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VCE Mathematical Methods ½
Graphs of Circular Function [0.19]
Workshop Solutions

Error Logbook:

New Ideas/Concepts	Didn't Read Question
Pg / Q #: ______________ Notes:	Pg / Q #: ______________ Notes:
Algebraic/Arithmetic/ Calculator Input Mistake	**Working Out Not Detailed Enough**
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Section A: Recap

The Exact Values Table

x	$0\ (0^o)$	$\frac{\pi}{6}\ (30^o)$	$\frac{\pi}{4}\ (45^o)$	$\frac{\pi}{3}\ (60^o)$	$\frac{\pi}{2}\ (90^o)$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

Particular Solutions

- Solving trigonometric equations **for finite solutions**.
- **Steps:**
  1. Make the trigonometric function the subject.
  2. Find the necessary angle for one period.
  3. Solve for x by equating the necessary angles to the inside of the trigonometric functions.
  4. Add and subtract the period to find all other solutions in the domain.

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Definition

General Solutions

- Solving infinite trigonometric equations.
- Steps:

1. Make the trigonometric function the subject.
2. Find the necessary angle for one period.
3. Solve for x by equating the necessary angles to the inside of the trigonometric functions.
4. Add $\text{period} \times n$ where $n \in Z$.

Definition

Period of a Trigonometric Function

$$\textit{Period of } \sin(nx) \textit{ and } \cos(nx) \textit{ functions} = \frac{2\pi}{n}$$

$$\textit{Period of } \tan(nx) \textit{ functions} = \frac{\pi}{n}$$

$$\textit{Where } n = \textit{coefficient of } x \textit{ and } n > 0$$

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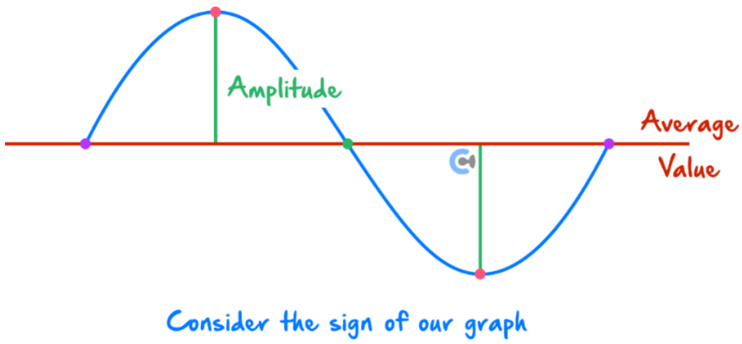
Definition

Amplitude, Period and Average Value

For $y = A\sin/\cos\,(nx + b) + k$



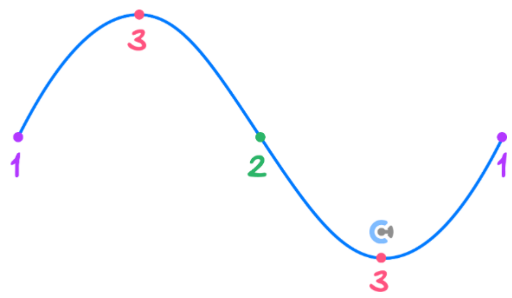
$$\text{Amplitude} = |A|$$

$$\text{Period} = \frac{2\pi}{|n|}$$

$$\text{Average Value} = k$$

Definition

Graphing of $\sin$ and $\cos$ Functions



1. Identify Amplitude, Period, Mean Value and Positive/Negative Shape.
2. Create a “mini-version” of the graph you are about to draw.
3. Start plotting the function from when the angle $= 0$.
4. Draw the start and end of the periods, and plot the halves (turning points).
5. Find any x-intercepts.
6. Join all the points!

Definition

Finding the Rule

$$\text{Amplitude } (A) = \frac{max - min}{2}$$

$$\text{Average } (k) = \frac{max + min}{2}$$

Definition

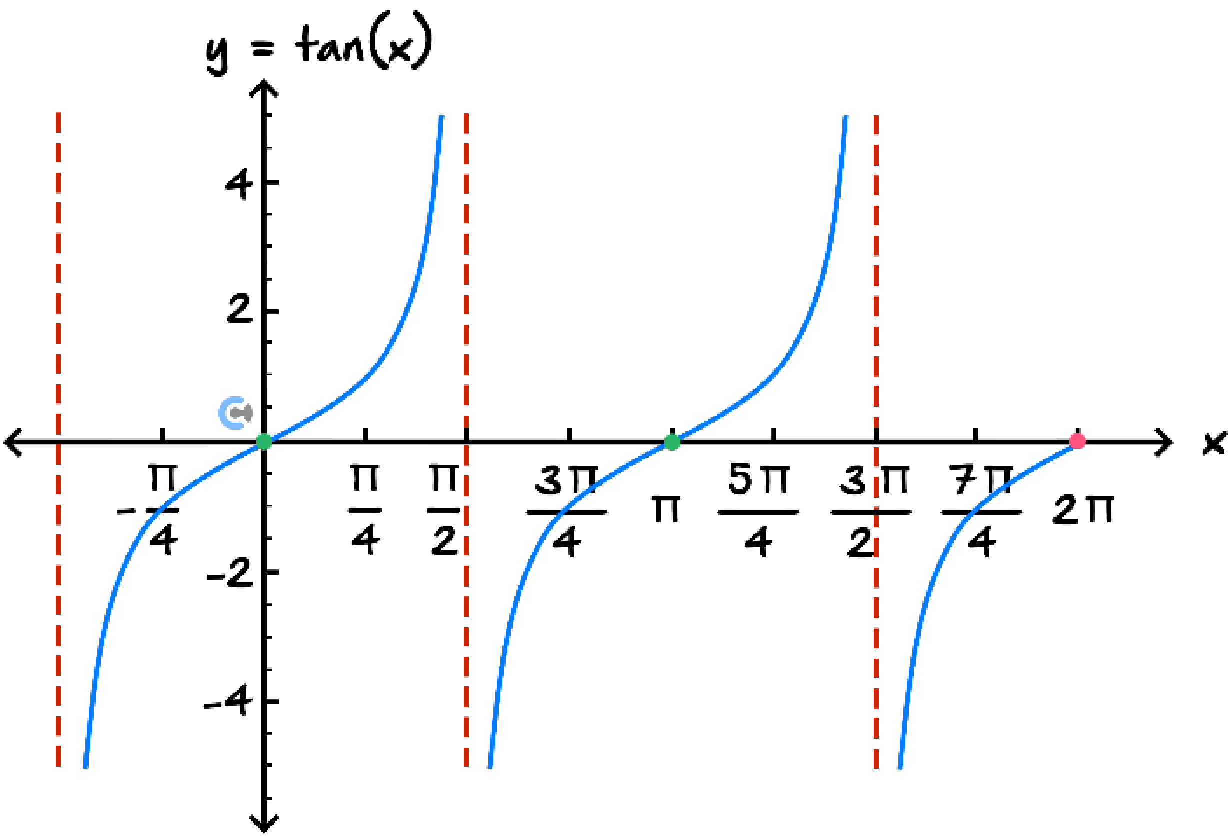
Fraction of Period

$$\textit{Fraction of Period} = \frac{\textit{Duration}}{\textit{Period}}$$

$$\% \textit{ of Period} = \frac{\textit{Duration}}{\textit{Period}} \times 100\%$$

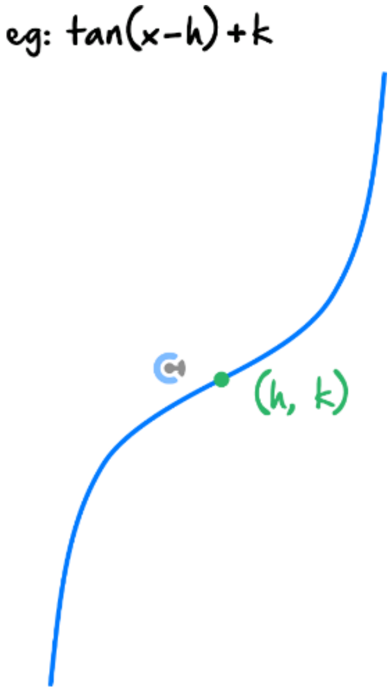
Definition

Graph of Tangent



Steps for Sketching tan Functions

1. Identify:
   - The period $= \frac{\pi}{n}$.
2. Find the vertical asymptotes by solving for angle $= \frac{\pi}{2}$.
3. Find other vertical asymptotes within the domain by adding the period to the answer from the previous step.
   - For instance, for $\tan\left(2x - \frac{\pi}{3}\right)$, solve $2x - \frac{\pi}{3} = \frac{\pi}{2}$ for x.
4. Plot the inflection point (h, k). (Midpoint of the two vertical asymptotes.)
   - x-value of inflection point $=$ x-value which makes angle $= 0$.
   - y-value of inflection point $=$ vertical translation of the function.



5. Find any x-intercepts.
6. Sketch a "cubic-like" shape.

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Section B: Warm Up (6 Marks)

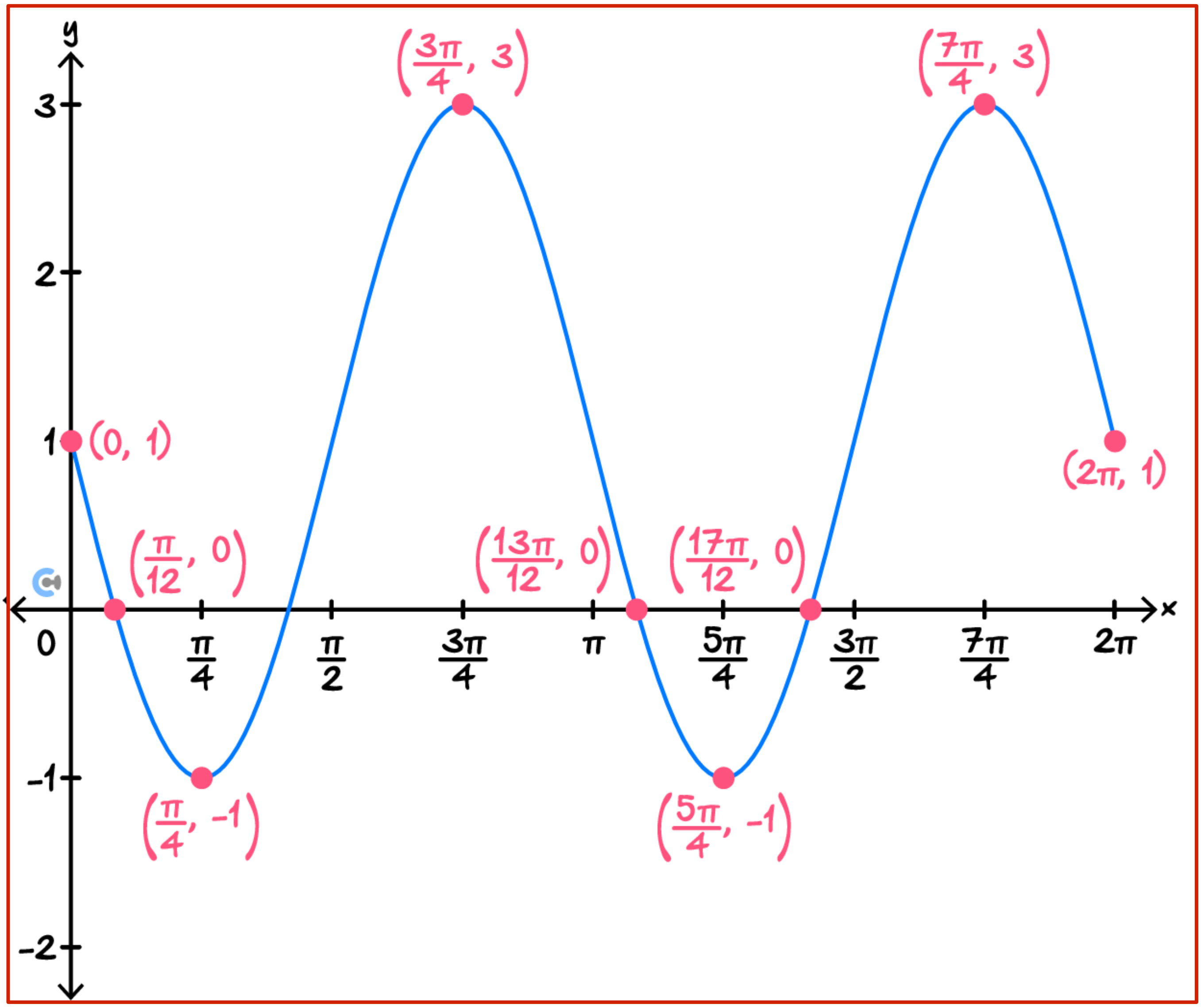
INSTRUCTION:

- Regular: 6 Marks. 9 Minutes Writing.
- Extension: Skip

Question 1 (3 marks)

Sketch the following function on the set of axes below, labelling all axes intercepts, endpoints, and turning points with their coordinates.

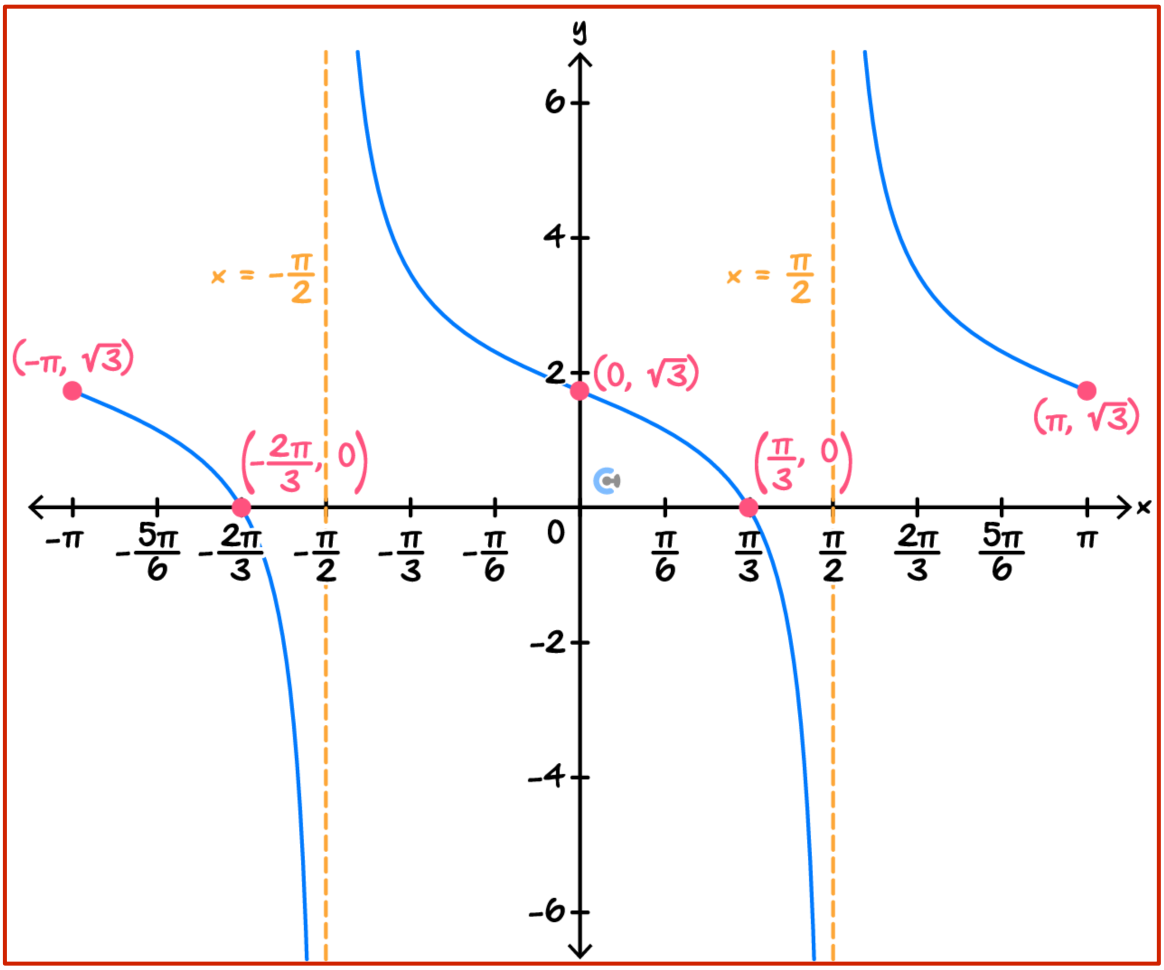
$$f(x) = 1 - 2\sin(2x), x \in [0, 2\pi]$$



Question 2 (3 marks)

Sketch the following function on the set of axes below, labelling all axes intercepts and endpoints with their coordinates, and the asymptotes with their equations.

$$h(x) = -\tan(x) + \sqrt{3}, x \in [-\pi, \pi]$$



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Section C: Exam 1 Questions (19 Marks)

INSTRUCTION:

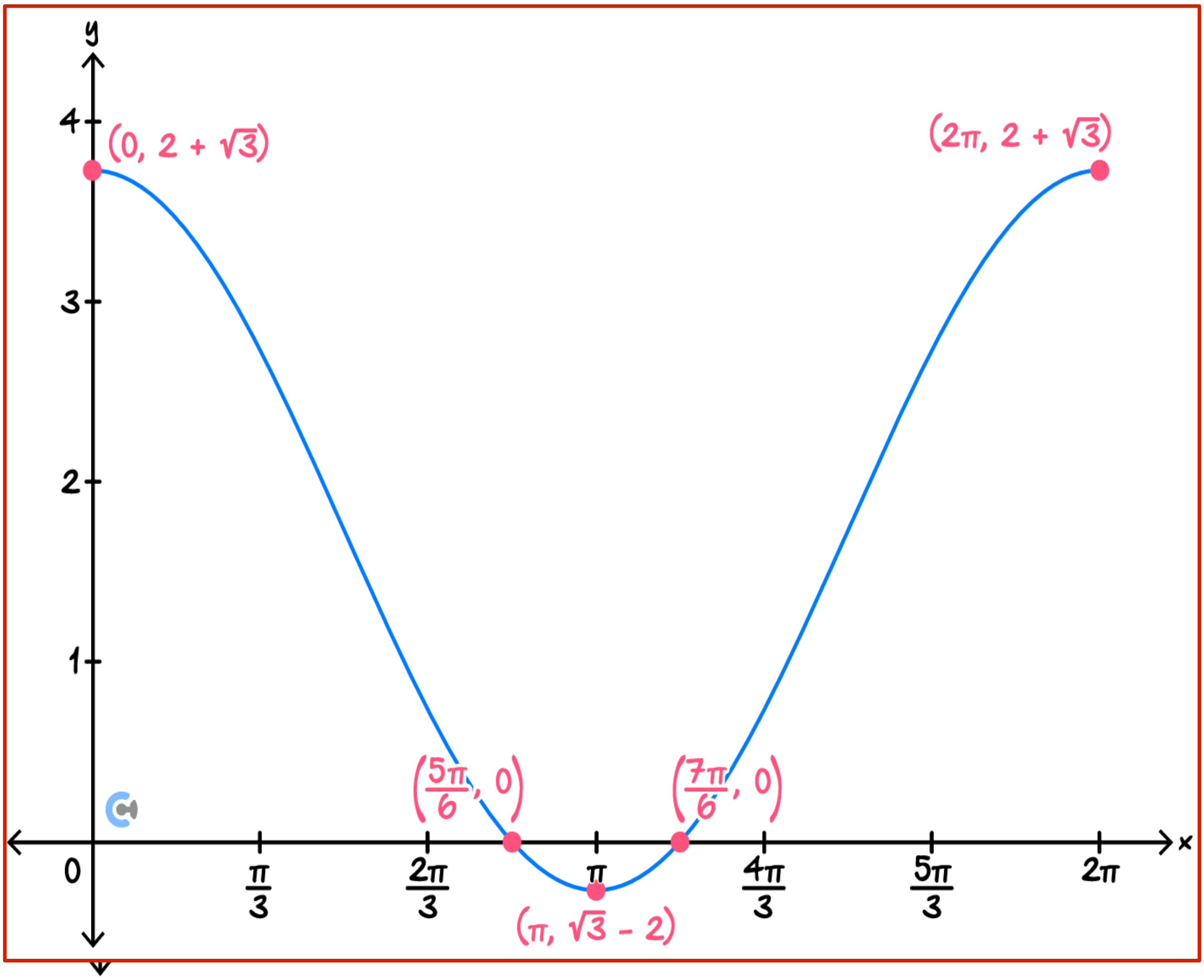
- Regular: 19 Marks. 5 Minutes Reading. 27 Minutes Writing.
- Extension: 19 Marks. 5 Minutes Reading. 19 Minutes Writing.

Question 3 (5 marks)

Consider the function:

$$g(x) = \sqrt{3} + 2\cos(-x), x \in [0, 2\pi]$$

a. Sketch the graph of $y = g(x)$ on the axes below, labelling all axes intercepts, endpoints and turning points with their coordinates. (3 marks)



b. What fraction of the graph of $g(x)$ lies **above** the x-axis? (2 marks)

Below for:

$$\frac{7\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{3} \text{ } (\mathbf{1M})$$

So, fraction below is $\frac{\pi/3}{2\pi} = \frac{1}{6}$.

Fraction above is:

$$\frac{5}{6} \, (\mathbf{1A})$$

Question 4 (3 marks)

A trigonometric function with rule $f(x) = a\sin(bx) + c$, satisfies the following properties:

- ran $= [-4, 6]$.
- $f(0) = 1$.
- The period is π.

Find a possible rule, for the function f, that satisfies **all** of these properties.

$$\frac{-4+6}{2} = 1 \text{ (Average value, } \mathbf{1M}\text{)}$$

$$\frac{6-(-4)}{2} = 5 \text{ (Amplitude, } \mathbf{1M}\text{)}$$

$f(0) = 1$ so, using a sine function is a smart choice
period $= \pi \Rightarrow$ coefficient in front of x is 2 .

$$f(x) = 5\sin(2x) + 1 \text{ } (\mathbf{1A})$$

Note: $f(x) = -5\sin(-2x) + 1$ *is also correct*

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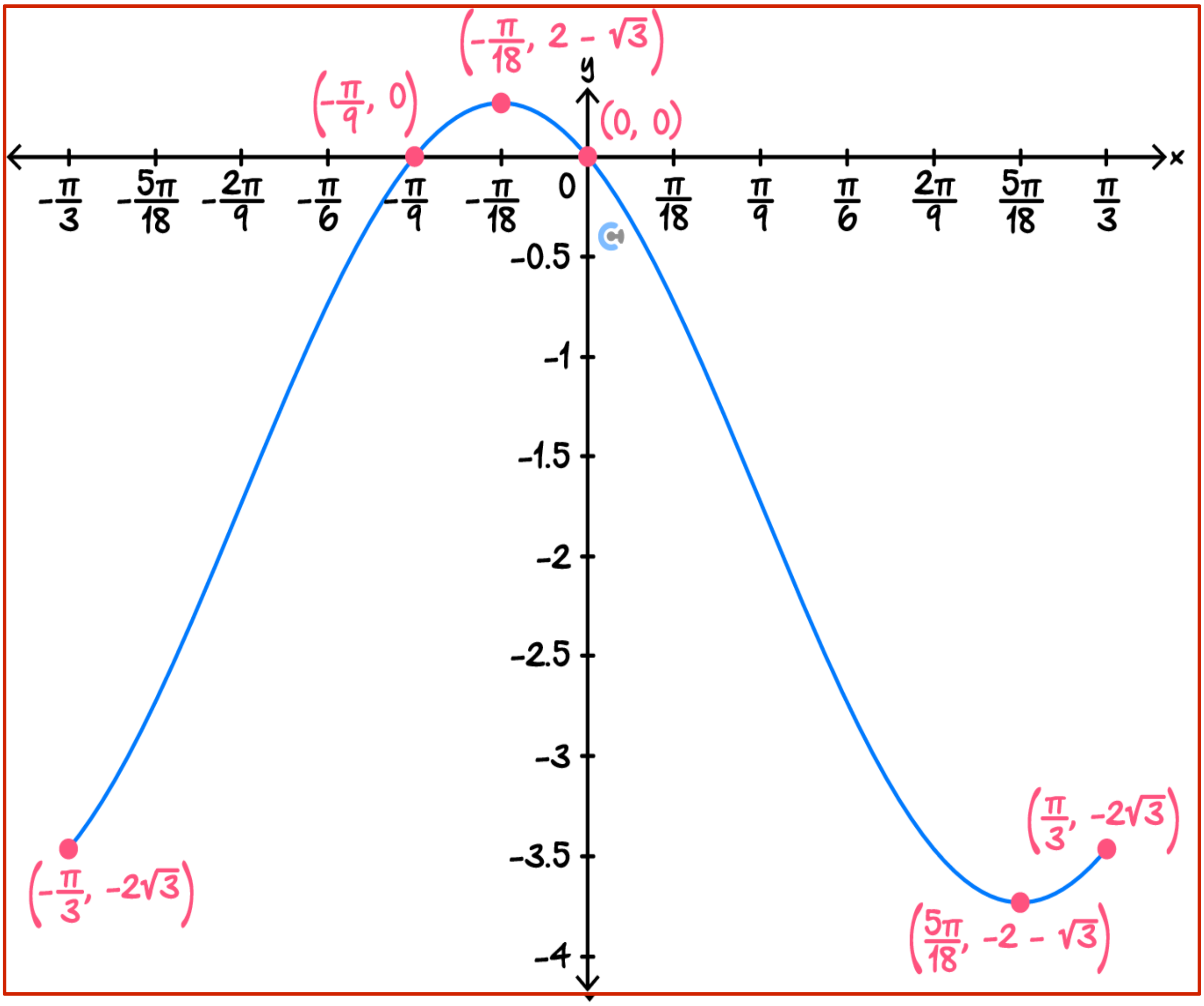
Question 5 (6 marks)

a. Sketch the following on the axes below, labelling all axes intercepts, endpoints, and turning points with their coordinates. (4 marks)

$$y = 2\sin\left(\frac{\pi}{3} - 3x\right) - \sqrt{3}, x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$$



b. What fraction of the graph lies **below** the x-axis? (2 marks)

Below for:

$\frac{\pi}{3} - \frac{\pi}{9} + \frac{\pi}{3} = \frac{5\pi}{9}$ **(1M)**

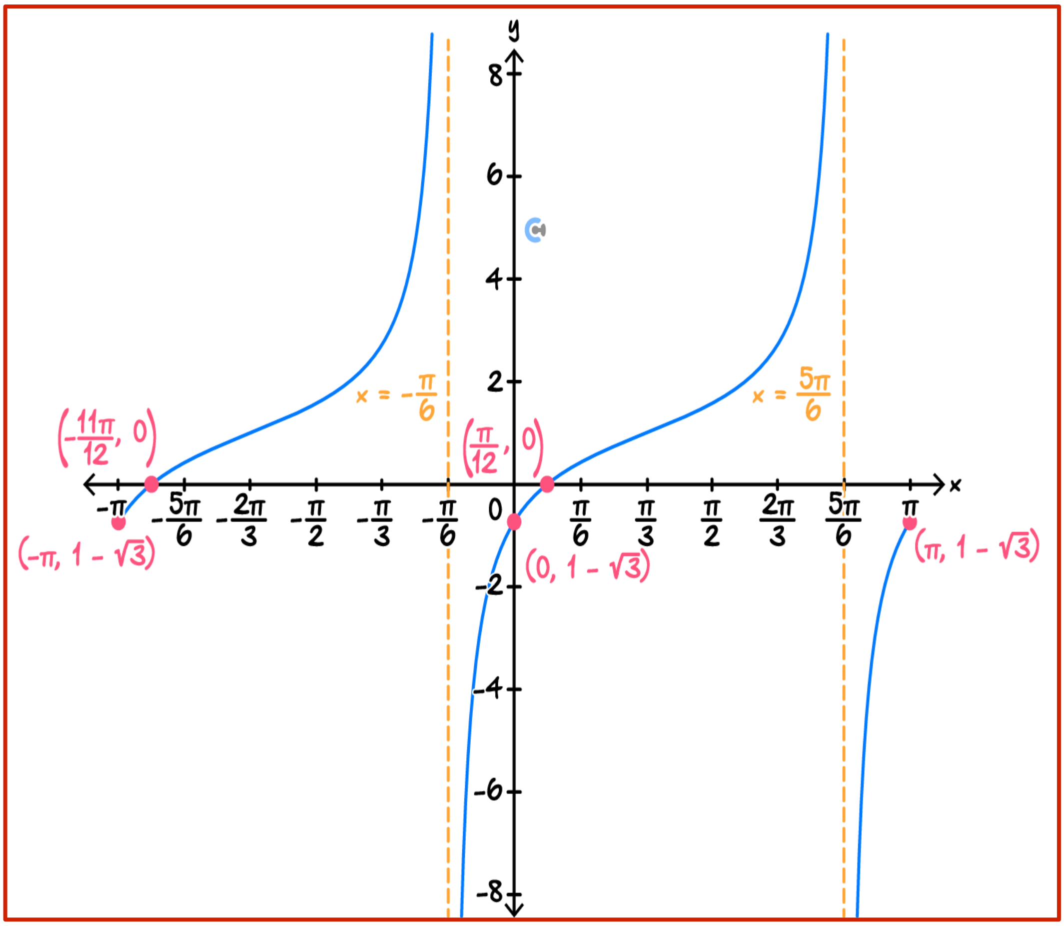
Therefore,

$\frac{5\pi/9}{\frac{2\pi}{3}} = \frac{5/9}{6/9} = \frac{5}{6}$ **(1A)**.

Question 6 (5 marks)

a. Sketch the following on the axes below, labelling all intercepts, and endpoints with their coordinates, and all asymptotes with their equations. (3 marks)

$$y = 1 - \tan\left(\frac{\pi}{3} - x\right), \quad x \in [-\pi, \pi]$$



b. What fraction of the graph lies **below** the x-axis? (2 marks)

Amount below is:

$$\frac{\pi}{12} + \frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{2} \text{ **(1M)**}$$

Therefore, fraction below is:

$$\frac{\pi/2}{2\pi} = \frac{1}{4} \text{ **(1A)**}$$

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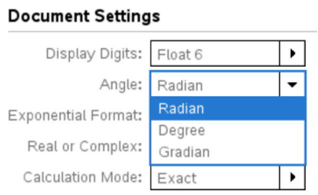
Section D: Tech Active Exam Skills

Calculator Commands: Degrees and Radians

- **TI**
  - Doc → 7 → 2



- **Casio**
  - Change at the bottom of the screen.

- **Mathematica**
  - In radians by default.
  - Write "Degree."

In[27]:= **Sin[30 Degree]**

Out[27]= $\frac{1}{2}$

Calculator Commands: Solving trigonometric functions.

- **TI**
  - solve$(\text{trig}(..) = a, x)$ | domain restriction.
  - | is under control equal.
- **Casio**
  - solve$(\text{trig}(..) = a, x)$ | domain restriction.
  - | is under maths 3.
- **Mathematica**
  - Solve[$\text{trig}[] == a$ && domain restriction, x].

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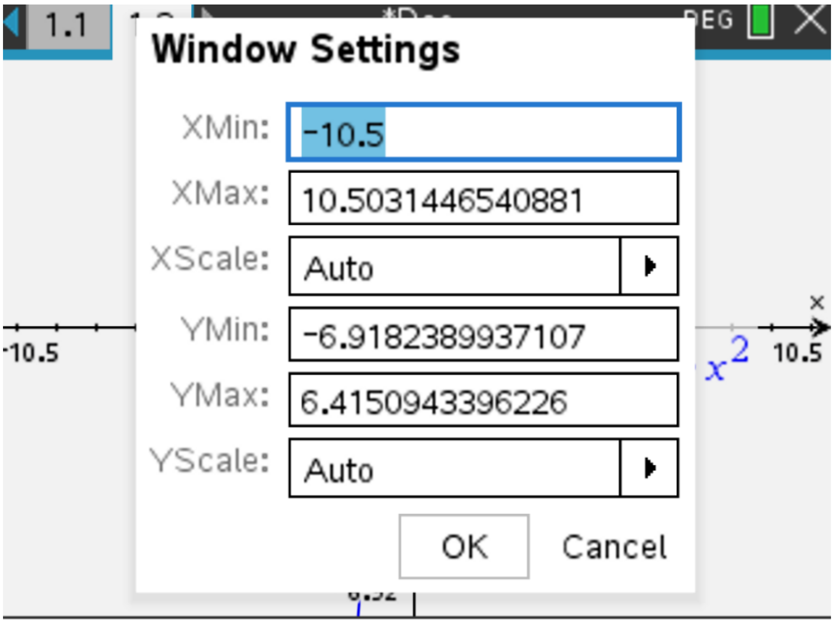
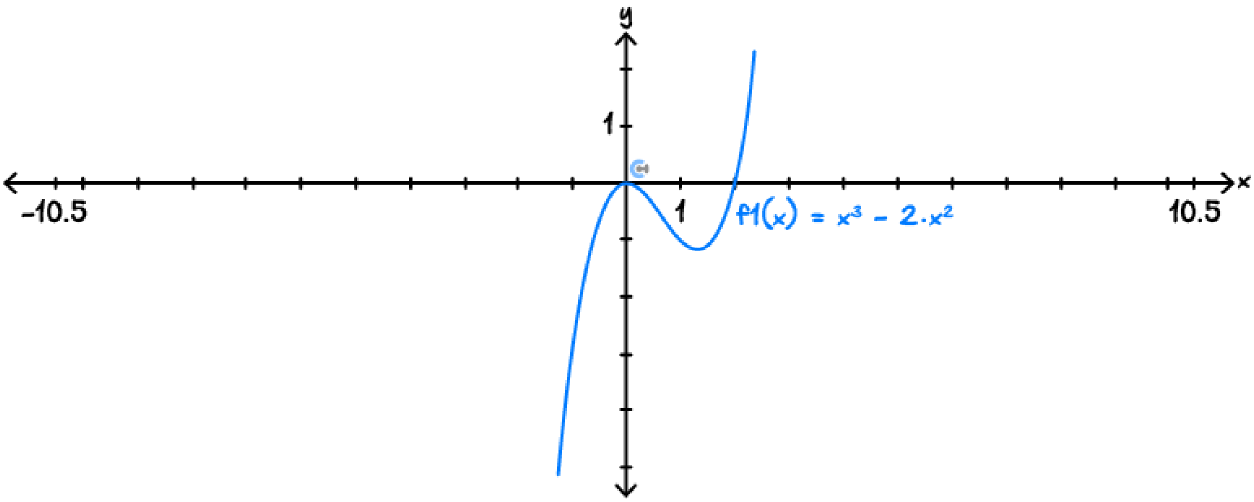
Calculator Commands: Graphing

- Open a graph page and plot your function.
- Zoom settings: Menu → 4 (window / zoom) → 1 enter your x and y-ranges.

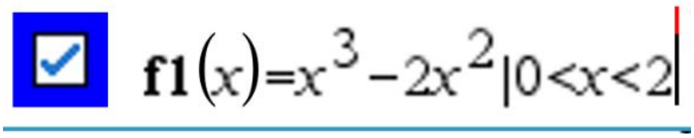


- Can also click the axis numbers on the graph and alter them directly.

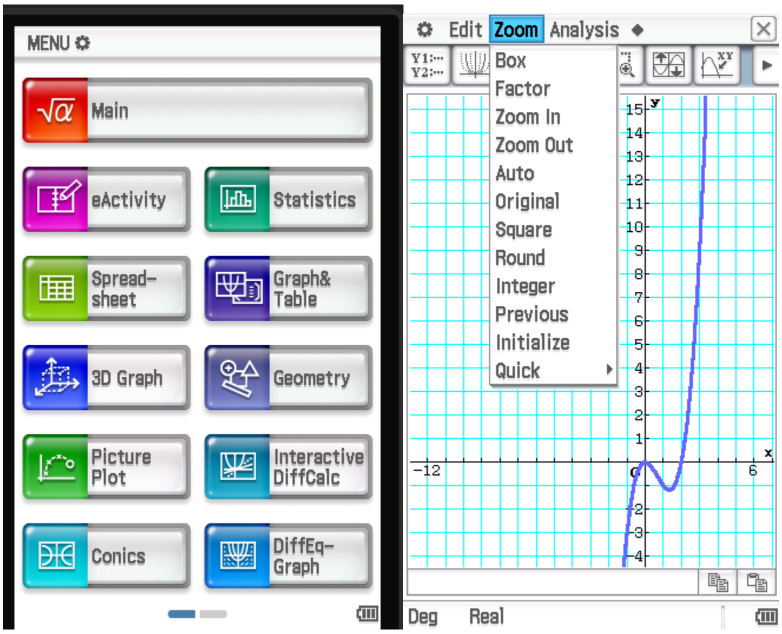


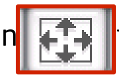
- Menu → 6 (Analyse) to find min / max x and y-intercepts.
- Restrict the domain to $0 < x < 2$, use the bar to get it

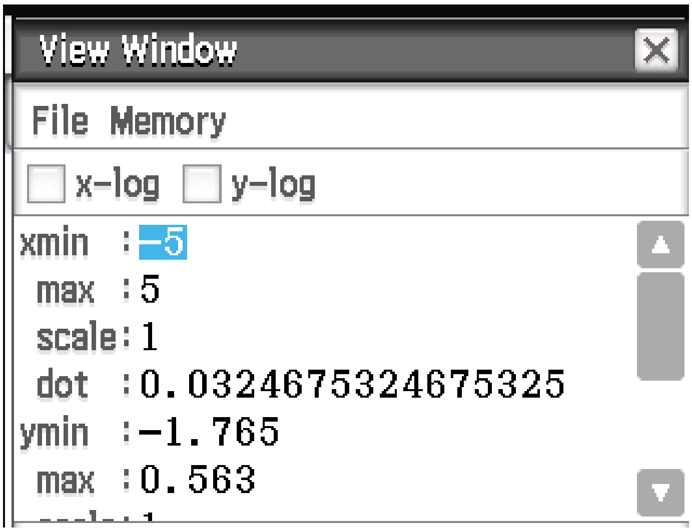
from ctrl+ =



- **Casio:** Click graph & table, and enter the function.



- Analysis → G-Solve to find intercepts.
- Use this button  to set the view window.



- Use | to restrict the domain → find it in Math 3.

☑ $y1=x^3-2\cdot x^2 \,|\, 0<x<2$

- **Mathematica:** Plot[function, $\{x, xmin, xmax\}$, PlotRange → $\{ymin, ymax\}$]
  - PlotRange is optional but can be used to make the scale appropriate for the question.

Section E: Exam 2 Questions (30 Marks)

INSTRUCTION:

- Regular: 30 Marks. 5 Minutes Reading. 45 Minutes Writing.
- Extension: 30 Marks. 5 Minutes Reading. 30 Minutes Writing.

Question 7 (1 mark)

What is the amplitude and period for the following function:

$$f(x) = 1 - 2\sin\left(4x - \frac{\pi}{2}\right)$$

A. $2, 4$

B. $2, 2\pi$

C. $2, \frac{\pi}{4}$

D. $2, \frac{\pi}{2}$

Question 8 (1 mark)

The general equation for all the asymptotes of $y = \tan(x)$, can be found by solving for x in the equation:

A. $\sin(x) = 0$

B. $\cos(x) = 1$

C. $\sin(x) = 1$

D. $\tan(x) = 1$

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Question 9 (1 mark)

Which of the following is false?

A. The amplitude of a trigonometric function does not depend on the dilation along the x-axis.

B. Some trigonometric functions can always satisfy the equation: $f(x) = k, k \in R$.

C. The tangent function is defined for the domain $x \in R$.

D. The sine function is defined for the domain $x \in R$.

Question 10 (1 mark)

Which of the following has a range of $[-3, 2]$?

A. $2.5\cos(x)$

B. $3\cos(x)$

C. $2.5\cos(x) - 0.5$

D. $2.5\sin(x) + 0.5$

Question 11 (1 mark)

What is the period and average value of the following trigonometric function?

$$y = \sin\left(\frac{\pi}{2}x + 2\right) - 1$$

A. $\pi, -1$

B. $4\pi, 1$

C. $4, -1$

D. $1, -1$

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Question 12 (1 mark)

What is the equation for the asymptotes of the following function?

$$f(x) = \tan\left(2x - \frac{\pi}{3}\right)$$

A. $x = \frac{n\pi}{2} + \frac{\pi}{12}$

B. $x = \frac{n\pi}{2} - \frac{\pi}{12}$

C. $x = \frac{(6n-1)\pi}{12}$

D. $x = \frac{(6n+1)\pi}{12}$

Question 13 (1 mark)

What fraction of the graph $y = \sin(x)$ lies above the x-axis?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. $\frac{1}{6}$

Question 14 (1 mark)

Which of the following is not the same as the rest, for $n \in Z$?

A. $(6n - 7)\pi$

B. $(6n + 1)\pi$

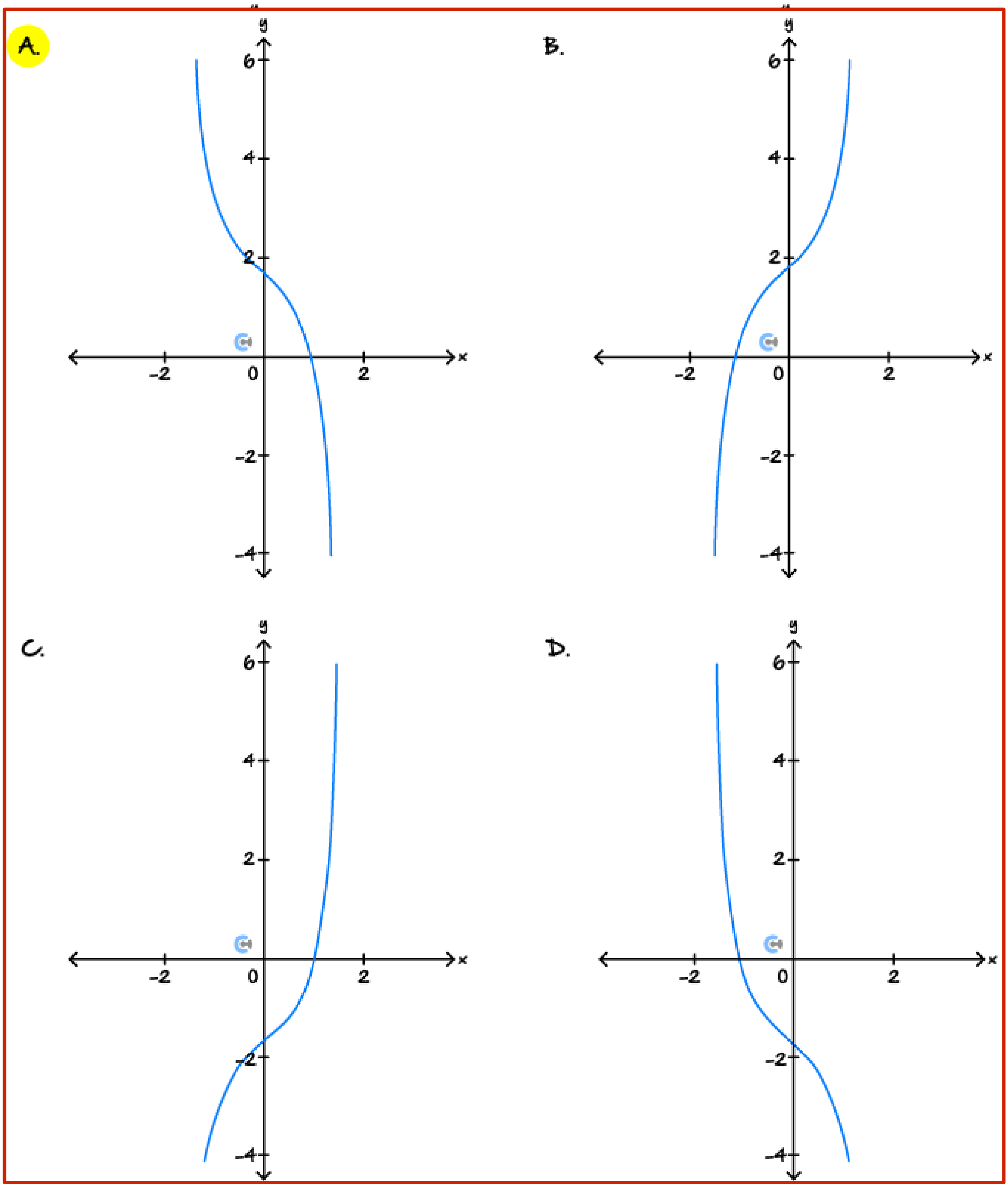
C. $(6n - 1)\pi$

D. $(6n - 5)\pi$

Question 15 (1 mark)

Which of the following best represents a part of the graph of:

$$y = \sqrt{3} - \tan(x)$$



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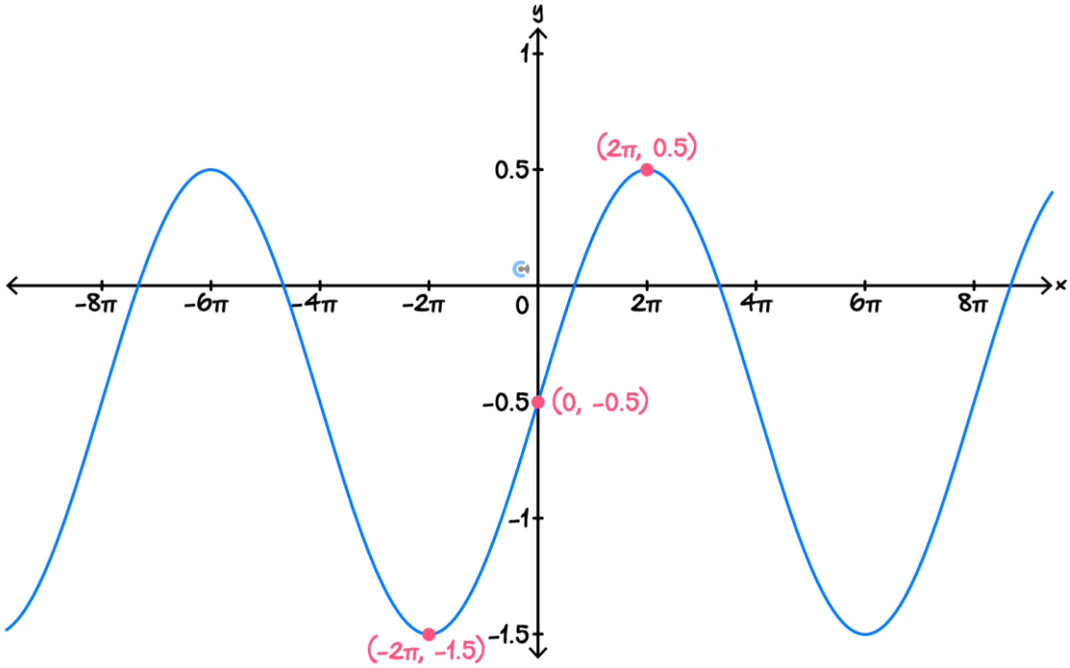
Question 16 (1 mark)



Which of the following functions could represent the graph above?

A. $y = 0.5\sin\left(\frac{x}{4}\right) - 1$

B. $y = -\sin\left(\frac{x}{8}\right) + 0.5$

C. $y = \sin\left(\frac{x}{8}\right) - 0.5$

D. $y = \sin\left(\frac{x}{4}\right) - 0.5$

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Question 17 (20 marks)

The height of a carriage above the ground on a Ferris wheel can be modelled by the following equation:

$$h(t) = 2\cos\left(\frac{\pi t}{4} + \frac{\pi}{3}\right) + 3$$

where $t \geq 0$ is the amount of time passed in minutes.

a. How long does it take for a carriage to go around the Ferris wheel once? (1 mark)

Period of cosine is given by:

$$\text{Period} = \frac{2\pi}{\frac{\pi}{4}} = 8 \text{ minutes} \quad [\mathbf{1A}]$$

b. What are the maximum height and minimum height of a carriage in the Ferris wheel? (2 marks)

Max height: $2 + 3 = 5$ [**1A**] Min height: $-2 + 3 = 1$ [**1A**]

c. Rewrite $h(t)$ in terms of the sine function. (2 marks)

Using the identity $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$ [**1M some appropriate working**]

$$h(t) = 2\sin\left(-\frac{\pi t}{4} + \frac{\pi}{6}\right) + 3$$ [**1A or any equivalent formulation**]

d. A carriage can only take passengers when the Ferris wheel is at its lowest point. find the time t, in minutes, when this occurs. (3 marks)

The minimum occurs when:

$$\cos\left(\frac{\pi t}{4} + \frac{\pi}{3}\right) = -1 \quad [1M]$$

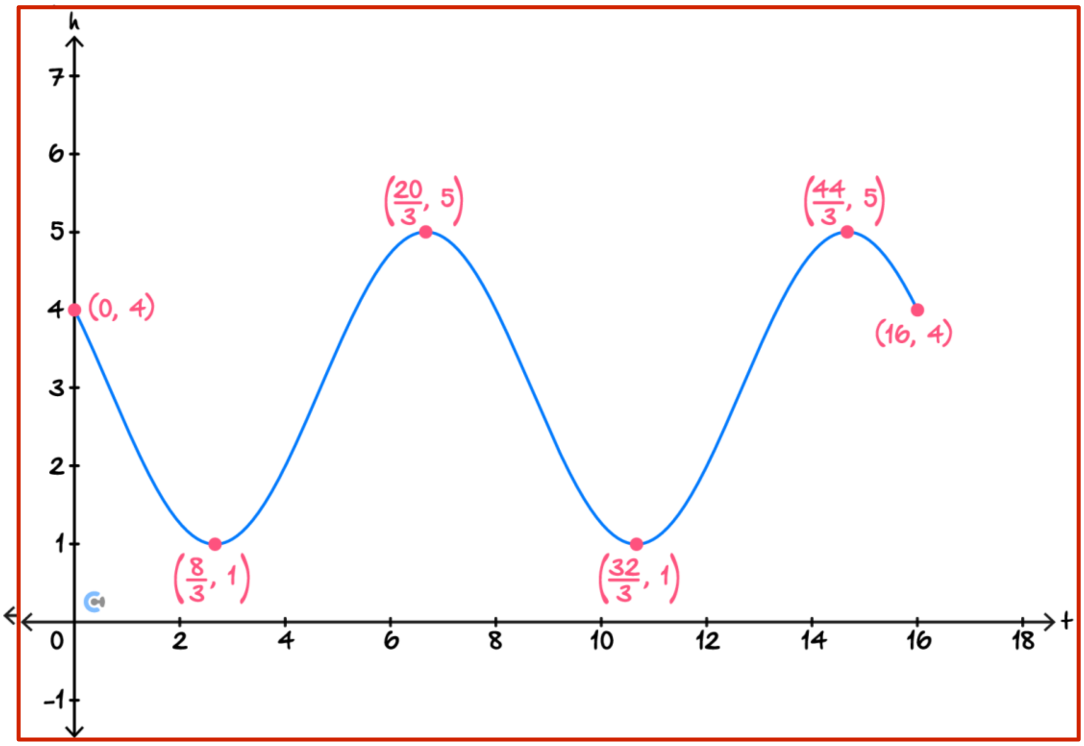
So:

$$\frac{\pi t}{4} + \frac{\pi}{3} = \pi + 2\pi n$$

$$t = \frac{8}{3} + 8n, \ n \in \mathbb{Z}^+ \cup \{0\}.$$

[1A for 8/3, 1A for general solution with correct domain restriction]

e. Sketch the graph of $h(t)$, for the first 2 periods on the set of axes below, labelling all turning points and axes-intercepts. (4 marks)



[1M shape, 1M over two periods, 1M turning points (exact values needed), 1M y-intercept]

f. A good view of the city can be seen when the carriage is more than 2 m above the ground.

i. For what fraction of the ride can passengers get to see a nice view? (2 marks)

Let us solve $h(t) = 2$ over the first cycle.

We get $t = \frac{4}{3}, 4$. **[1M]**

Then by the shape, we have the portion below as

$$\frac{4 - 4/3}{8} = \frac{1}{3}$$

so the portion above is $\frac{2}{3}$. **[1A]**

ii. At what times is the carriage at exactly 2 m and travelling upwards? (2 marks)

From above and the shape of the graph we have the first time is $t = 4$. **[1M]**

The period is 8 minutes. So all values are

$$t = 4 + 8n, \quad n \in \mathbb{Z}^+ \cup \{0\} \quad \textbf{[1A]}$$

iii. At what times is the carriage at exactly 2 m and travelling downwards? (2 marks)

First time occurs at $t = \frac{4}{3}$. **[1M]**

The period is 8 minutes. So all values are

$$t = \frac{4}{3} + 8n, \quad n \in \mathbb{Z}^+ \cup \{0\} \quad \textbf{[1A]}$$

g. To make the ride more exciting, the Ferris wheel has been reprogrammed to follow a new path such that the height can be modelled by:

$$h_{\text{new}}(t) = 2\cos\left(\frac{\pi t}{4} + \frac{\pi}{3}\right) - \sin\left(\frac{\pi t}{3} + \frac{\pi}{3}\right) + 4$$

What is the total time for one ride (one ride is one complete cycle) now? (2 marks)

Find LCM of periods:

$$\text{Period of cosine} = \frac{2\pi}{\pi/4} = 8, \quad \text{Period of sine} = \frac{2\pi}{\pi/3} = 6 \quad [\mathbf{1M}]$$

LCM of 8 and 6 is 24 minutes which is now the time of the ride. [**1A**]
Alternatively the answer could be found by just sketching the function on CAS and seeing when it starts to repeat.

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Section F: Extension Exam 1 (16 Marks)

INSTRUCTION:

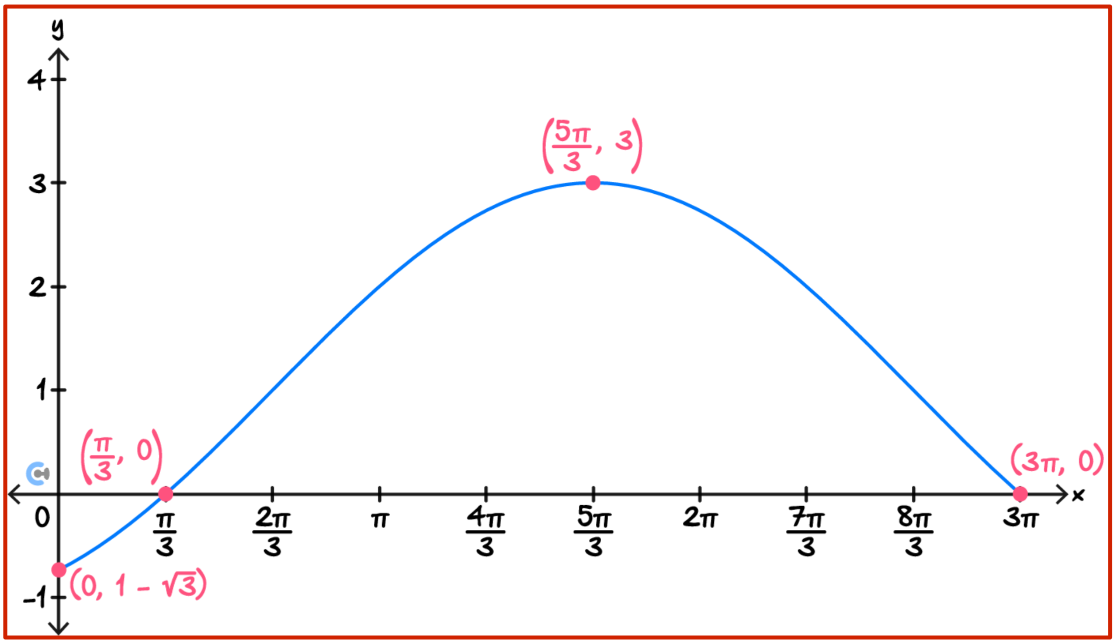
- Regular: Skip
- Extension: 16 Marks. 2 Minutes Reading. 20 Minutes Writing.

Question 18 (7 marks)

Consider the following function:

$$f(x) = 2\sin\left(\frac{x}{2} - \frac{\pi}{3}\right) + 1, x \in [0, 3\pi]$$

a. Sketch $f(x)$ on the set of axes below. Label all axes intercepts and turning points with their coordinates. (3 marks)



b. What fraction of $f(x)$ lies **above** the x-axis? (1 mark)

$\frac{3\pi - \pi/3}{3\pi} = \frac{8}{9}$ **(1A)**

c. Consider another function,

$$g: R \to R, g(x) = 2\sin\left(\frac{x}{2} - \frac{\pi}{3}\right) + 1.$$

What fraction of $g(x)$ lies **above** the x-axis? (1 mark)

$\frac{3\pi - \pi/3}{4\pi} = \frac{2}{3}$ **(1A)**

d. Now consider the function

$$h: R \to R, h(x) = g(x) + k$$

Find the value of k such that $\frac{1}{3}$ of $h(x)$ lies **above** the x-axis. (2 marks)

By symmetry we will have that $\frac{2}{3}$ of

$$y = 2\sin\left(\frac{x}{2} - \frac{\pi}{3}\right) - 1$$

lies **below** the x-axis. So $\frac{1}{3}$ will lie above. (**1M** any reasonable method that leads to answer)

Thus $k = -2$. **(1A)**

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Question 19 (9 marks)

Consider the function:

$$f(x) = 2\sin^2(x) - \cos(x) - 1$$

a. Solve the equation $f(x) = 0$. (3 marks)

We can write

$$f(x) = 2(1-\cos^2(x)) - \cos(x) - 1 = -2\cos^2(x) - \cos(x) + 1 \quad \textbf{[1M]}$$

Let $u = \cos(x)$, then:

$$\begin{aligned} -2u^2 - u + 1 &= 0 \\ 2u^2 + u - 1 &= 0 \\ (2u-1)(u+1) &= 0 \\ u &= \frac{1}{2}, -1 \end{aligned}$$

So $\cos(x) = \frac{1}{2}$ or $\cos(x) = -1$ **[1M]**

$$x = \frac{\pi}{3} \pm 2n\pi \quad \text{or} \quad x = (2n+1)\pi \quad n \in \mathbb{Z} \quad \textbf{[1A]}$$

b. Consider the function:

$$g: [-1, 1] \to \mathbb{R}, g(x) = -2x^2 - x + 1$$

i. Find the range of g. (2 marks)

Check values at endpoints:

$$\begin{aligned} g(-1) &= -2(1) + 1 + 1 = 0 \\ g(1) &= -2(1) - 1 + 1 = -2 \quad \textbf{[1M]} \end{aligned}$$

Now also note that

$$g(x) = -2\left(x - \frac{1}{4}\right)^2 + \frac{9}{8}$$

so there is a maximum at $\left(\frac{1}{4}, \frac{9}{8}\right)$. Therefore,

$$\text{ran } g = \left[-2, \frac{9}{8}\right] \quad \textbf{[1A]}$$

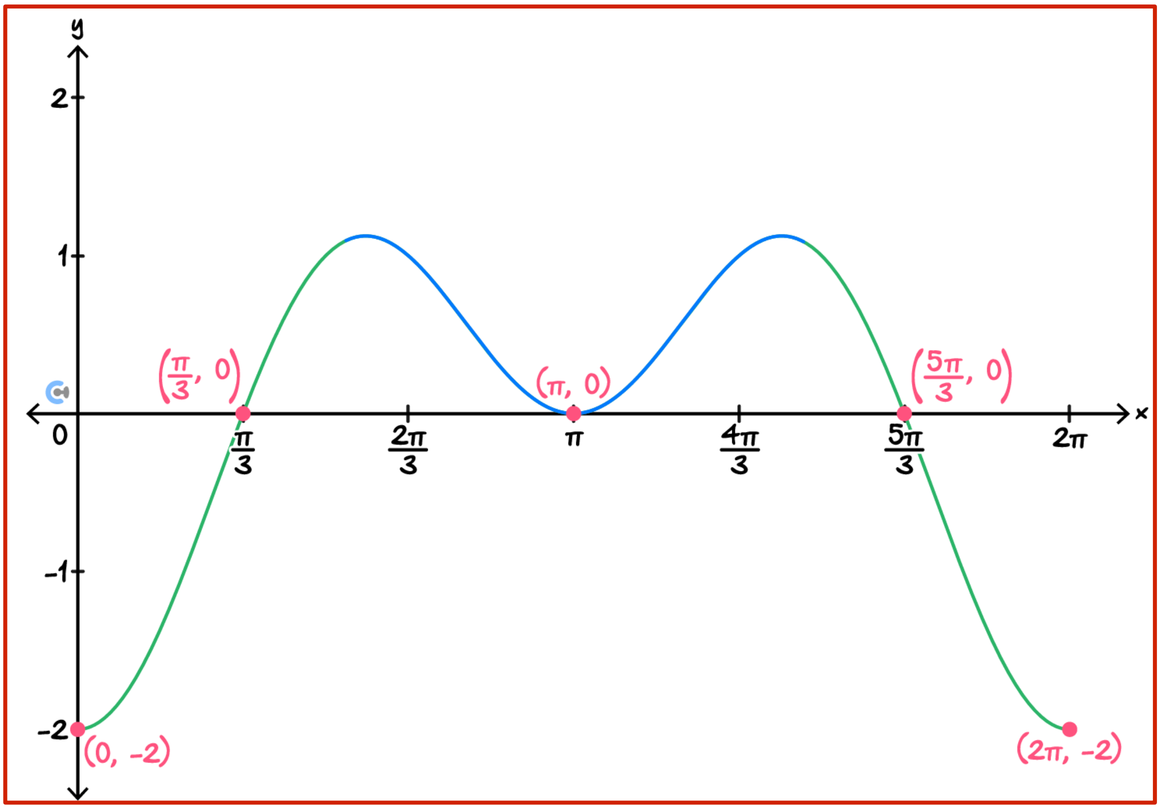
ii. Hence, state the range of f. (1 mark)

Note that $f(x) = g(\cos(x))$, and $\cos(x) \in [-1, 1]$
So range of f is the same as range of g over $[-1, 1]$:

$$\text{ran } f = \left[-2, \frac{9}{8}\right] \quad \textbf{[1A]}$$

c. Part of the graph of $y = f(x)$ is shown on the axes below.

Use your answers to the previous parts to sketch the rest of the graph of $y = f(x)$ for $x \in [0, 2\pi]$. Label any axes intercepts and endpoints with coordinates. (3 marks)



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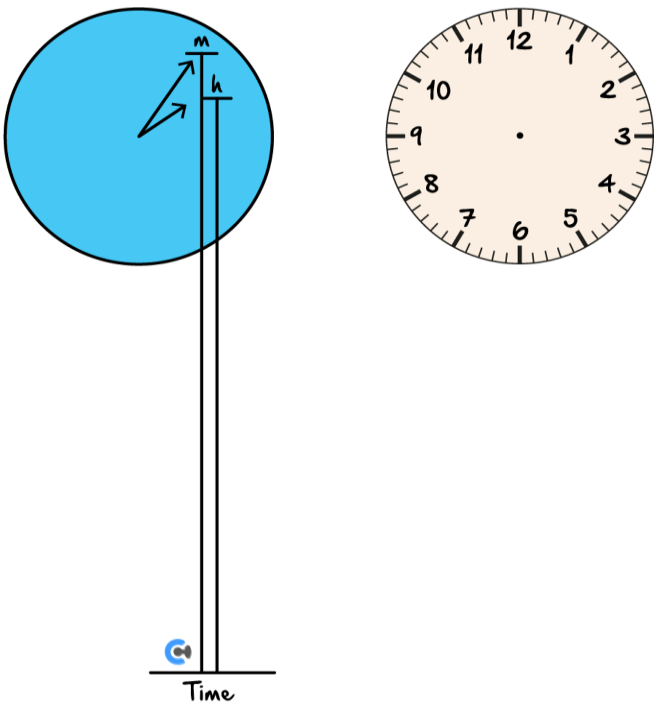
Section G: Extension Exam 2 (15 Marks)

INSTRUCTION:

- Regular: Skip
- Extension: 15 Marks. 2 Minutes Reading. 18 Minutes Writing.

Question 20 (15 marks)



The height of the tip of a minute hand of a clock on the wall is modelled by the following equation:

$$m(t) = 2.3 + 0.4 \sin\left(\frac{\pi}{6} - 120\pi t\right)$$

where $m(t)$ is the height t **hours** after a certain moment.

a. How far above the ground in the centre of the clock? (1 mark)

The centre of the circular motion is the vertical translation, which is:

$$2.3 \text{ m} \quad [1\text{A}]$$

b. How long does it take for the minute hand to go around the clock once? (1 mark)

$$\text{Period} = \frac{2\pi}{120\pi} = \frac{1}{60} \text{ hours } = 1 \text{ minute} \quad [\mathbf{1A}]$$

c.

i. What was the initial height of the minute hand? (1 mark)

$$m(0) = 2.3 + 0.4\sin\left(\frac{\pi}{6}\right) = 2.3 + 0.4 \cdot \frac{1}{2} = 2.3 + 0.2 = 2.5 \text{ m} \quad [\mathbf{1A}]$$

ii. Determine the initial trajectory. That is, is the minute hand moving down or up? (1 mark)

Since the sine function has a negative coefficient for t, the graph is decreasing initially.

Down [**1A**]

iii. After how many seconds is the minute hand first at the same height that it started at? (2 marks)

$$\text{Solve } m(t) = 2.5 \implies t = \frac{1}{90} \text{ of an hour} \quad [\mathbf{1M}]$$

$$\frac{1}{90} \times 60 = \frac{2}{3} \text{ minute} = 40 \text{ seconds.} \quad [\mathbf{1A}]$$

iv. Hence, determine the digit that the minute hand was pointing to when $t = 0$. (1 mark)

2 [**1A**]

d. For what fraction of the time is the minute hand above 2.5 m. (1 mark)

At 2.5m when the minute hand is at 2pm.
Above 2.5 when hand is between 10 and 2. Thus above for $\frac{4}{12} = \frac{1}{3}$ of the time **[1A]**

e. The height of the tip of the hour hand can be modelled by the following:

$$h(t) = 2.3 + 0.15 \sin\left(\frac{\pi}{6} - at\right)$$

where $h(t)$ is the height t **hours** after a certain moment.

i. What is the length of the hour hand? (1 mark)

The amplitude gives the length:

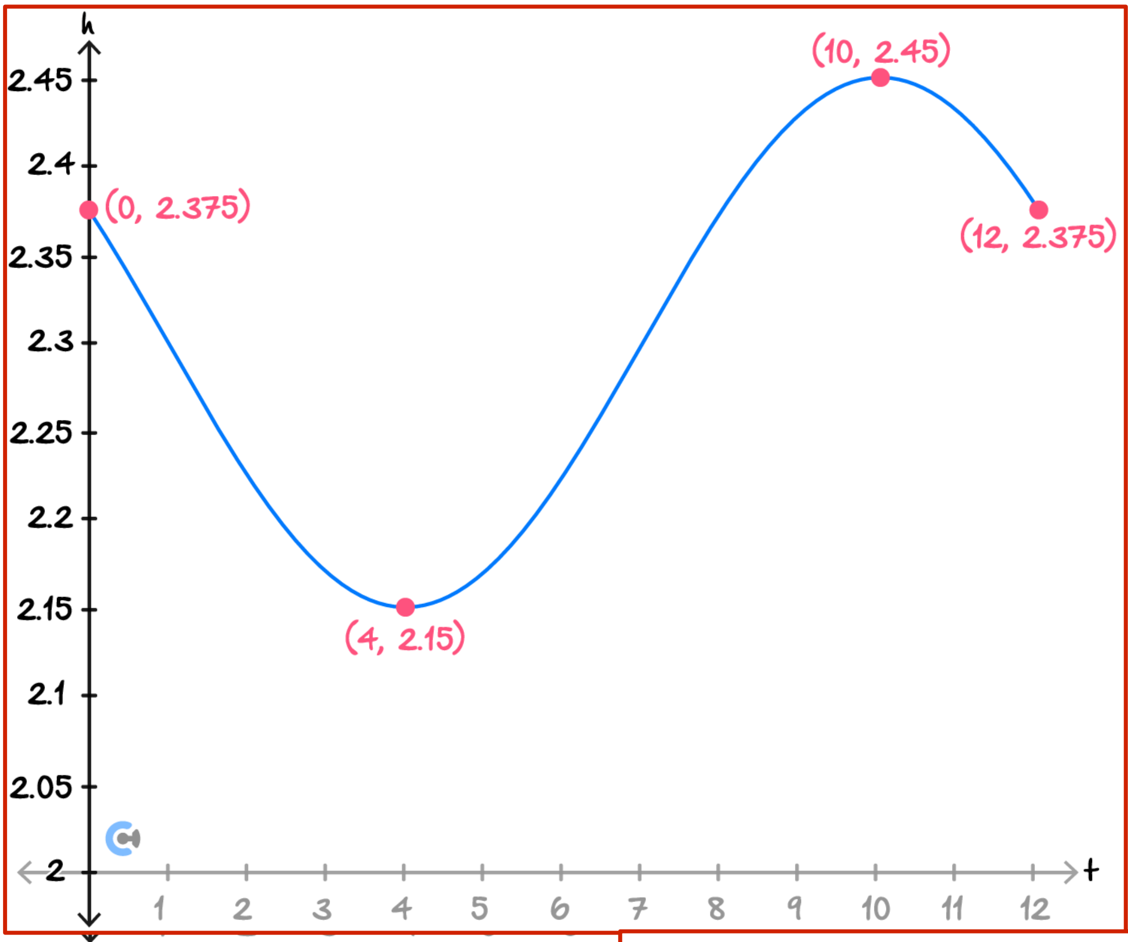
0.15 m **[1A]**

ii. Find the value of a. (1 mark)

Hour hand completes one cycle every 12 hours, so:

$\alpha = \frac{2\pi}{12} = \frac{\pi}{6}$ **[1A]**

f. On the set of axes below, draw one period of $h(t)$. Label all turning points and endpoints. (3 marks)



1M shape, 1M endpoints, 1M turning points.

g. Find the height H metres, such that the hour hand of the clock is **above** this value exactly 1 hour over a full cycle. Give your answer correct to three decimal places. (2 marks)

By symmetry the clock is highest from 11:30 AM to 12:30 PM. **(1M)**

Our clock starts at 2 PM when $t = 0$.

So, our desired height will be $h(9.5) = h(10.5) = 2.445$. **(1A)**

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