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# VCE Mathematical Methods ½
# Polynomials [0.5]
## Workshop

## Section A: Recap

### Degree of Polynomial Functions

Definition

$$Degree = Highest\ Power\ of\ the\ Polynomial$$

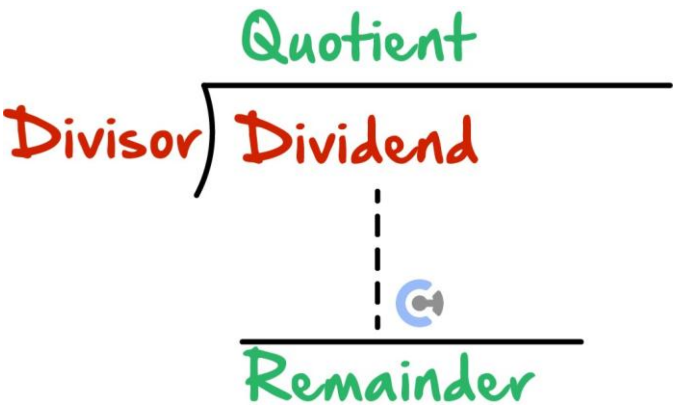
### Roots of Polynomial Functions

Definition

$$Roots = x - intercept(s)$$

### Polynomial Long Division

- Division of polynomials




$$\frac{Dividend}{Divisor} = Quotient + \frac{Remainder}{Divisor}$$

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## Remainder Theorem

- **Definition**:
  - Finds the remainder of long division without the need of long division,

$$\text{when } P(x) \text{ is divided by } (x-\alpha), \text{the remainder is } P(\alpha).$$

- **Steps**
  1. Find $x$-values which make the divisor equal to $0$.
  2. Substitute it into the dividend function.

## Factor Theorem

- For every $x$-intercept, there is a factor:

$$\text{If } P(\alpha) = 0 \text{ then, } (x-\alpha) \text{ is a factor of } P(x).$$

## Factorising Polynomials

- The steps are:
  - Find a single root by trial and error.
    - (Factor Theorem: Substitute into the function and see if we get zero.)
  - Use long division to find the quadratic factor.
  - Factorise the quadratic.

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### Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$\text{Potential root} = \pm\frac{\text{Factors of constant term } a_0}{\text{Factors of leading coefficient } a_n}$$

- If the roots are rational numbers, the roots can only be $\pm\frac{\text{Factors of constant term } a_0}{\text{Factors of leading coefficient } a_n}$.

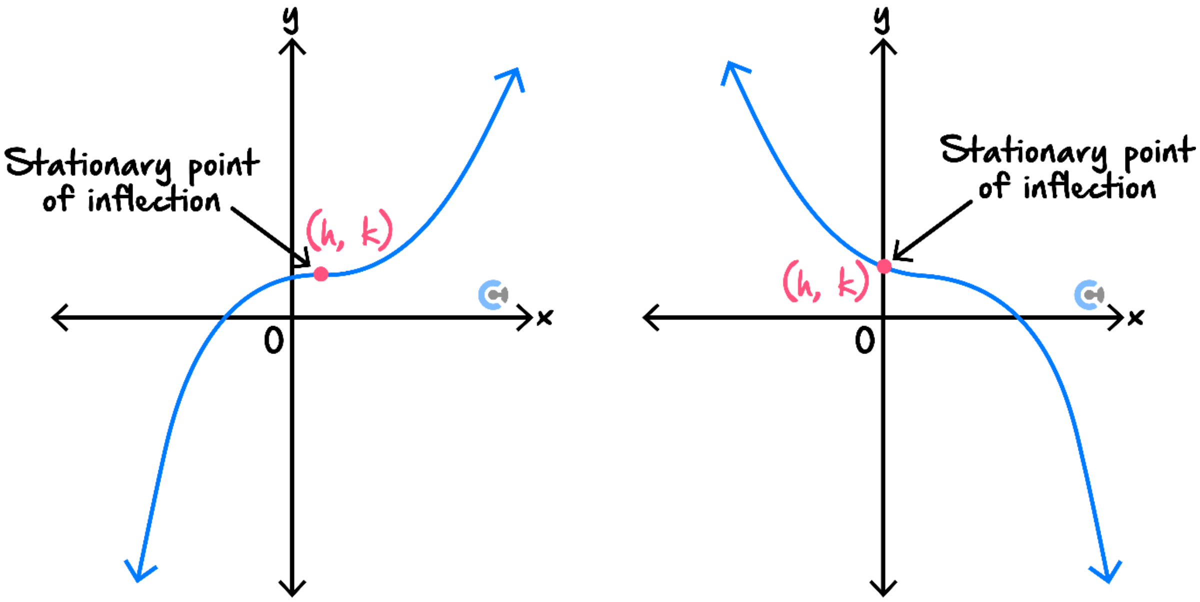
### Sum and Difference of Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

### Graphs of $a(x-h)^n + k$, where $n$ is an Odd Positive Integer

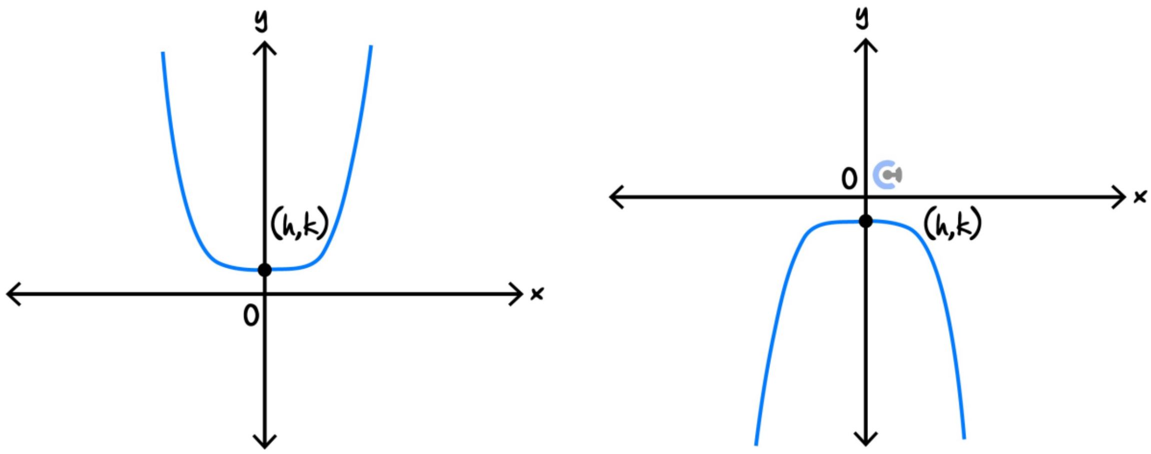
- All graphs look like a "cubic".




- The point $(h, k)$ gives us the stationary point of inflection.
- $n$ cannot be $1$ for this shape to occur!

## Graphs of $a(x-h)^n + k$, Where $n$ is an Even Positive Integer

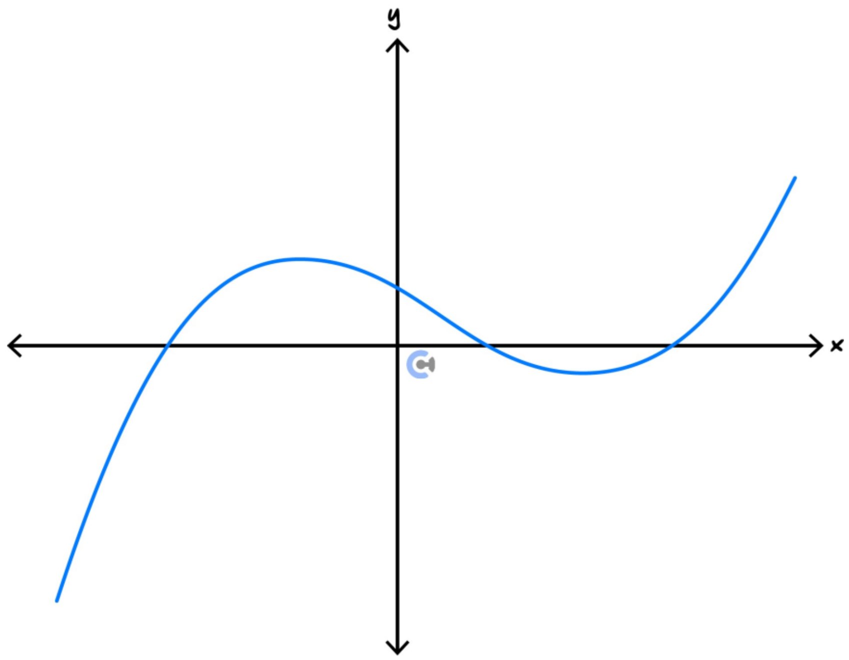
- All graphs look like a "quadratic".




- The point $(h, k)$ gives us the turning point.

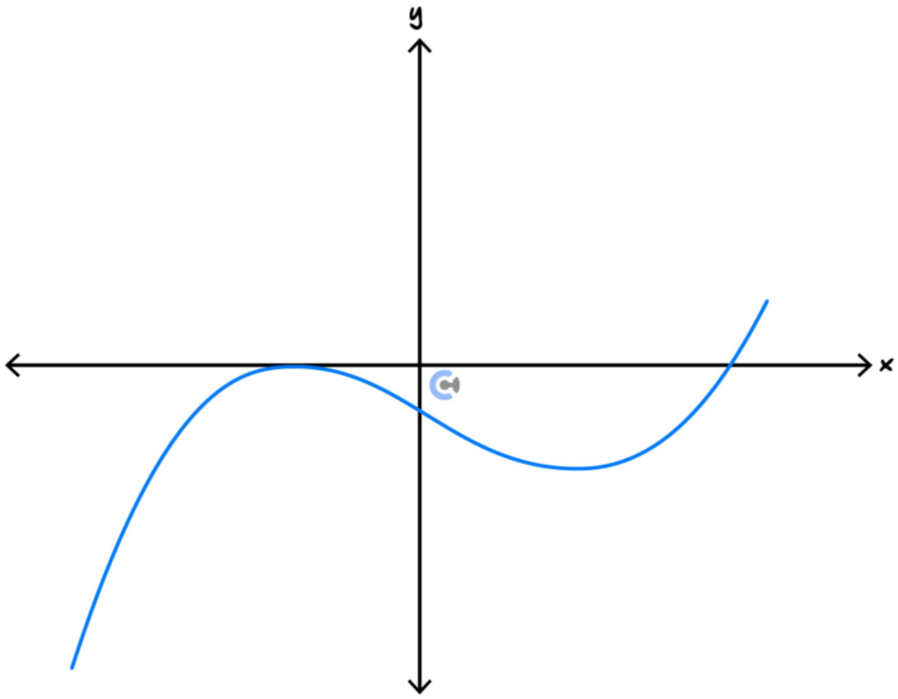
## Graphs of Factorised Polynomials

- **All non-repeated linear factors correspond to $x$-intercepts of the graph.**



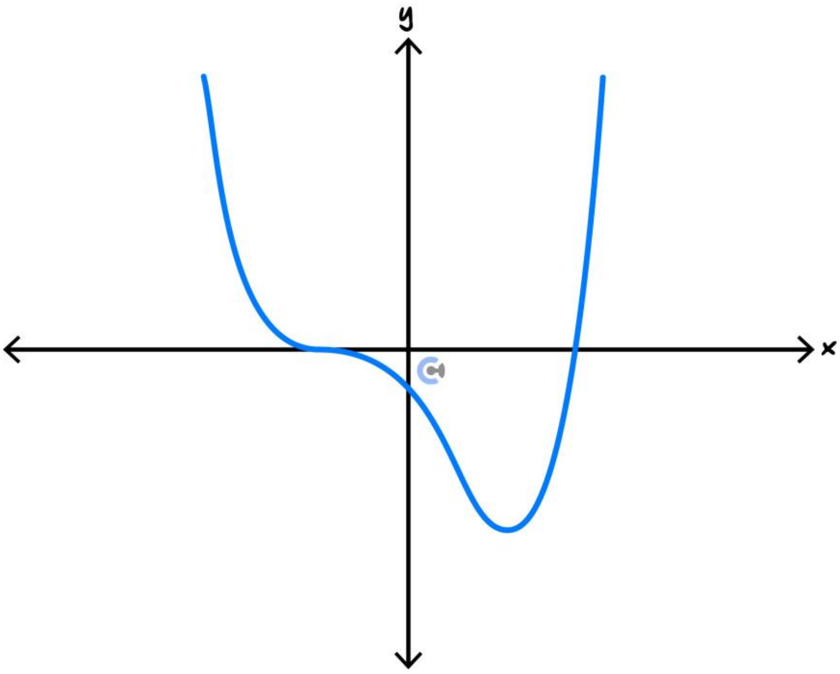

- E.g., $f(x) = (x-a)(x-b)(x-c)$ results in $x$-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$.

- **All squared linear factors correspond to $x$-intercepts and T.P. of the graph.**




- E.g., $f(x) = (x-a)^2(x-b)$ will have an $x$-intercept $(a, 0)$ which is also a local minimum/maximum.

- **All cubed linear factors correspond to $x$-intercepts and SPI of the graph.**




- E.g., $f(x) = (x-a)^3(x-b)$ has an $x$-intercept $(a, 0)$ which is also a stationary point of inflection.

Definition

## Steps to Graphing Factorised Polynomials

- **Steps:**

1. Plot $x$-intercepts.

2. Determine whether the polynomial is positive or negative.

3. Use the repeated factors to deduce the shape.

   - Non - Repeated: Only $x$-intercept.

   - Even Repeated: $x$-intercept and a turning point.

   - Odd Repeated: $x$-intercept and a stationary point of inflection.

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## Section B: Warmup

**Question 1**

**a.** Use polynomial long division to write $f(x) = \frac{x^3+2x^2+3}{x-2}$ in the form $f(x) = Q(x) + \frac{a}{x-2}$, for quadratic function $Q$ and integer $a$.

**b.** Find the remainder of the division $\frac{f(x)}{g(x)}$ where $f(x) = x^3 + 3x^2 + 2$ and $g(x) = x + 1$.

**c.** Find all the roots of $f(x) = x^3 + 2x^2 - x - 2$.

**d.** Factorise the expression $8x^3 - 27$.

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## Section C: Exam 1 (18 Marks)

**Question 2** (5 marks)

We know that $f(x) = \frac{12}{2x+3}$ for all positive values of $x$.

**a.** Simplify $\frac{x^2+4x-5}{12} \times f(x)$. Give your answer in the form of $\frac{ax+5}{b} - \frac{c}{2x+3}$ where $a$ and $b$ are positive integers and $c$ is a rational number. (3 marks)

Consider $g(x) = \frac{5-81x^2}{4}$ for all values of $x$.

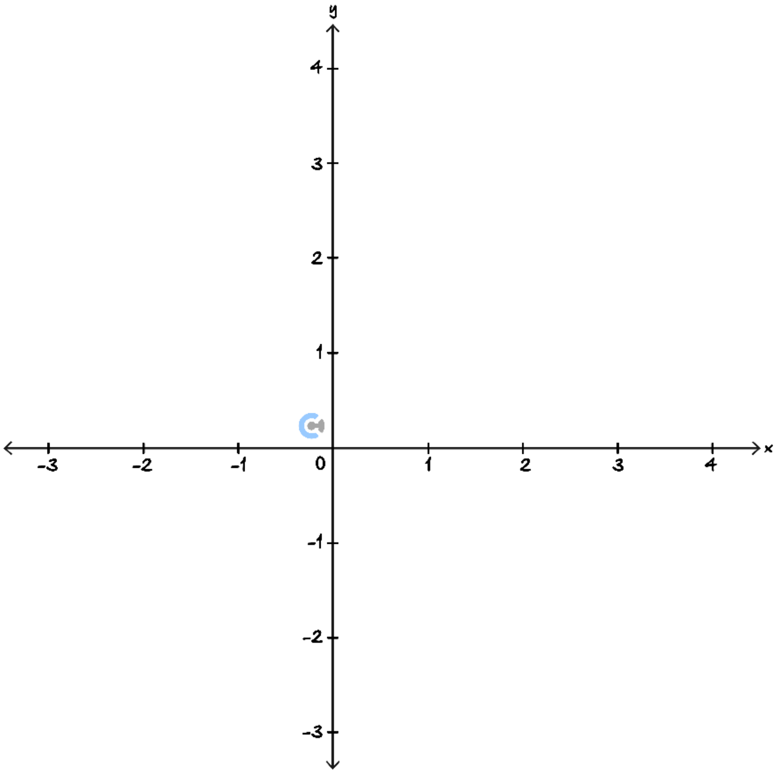
**b.** Solve $g(x) = -13$. (2 marks)

**Question 3** (3 marks)

Consider the function:

$$f(x) = (2 - x)^2(x + 1)\left(\frac{1}{2} - x\right)$$

It is known that $f$ has turning points at approximately $(-0.5,3.1)$ and $(1.1,-1)$
Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts with coordinates.

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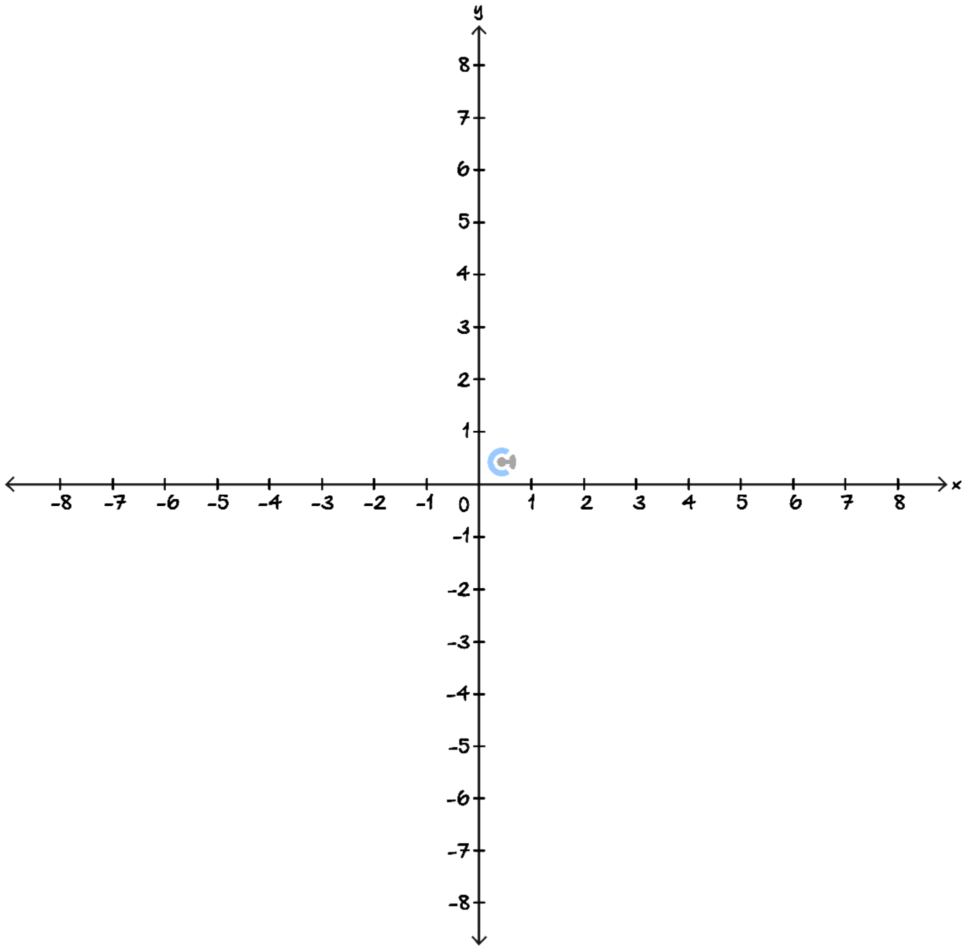
**Question 4** (4 marks)

Consider the functions $f(x) = (x-2)^3$ and $g(x) = -x$.

**a.** Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the axes below. Label all axes intercepts. (3 marks)



**b.** Hence, solve the equation $(x-2)^3 = -x$ for $x$. (1 mark)

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**Question 5** (6 marks)

**a.** Consider the function $f(x) = 3x^3 + ax^2 + bx - 12$. If $x - 2$ is a factor of $f(x)$ and the remainder of $f(x) \div x - 1 = -18$, find the value(s) of $a$ and $b$. (3 marks)

**b.** Hence, simplify the following using polynomial long division: $\frac{3x^3+3x^2-12x-12}{x^2+4x+1}$. (3 marks)

## Section D: Tech Active Exam Skills

**Calculator Commands: Turning Point**

- ALWAYS sketch the graph to find approximate bounds for where the turning point you want is located.
- To find a local maximum we maximise the function over a specific domain.
- To find a local minimum we minimise the function over a specific domain.
- **TI and Casio:** Use fmin(expression, variable, lower (optional), upper (optional)) or fmax(expression, variable, lower (optional), upper (optional)).
- **TI:** Menu→ $4 \rightarrow \frac{7}{8}$.

| | |
|---|---|
| $\text{Define } f(x)=x^3-4\cdot x$ | *Done* |
| $\text{fMin}(f(x),x,0,2)$ | $x=\frac{2\cdot\sqrt{3}}{3}$ |
| $f\left(\frac{2\cdot\sqrt{3}}{3}\right)$ | $\frac{-16\cdot\sqrt{3}}{9}$ |

- **Casio:** Action→Calculation→ fmin/fmax

$$\text{fmin}(x\text{^}3-4x, x, 0, 2)$$

$$\left\{\text{MinValue}=\frac{-16\cdot\sqrt{3}}{9}, x=\frac{2\cdot\sqrt{3}}{3}\right\}$$

.

**NOTE:** TI only gives the $x$-value for the min/max so we then need to sub it back into our function. Casio gives us both!

<u>Calculator Commands</u>

- **Mathematica:** Minimize[ ] and Maximize[ ] commands.
- Minimize[$f[x], x$] will minimize $f[x]$ over its whole domain.
- To restrict the domain, we must use Minimize[$\{f[x], a \leq x \leq b\}, x$].

```
In[34]:= Minimize[{x^3 - 4 x, 0 < x < 2}, x]
```

Out[34]= $\left\{-\frac{16}{3\sqrt{3}}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$

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## Section E: Exam 2 (26 Marks)

**Question 6** (1 mark)

The stationary point of inflection of the cubic, $y = (x-2)^3 + 4$ occurs at:

**A.** $(-2, 4)$

**B.** $(2, 4)$

**C.** $(-2, -4)$

**D.** $(4, 2)$

**Question 7** (1 mark)

The data $(3, 3)$, $(4, 0)$, $(5, 3)$, $(6, 48)$ and $(7, 243)$ can be modelled by the equation $y = a(x-b)^4$. The values of $a$ and $b$ respectively are:

**A.** $3$ and $4$.

**B.** $-3$ and $4$.

**C.** $4$ and $3$.

**D.** $-3$ and $-4$.

**Question 8** (1 mark)

The values of $x$ that satisfy $x^3 - 6x^2 - 27x + 140 = 0$ are:

**A.** $x = -5,4,5$

**B.** $x = -5,-4,6$

**C.** $x = -5,4,7$

**D.** $x = 2,3,7$

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**Question 9** (1 mark)

Using the Rational Root Theorem, count the number of possible roots that $y = 3x^3 + 7x^2 - 5x + 11$ has.

**A.** 4

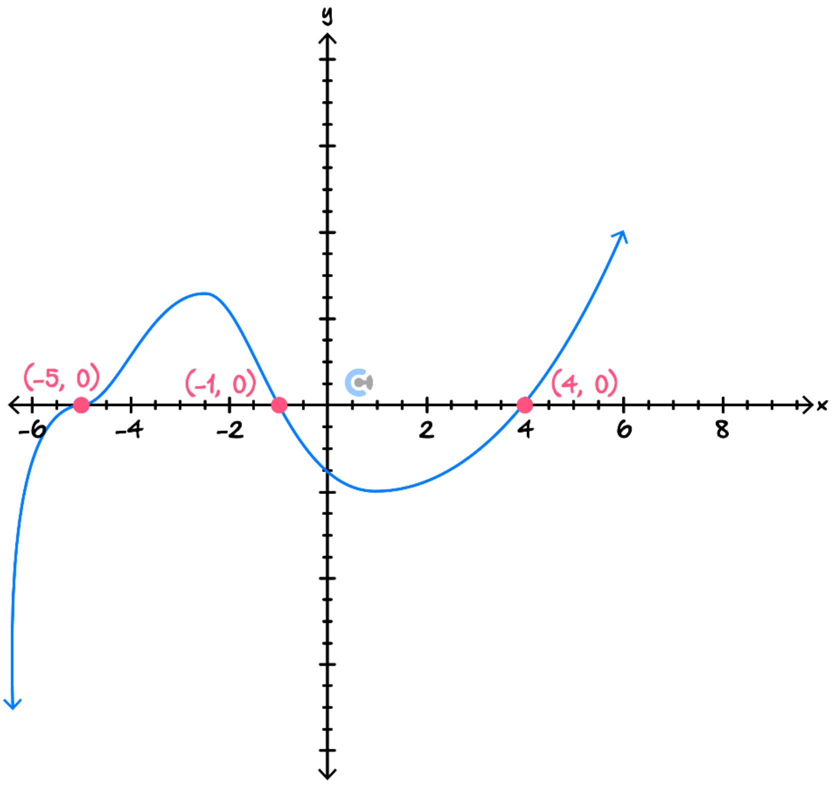
**B.** 6

**C.** 7

**D.** 8

**Question 10** (1 mark)

The equation that best represents the graph below is:




**A.** $y = (x - 4)^3(x + 5)(x - 1)$

**B.** $y = (x + 5)^3(x + 1)(x - 4)$

**C.** $y = (x + 1)^3(x + 5)^2(x - 4)$

**D.** $y = (x + 4)(x + 5)(x - 1)$

**Question 11** (1 mark)

Find the remainder of the division, $f(x) \div g(x)$, where $f(x) = 7x^3 - 4x^2 + 5x - 11$ and $g(x) = 3x - 2$.

**A.** $0$

**B.** $13$

**C.** $\frac{233}{17}$

**D.** $-199$

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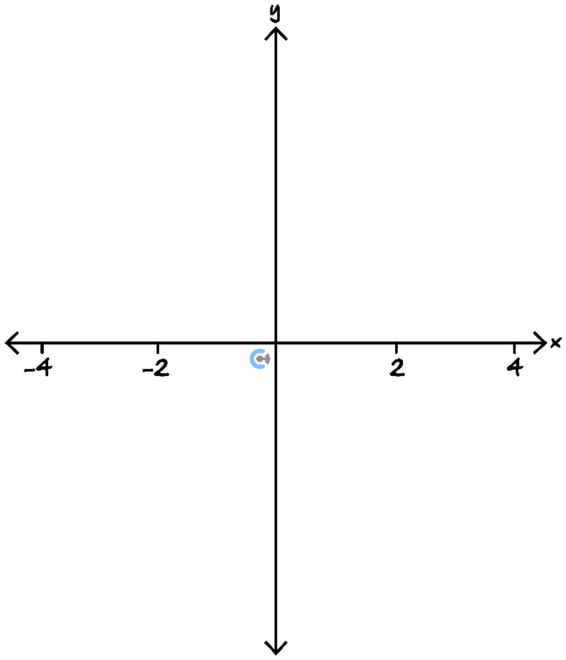
**Question 12** (10 marks)

Consider the quintic polynomial $f(x) = (x+3)(x-1)^2(x-2)^2$. The function has turning points at $(-2.11, 145.41), (a, 0), (1.51, 0.28), (b, 0)$, where $a > b$.

**a.** State the values of $a$ and $b$. (1 mark)

**b.** Sketch the graph on the axes below labelling all $x$-intercepts and turning points. Ignore the $y$-axis scale. (3 marks)




**c.** Find all solutions to $f(x) = 0.28$ for $x$ correct to 2 decimal places. (2 marks)

**d.** Find the values of $k$ (to 2 decimal places) when:

**i.** $f(x) = k$ has 1 solution. (1 mark)

**ii.** $f(x) = k$ has 5 solutions. (2 marks)

**iii.** $f(x) = k$ has 2 solutions. (1 mark)

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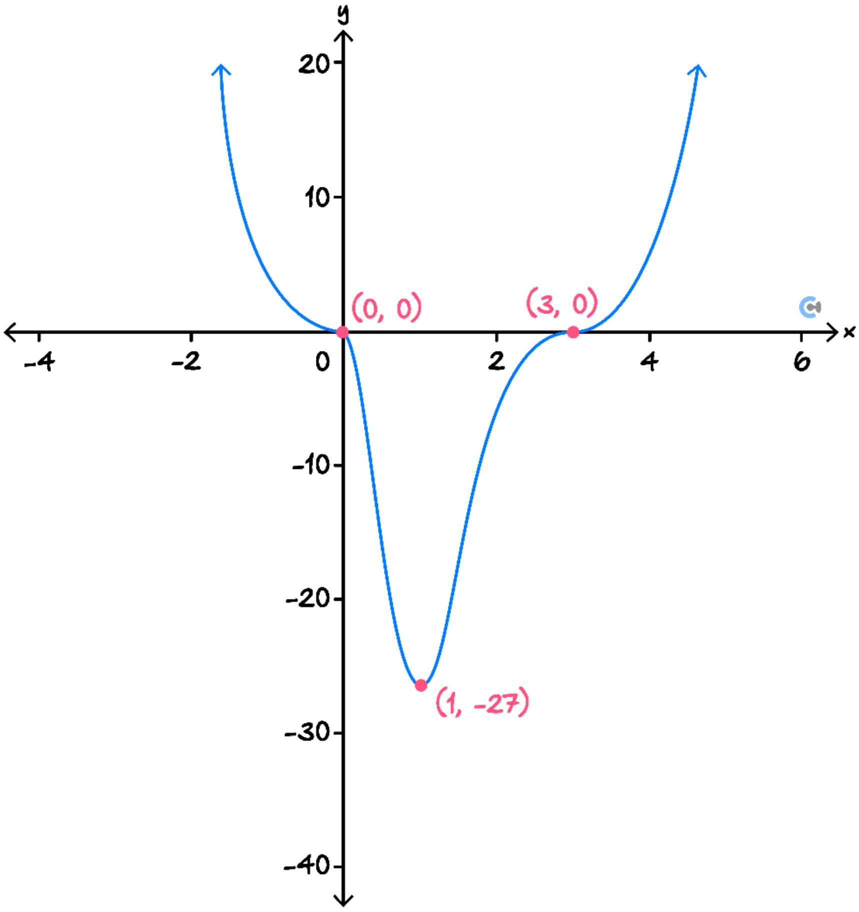
**Question 13** (9 marks)

Consider the function of the form $f(x) = a(x-b)^3(x-c)^3$ where $b > c$, shown on the axes below.

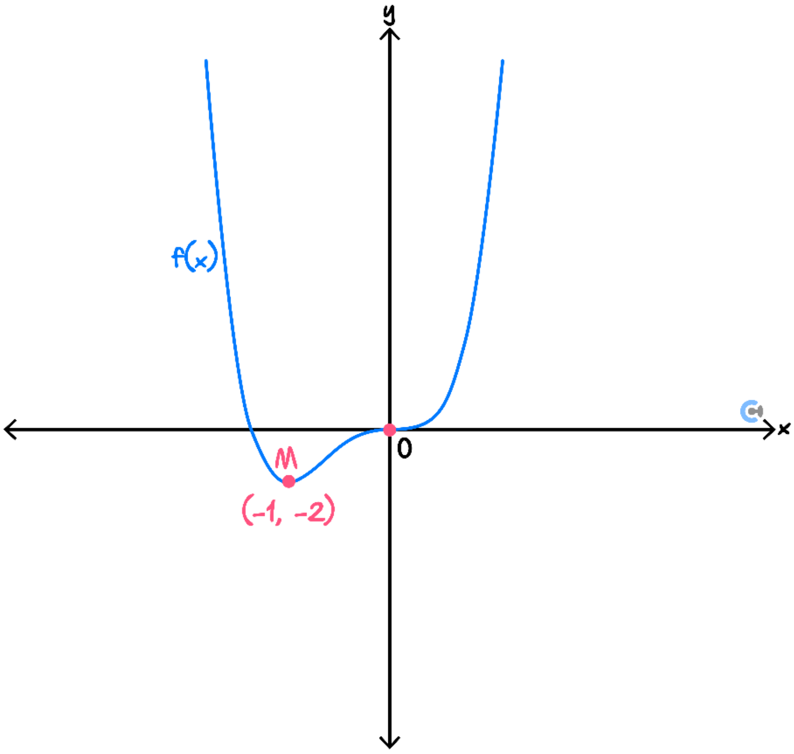



**a.** Find the values of $a$, $b$ and $c$. (3 marks)

**b.** Hence, simplify $\frac{f(x)}{(x-3)^3}$, leaving your answer in the form of $\frac{px^q}{m}$, where $p$, $q$ and $m$ are positive integers. Show all working. (2 marks)

**c.** It is known that the division, $f(x) \div (x-k)$, leaves a remainder of $0$. Find the possible values of $k$. (2 marks)

Consider the quartic $f: R \to R, f(x) = 5x^4 + 6x^3 - x^2$. Part of the graph of $f(x)$ is shown below.




**d.** State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no $x$-intercepts. (1 mark)

Let $p: R \to R, p(x) = 5x^4 + 3(a+2)x^3 - x^2 - 2ax + 3a^2, a \in R$.

**e.** State the value of $a$ for which $p(x) = f(x)$ has infinite solutions. (1 mark)

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## Section F: Extension Exam 1 (14 Marks)

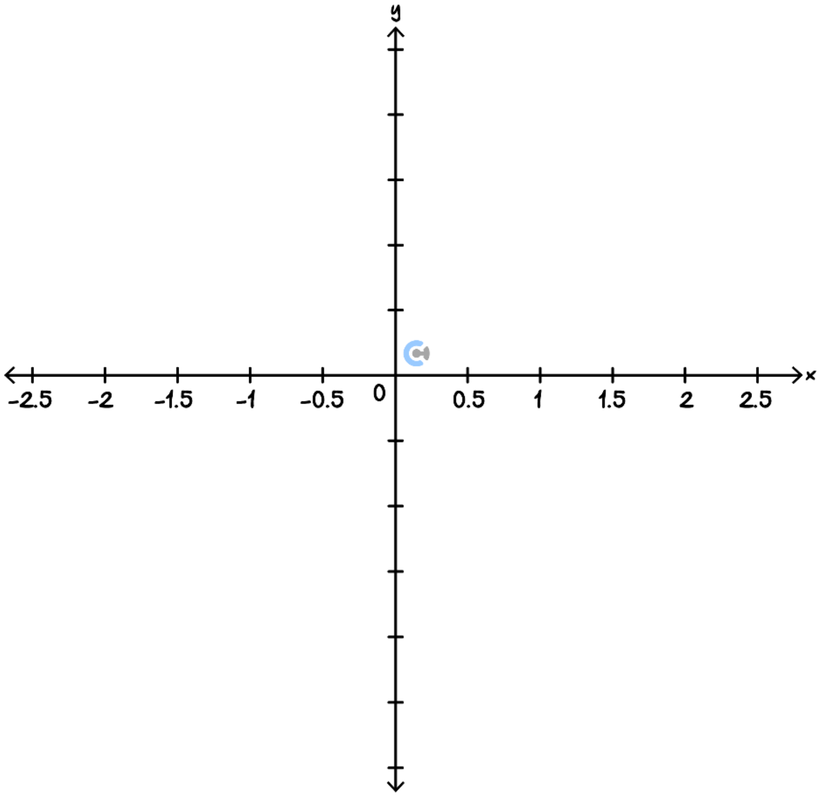
**Question 14** (8 marks)

Consider the function $f(x) = x^5 + x^4 - 5x^3 - 5x^2 + 4x + 4$.

**a.** Show that $(x^2 - 4)$ is a factor of $f$. (2 marks)

**b.** Find all roots of $f$. (3 marks)

**c.** It is known that $f$ has two turning points when $x > 0$.
Sketch the graph of $y = f(x)$ on the axes below. Label all axes intercepts and ignore the $y$-scale. (3 marks)

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**Question 15** (6 marks)

A cubic function $f(x) = ax^3 + bx^2 + cx + d$ passes through the points $(-2, -22)$, $(0,4)$ $(1,2)$ and $(2,6)$.

**a.** Show that $a = 2, b = -3, c = -1$ and $d = 4$. (3 marks)

**b.** Write the function $g(x) = \frac{(x+1)f(x)}{x-1}$ in the form $g(x) = C(x) + \frac{A}{x-1}$, for a cubic function and an integer $A$. (3 marks)

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## Section G: Extension Exam 2 (15 Marks)

**Question 16** (1 mark)

How many **rational** roots does the polynomial $4x^3 - x^2 - 12x + 3$ have?

**A.** 0

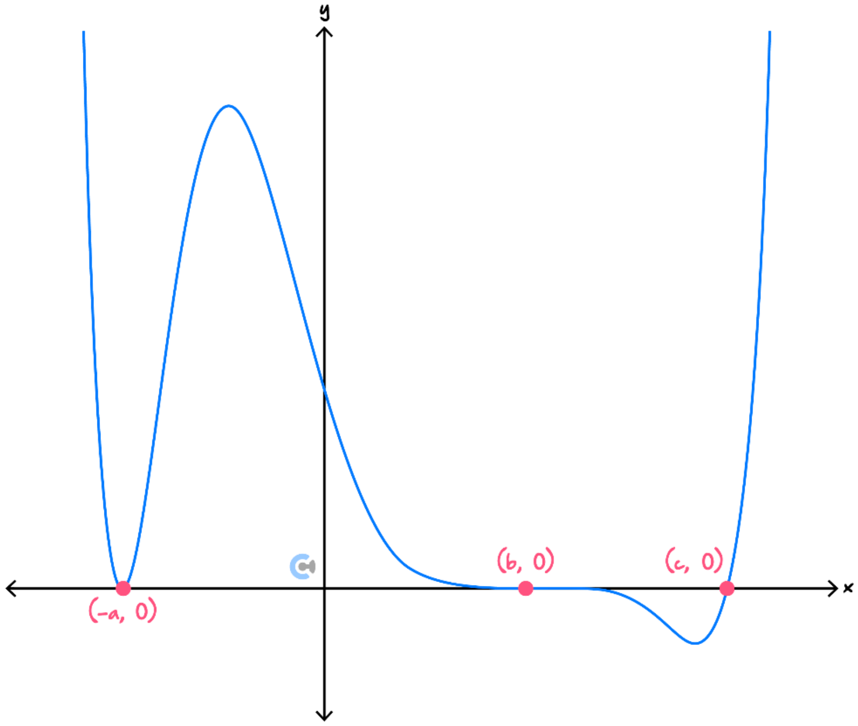
**B.** 1

**C.** 2

**D.** 3

**Question 17** (1 mark)

If $a, b, c > 0$, which of the following could describe the graph below?




**A.** $y = (x - a)^2(x - b)^3(x - c)$

**B.** $y = (x + a)^2(x - b)^5(x - c)$

**C.** $y = (x - a)^2(x - b)^3(x - c)^2$

**D.** $y = (x + a)^2(x - b)(x - c)^3$

**Question 18** (1 mark)

A set of three numbers that could be solutions to the equation $x^3 + ax^2 - 17x + 60 = 0$ is:

**A.** $\{1,4,7\}$

**B.** $\{-4,3,5\}$

**C.** $\{-5,2,3\}$

**D.** $\{-4,-5,3\}$

**Question 19** (1 mark)

If $x + a$ is a factor of $x^3 + (1 - a)x^2 - 8x + 21$, where $a > 0$, then the value of $a$ is:

**A.** 1

**B.** 2

**C.** 3

**D.** 4

**Question 20** (1 mark)

The graph of $y = x^4 - 4kx^2 + 4$, where $k$ is a real number, has four $x$-intercepts when:

**A.** $k < 1$

**B.** $k > 1$

**C.** $-1 \leq k \leq 1$

**D.** $k \leq 1$

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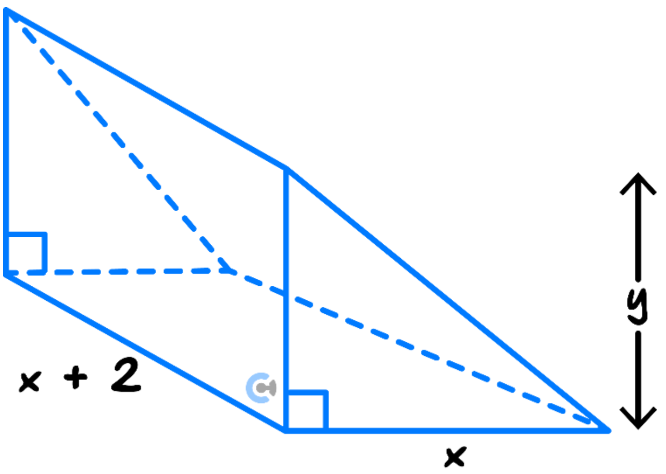
**Question 21** (5 marks)

The triangular box below consists of sides $x\ cm$ and $x + 2\ cm$, in length and width, and a height of $y\ cm$.

It is known that the width, length and height of the box added together equals $30\ cm$.



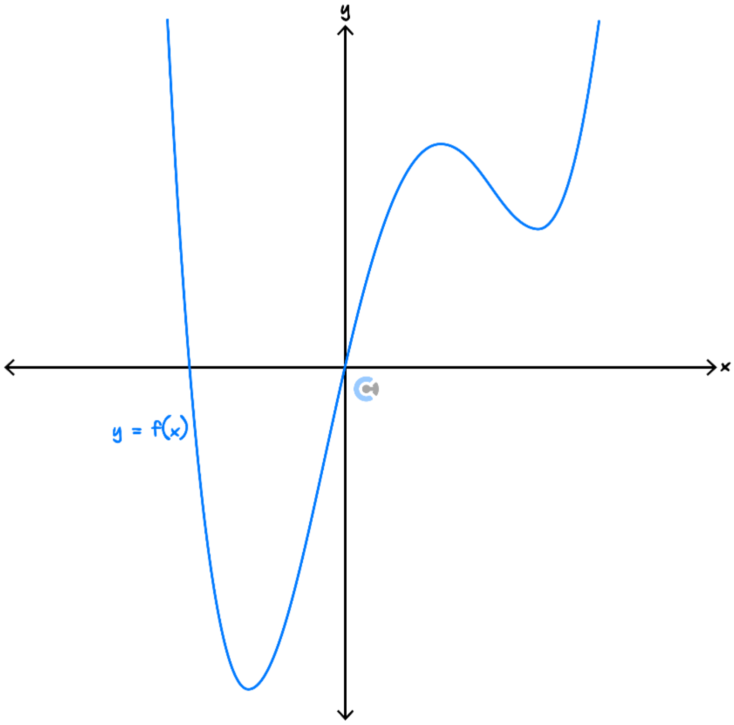

**a.** Find an expression for $V$, the volume of the box, in terms of $x$. (2 marks)

**b.** State the possible values that $x$ can take. (2 marks)

**c.** Find the maximum value of the box correct to the nearest cubic centimetre. (1 mark)

**Question 22** (5 marks)

Consider the function $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x$. Part of the graph of $y = f(x)$ shown below.




**a.** Find the distance between the $x$-intercepts. Give your answer correct to two decimal places. (1 mark)

**b.** It is known that $f$ has turning points at $x$-values that are roots to the function $g(x) = 12x^3 - 24x^2 - 12x + 24$.

Find the coordinates of all turning points of $f$. (2 marks)

**c.** Hence, find the values of $k$ such that $f(x) = k$ has two real solutions. (2 marks)

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