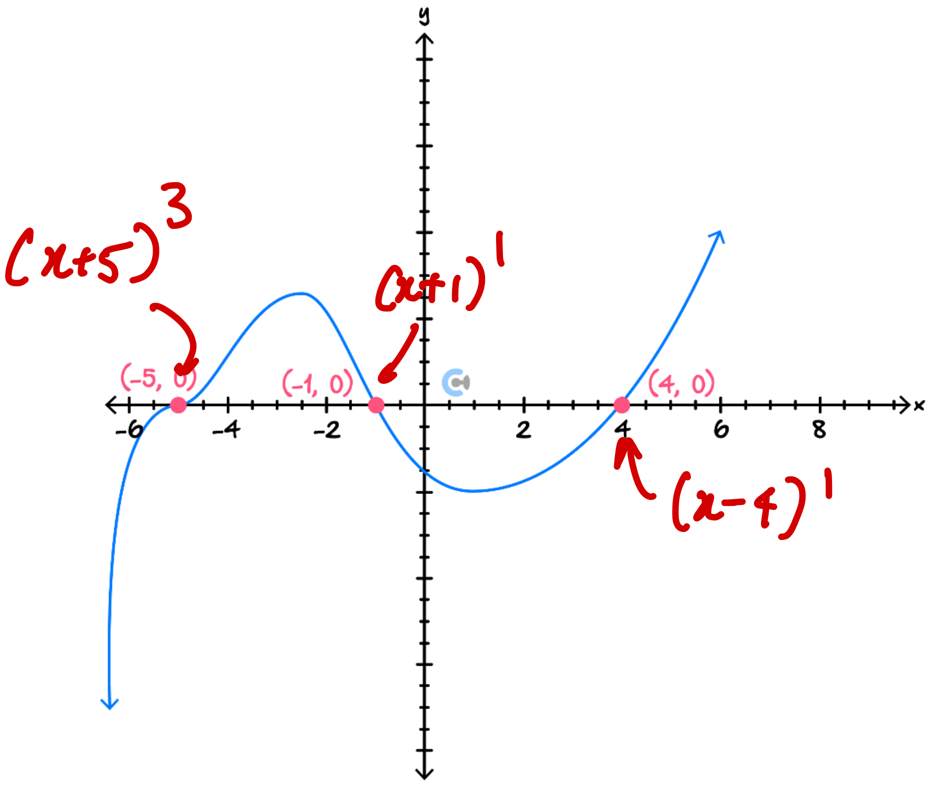
D. 8

$\text{RRT} = \pm \dfrac{\{1, 11\}}{\{1, 3\}} = \pm \left\{1, \frac{1}{3}, 11, \frac{11}{3}\right\}$

$2 \times 4 = 8$

Question 10 (1 mark)

The equation that best represents the graph below is:



A. $y = (x-4)^3(x+5)(x-1)$

B. $y = (x+5)^3(x+1)(x-4)$

C. $y = (x+1)^3(x+5)^2(x-4)$

D. $y = (x+4)(x+5)(x-1)$

Question 11 (1 mark)

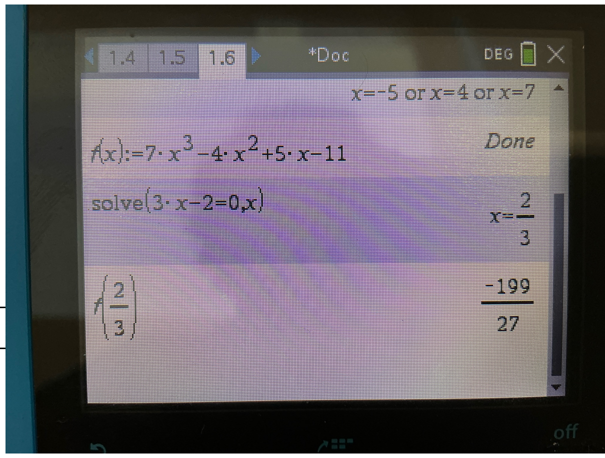
Find the remainder of the division, $f(x) \div g(x)$, where $f(x) = 7x^3 - 4x^2 + 5x - 11$ and $g(x) = 3x - 2$.

A. 0

B. 13

C. $\frac{233}{17}$

D. $-\frac{199}{27}$



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x-int (-3,0)   x-int TP (1,0)   x-int TP (2,0)

Question 12 (10 marks)

Consider the quintic polynomial $f(x) = (x+3)(x-1)^2(x-2)^2$. The function has turning points at $(-2.11, 145.41)$, $(a, 0)$, $(1.51, 0.28)$, $(b, 0)$, where $a > b$.

← $x \cdot x^2 \cdot x^2 = +x^5$

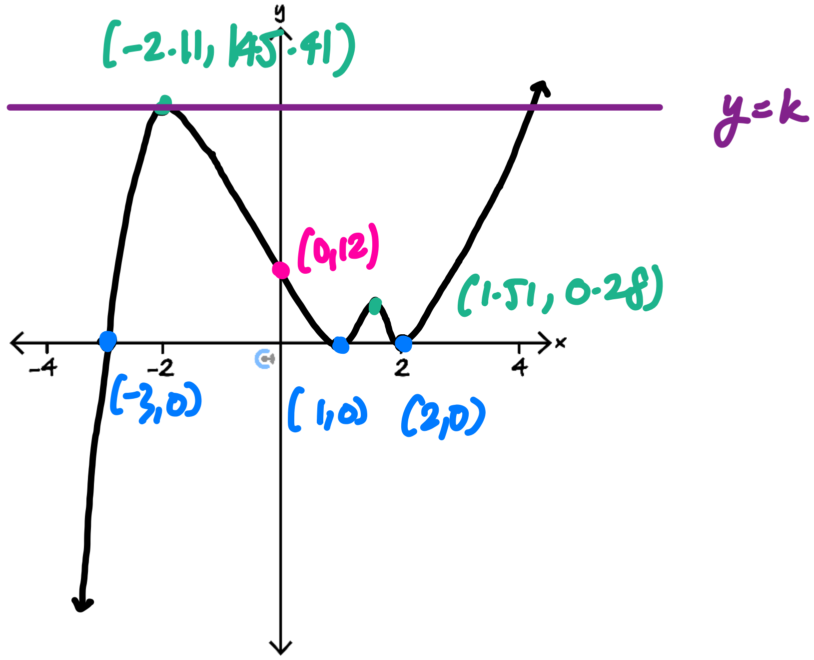
+ve cubic

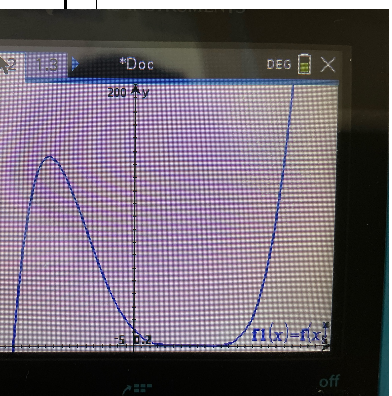
a. State the values of a and b. (1 mark)

$a = 2,\ b = 1$

b. Sketch the graph on the axes below labelling all x-intercepts and turning points. Ignore the y-axis scale. (3 marks)

$y = (3)(-1)^2(-2)^2 = 12$

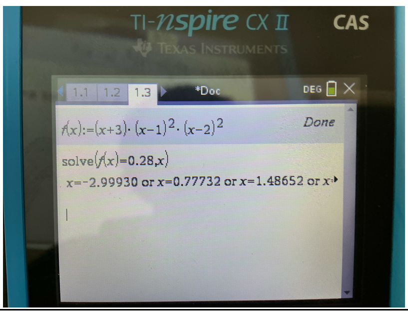


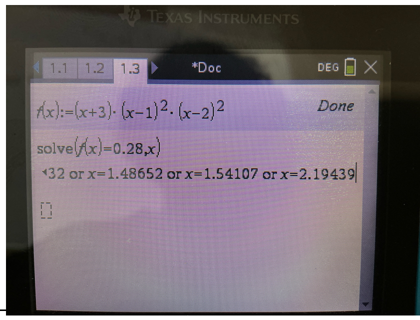


c. Find all solutions to $f(x) = 0.28$ for x correct to 2 decimal places. (2 marks)

$f(x) = 0.28$

$\therefore x = -3.00, 0.78, 1.49, 1.54, 2.19$





$y = f(x)$ $y = k$

$f(x) = k$

d. Find the values of k (to 2 decimal places) when:

i. $f(x) = k$ has 1 solution. (1 mark)

1 intersection

$\therefore k < 0$ or $k > 145.41$

ii. $f(x) = k$ has 5 solutions. (2 marks)

$\therefore 0 < k < 0.28$

iii. $f(x) = k$ has 2 solutions. (1 mark)

$\therefore k = 145.41$

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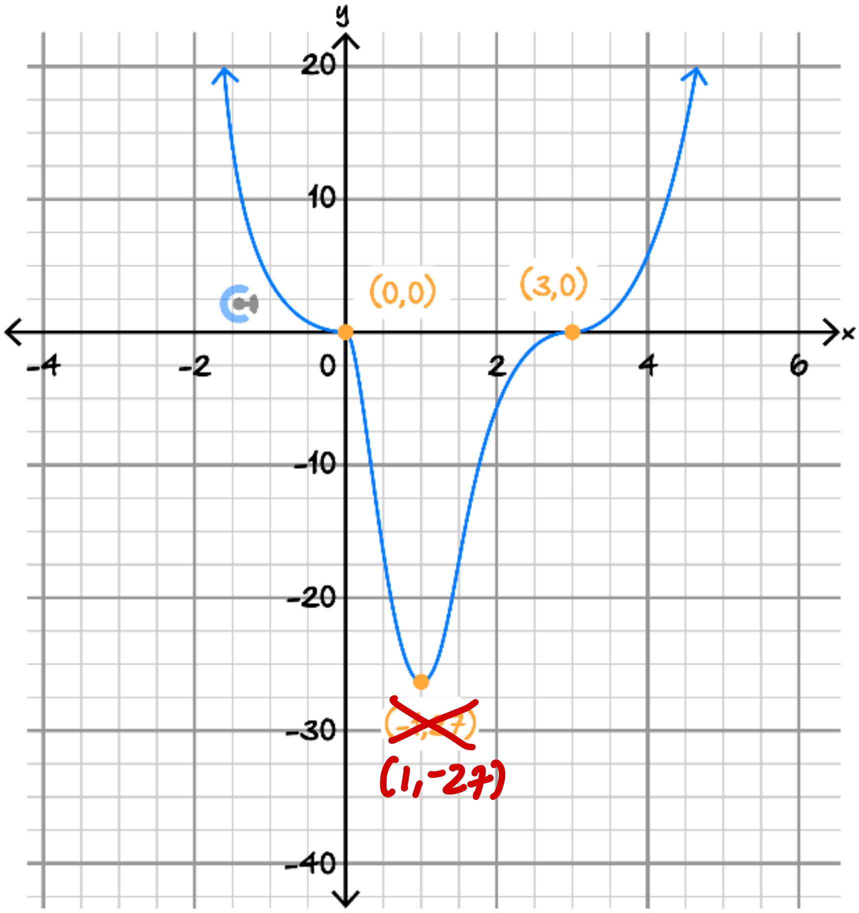
Question 13 (9 marks)

Consider the function of the form $f(x) = a(x-b)^3(x-c)^3$ where $b > c$, shown on the axes below.

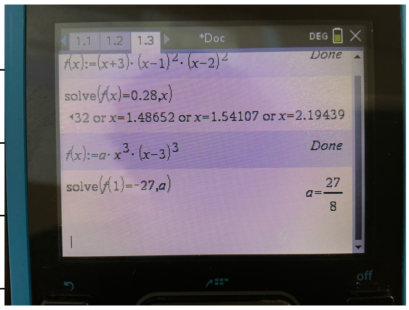


a. Find the values of a, b and c. (3 marks)

$\therefore b = 3,\ c = 0$

$f(1) = -27$

$\therefore a = \frac{27}{8}$



b. Hence, simplify $\frac{f(x)}{(x-3)^3}$, leaving your answer in the form of $\frac{px^q}{m}$, where p, q and m are positive integers. Show all working. (2 marks)

$$= \frac{\frac{27}{8}x^3\cancel{(x-3)^3}}{\cancel{(x-3)^3}} = \frac{27}{8}x^3$$

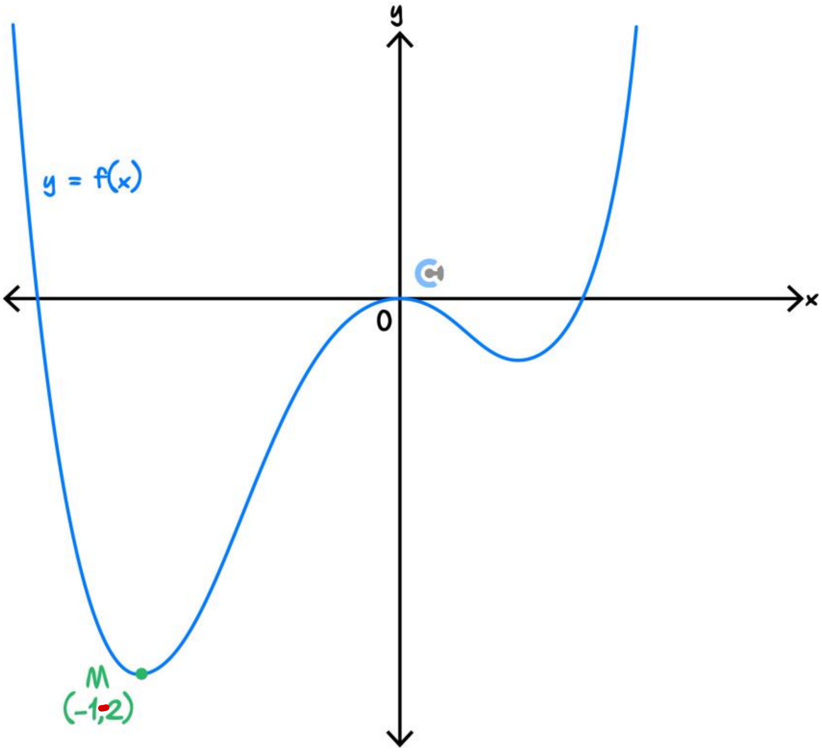
$$= \frac{27x^3}{8}$$ //

c. It is known that the division, $f(x) \div (x-k)$, leaves a remainder of 0. Find the possible values of k. (2 marks)

$(x-k)$ is a factor

$\therefore k = 3, 0$

Consider the quartic $f: R \to R, f(x) = 5x^4 + 6x^3 - x^2$. Part of the graph of $f(x)$ is shown below.



d. State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts. (1 mark)

y-translation (up or down)

$\therefore b > 2$

Let $p: R \to R, p(x) = 5x^4 + 3(a+2)x^3 - x^2 - 2ax + 3a^2, a \in R$.

e. State the value of a for which $p(x) = f(x)$ has infinite solutions. (1 mark)

∞ intersections

$\therefore a = 0$

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Section F: Extension Exam 1 (14 Marks)

Question 14 (8 marks)

Consider the function $f(x) = x^5 + x^4 - 5x^3 - 5x^2 + 4x + 4$.

a. Show that $(x^2 - 4)$ is a factor of f. (2 marks)

$(x^2-4) = (x+2)(x-2)$

$f(-2) = -32+16+40-20-8+4$
$= 0 \Rightarrow \therefore (x+2)$ is a factor

$f(2) = 32+16-40-20+8+4$
$= 0 \Rightarrow (x-2)$ is a factor

$\therefore (x^2-4)$ is a factor

b. Find all roots of f. (3 marks)

$$\begin{array}{r}
x^3+x^2-x-1 \\
(x^2-4)\overline{)\,x^5+x^4-5x^3-5x^2+4x+4} \\
-(x^5 \quad -4x^3) \\
\hline
x^4-x^3-5x^2+4x+4 \\
-(x^4 \quad -4x^2) \\
\hline
-x^3-x^2+4x+4 \\
-(-x^3 \quad +4x) \\
\hline
-x^2+4 \\
-(-x^2+4) \\
\hline
0
\end{array}$$

$= (x^2-4)(x^3+x^2-x-1)$

$= (x+2)(x-2)(x+1)^2(x-1)$ //

$x^2(x+1) - 1(x+1)$

$(x+1)[x^2-1]$

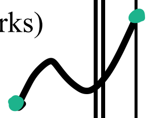
$= (x+1)(x+1)(x-1)$

$\therefore x = \pm 2, \pm 1$

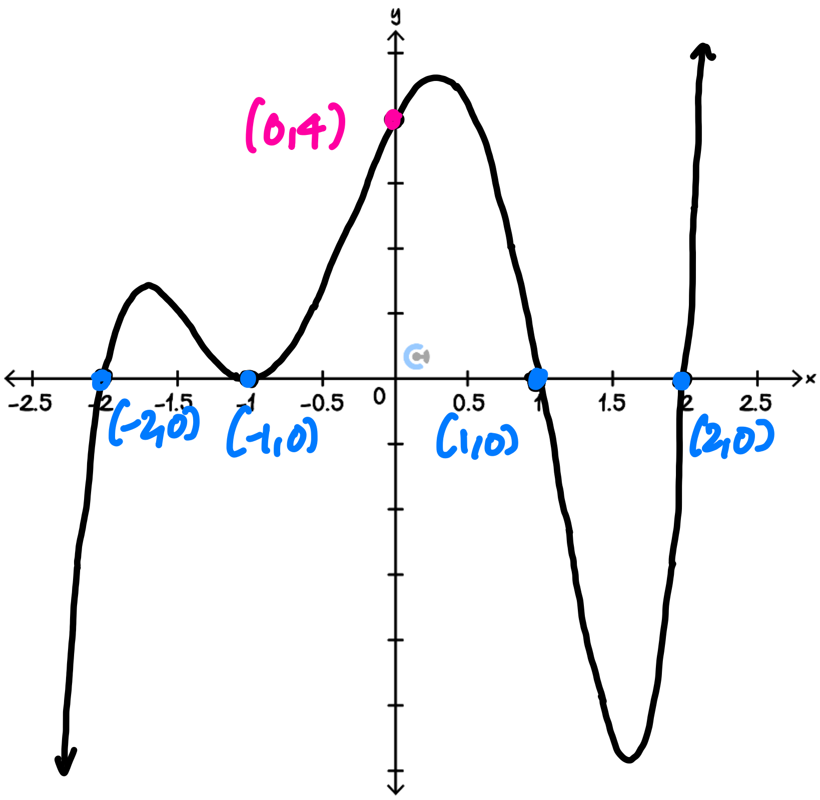
$x = \pm 2, \pm 1$

$y = 4$

$+x^5 \Rightarrow$ Shape $\rightarrow$ +ve cubic



c. It is known that f has two turning points when $x > 0$.
Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts and ignore the y-scale. (3 marks)



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Question 15 (6 marks)

A cubic function $f(x) = ax^3 + bx^2 + cx + d$ passes through the points $(-2, -22)$, $(0,4)$ $(1,2)$ and $(2,6)$.

a. Show that $a = 2, b = -3, c = -1$ and $d = 4$. (3 marks)

b. Write the function $g(x) = \frac{(x+1)f(x)}{x-1}$ in the form $g(x) = C(x) + \frac{A}{x-1}$, for a cubic function and an integer A.
(3 marks)

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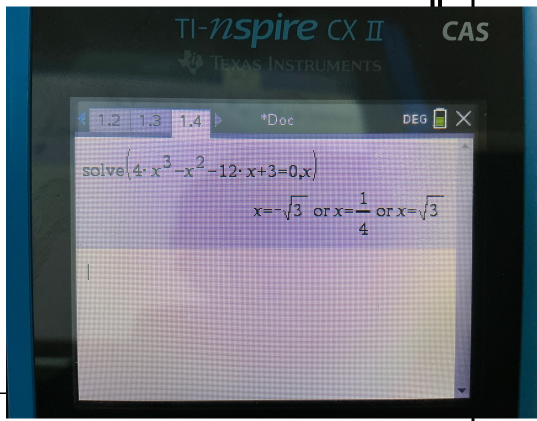
Section G: Extension Exam 2 (15 Marks)

Question 16 (1 mark)

How many **rational** roots does the polynomial $4x^3 - x^2 - 12x + 3$ have?



A. 0

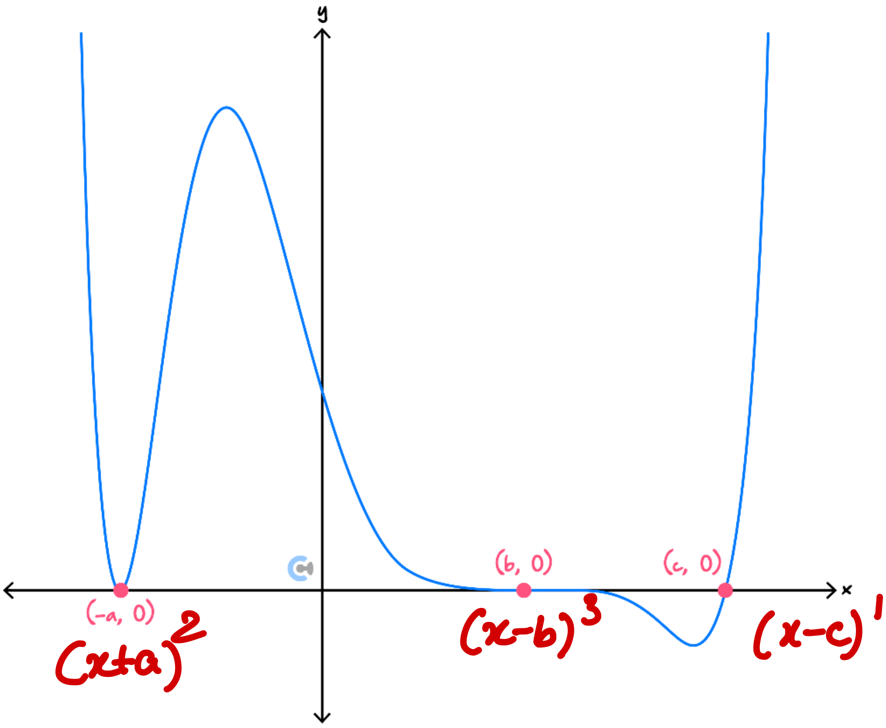
B. 1

C. 2

D. 3

Question 17 (1 mark)

If $a, b, c > 0$, which of the following could describe the graph below?



A. $y = (x - a)^2(x - b)^3(x - c)$

B. $y = (x + a)^2(x - b)^5(x - c)$

C. $y = (x - a)^2(x - b)^3(x - c)^2$

D. $y = (x + a)^2(x - b)(x - c)^3$

Question 18 (1 mark)

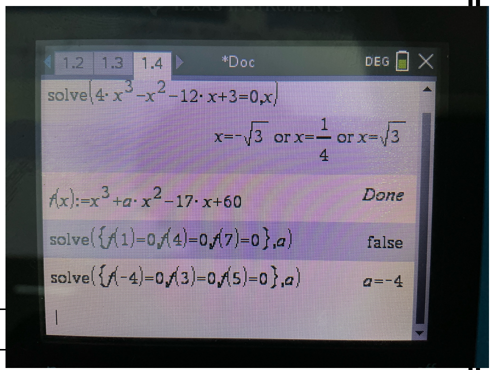
A set of three numbers that could be solutions to the equation $x^3 + ax^2 - 17x + 60 = 0$ is:

A. $\{1,4,7\}$

B. $\{-4,3,5\}$

C. $\{-5,2,3\}$

D. $\{-4,-5,3\}$



Question 19 (1 mark)

If $x + a$ is a factor of $x^3 + (1 - a)x^2 - 8x + 21$, where $a > 0$, then the value of a is:

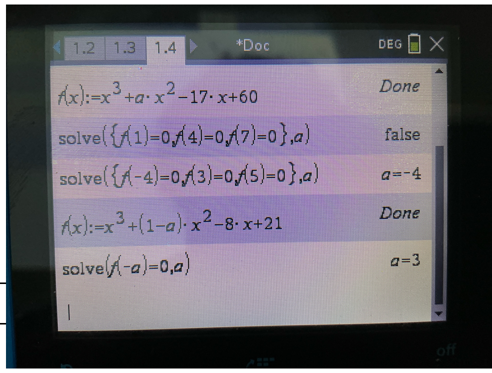
$f(-a) = 0$

A. 1

B. 2

C. 3

D. 4



Question 20 (1 mark)

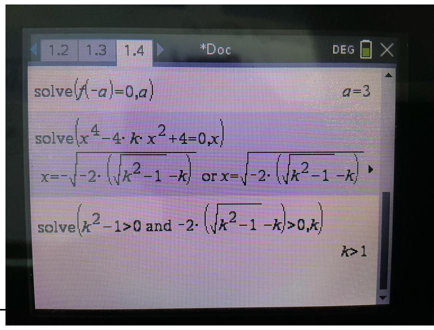
The graph of $y = x^4 - 4kx^2 + 4$, where k is a real number, has four x-intercepts when:

A. $k < 1$

B. $k > 1$

C. $-1 \le k \le 1$

D. $k \le 1$



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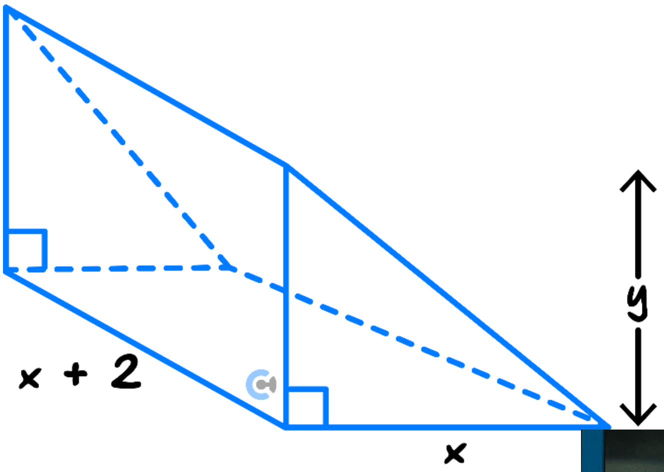
$$x = \pm\sqrt{\underbrace{-2\left(\pm\sqrt{\underbrace{k^2-1}_{>0}} - k\right)}_{>0}}$$

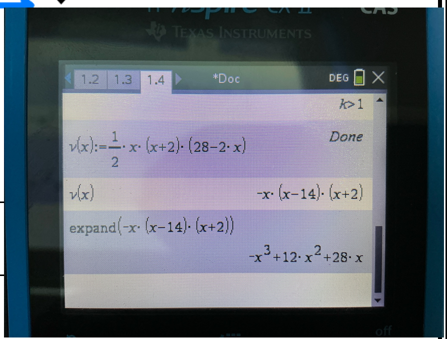
Question 21 (5 marks)

The triangular box below consists of sides $x\ cm$ and $x + 2\ cm$, in length and width, and a height of $y\ cm$.

It is known that the width, length and height of the box added together equals 30 cm.



a. Find an expression for V, the volume of the box, in terms of x. (2 marks)



$x + (x+2) + y = 30$

$\therefore y = 28 - 2x$

$V = \frac{1}{2} \cdot x \cdot y \cdot (x+2)$

$= \frac{1}{2} \cdot x \cdot (28-2x) \cdot (x+2)$

$\therefore V(x) = -x^3 + 12x^2 + 28x$

b. State the possible values that x can take. (2 marks)

$V > 0$

$x > 0 \Rightarrow x > 0$

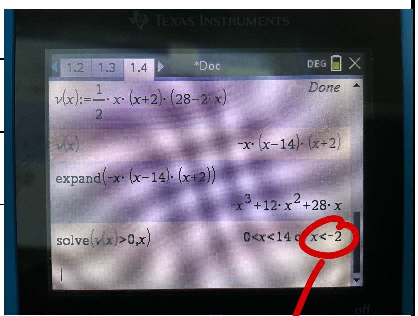
$x + 2 > 0 \Rightarrow x > -2$

$y > 0 \Rightarrow 28 - 2x > 0$

$2x < 28$

$x < 14$

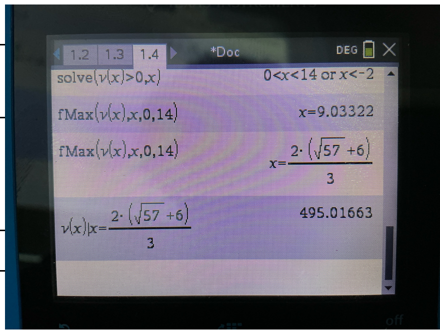
$\therefore 0 < x < 14$



reject as $x > 0$

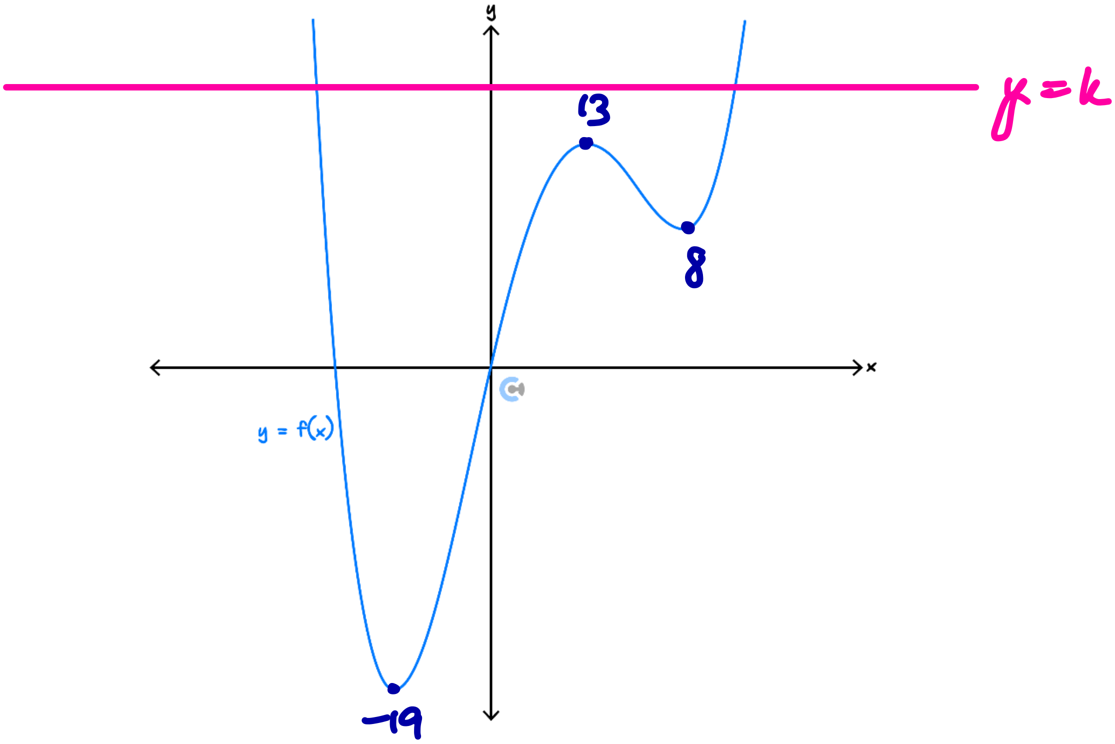
c. Find the maximum value of the box correct to the nearest cubic centimetre. (1 mark)

Max Volume = 495 cm^3 //



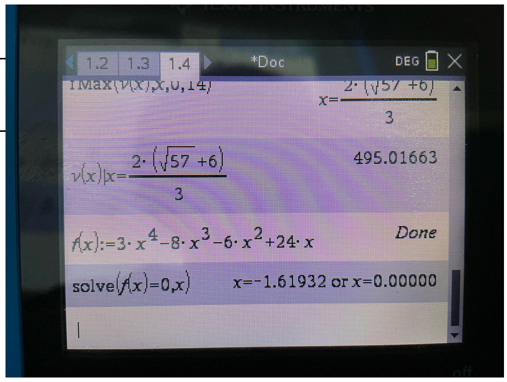
Question 22 (5 marks)

Consider the function $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x$. Part of the graph of $y = f(x)$ shown below.



a. Find the distance between the x-intercepts. Give your answer correct to two decimal places. (1 mark)

∴ Distance = 1.62 units //



b. It is known that f has turning points at x-values that are roots to the function $g(x) = 12x^3 - 24x^2 - 12x + 24$.

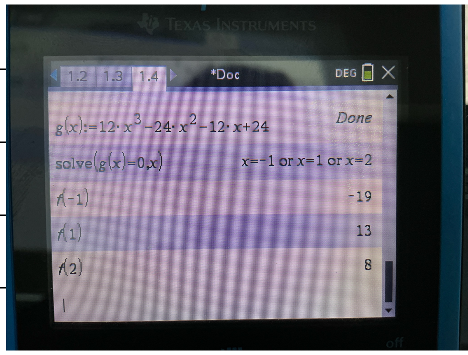
Find the coordinates of all turning points of f. (2 marks)

at $g(x) = 0$:

$x = -1, 1, 2$

$f(-1) = -19$ $\quad f(1) = 13$ $\quad f(2) = 8$

$\therefore (-1, -19)$ $\quad \therefore (1, 13)$ $\quad \therefore (2, 8)$



c. Hence, find the values of k such that $f(x) = k$ has two real solutions. (2 marks)

2 intersections

$y = f(x)$ $\quad y = k$

$\therefore -19 < k < 8$ or $k > 13$

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