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VCE Mathematical Methods ½
Circular Function I [0.16]
Workshop Solutions

Error Logbook:

New Ideas/Concepts	Didn't Read Question
Pg / Q #: ______________________ Notes:	Pg / Q #: ______________________ Notes:
Algebraic/Arithmetic/ Calculator Input Mistake	**Working Out Not Detailed Enough**
Pg / Q #: ______________________ Notes:	Pg / Q #: ______________________ Notes:

Section A: Recap

Radians and Degrees

Definition

$$1^c = \left(\frac{180}{\pi}\right)^o$$

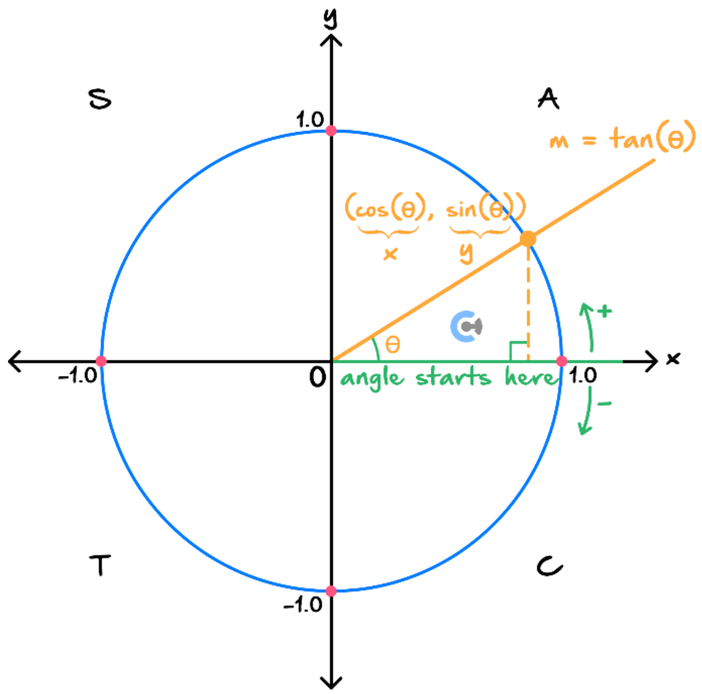
$$1^o = \left(\frac{\pi}{180}\right)^c$$

$$180^o = \pi^c$$

Unit Circle

Definition

- The unit circle is simply a circle of radius 1.



$$\sin(\theta) = y$$

$$\cos(\theta) = x$$

$$\tan(\theta) = gradient$$

Definition

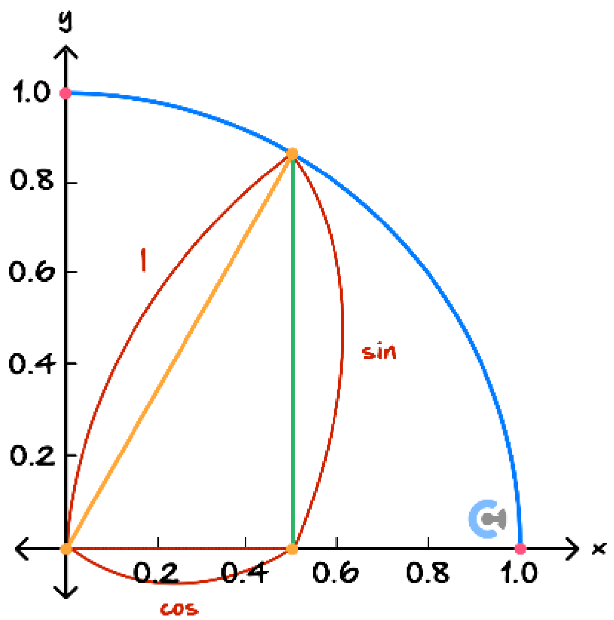
Period of a Trigonometric Function

$$\textit{period of } \sin(nx) \textit{ and } \cos(nx) \textit{ functions} = \frac{2\pi}{n}$$

$$\textit{period of } \tan(nx) \textit{ functions} = \frac{\pi}{n}$$

$$\textit{where } n = \textit{coefficient of } x \textit{ and } n > 0$$

Pythagorean Identities



$$\sin^2(\theta) + \cos^2(\theta) = 1$$

- Can be used for finding one trigonometry function by using the other.

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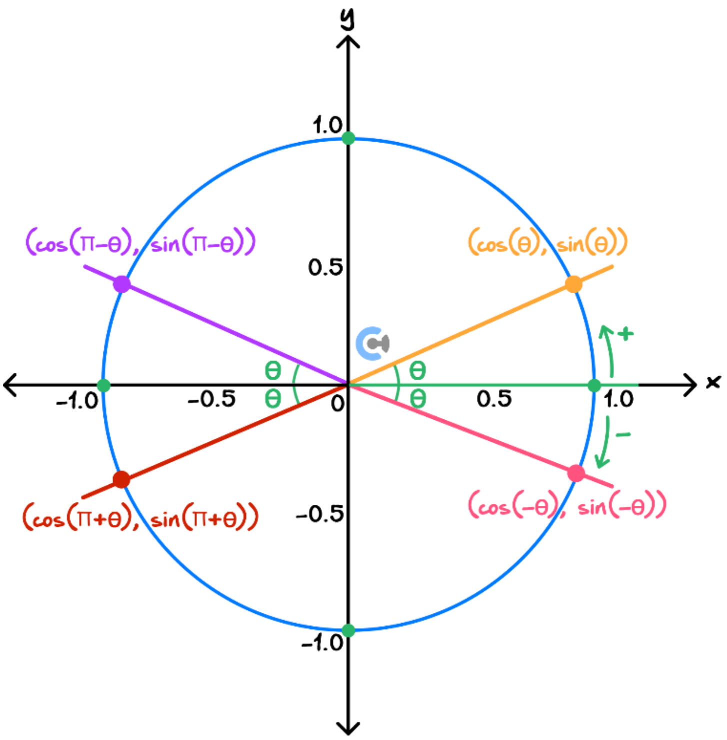
The Exact Values Table

x	$0\ (0^{o})$	$\frac{\pi}{6}\ (30^{o})$	$\frac{\pi}{4}\ (45^{o})$	$\frac{\pi}{3}\ (60^{o})$	$\frac{\pi}{2}\ (90^{o})$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

Supplementary Relationships



- Simply look at the quadrant to find the correct sign.
  - Second Quadrant $(\pi - \theta)$

$$\cos(\pi - \theta) = -\cos(\theta)$$

$$\sin(\pi - \theta) = +\sin(\theta)$$

$$\tan(\pi - \theta) = -\tan(\theta)$$

- Third Quadrant $(\pi + \theta)$

$$\cos(\pi + \theta) = -\cos(\theta)$$

$$\sin(\pi + \theta) = -\sin(\theta)$$

$$\tan(\pi + \theta) = +\tan(\theta)$$

- Fourth Quadrant $(-\theta)$

$$\cos(-\theta) = +\cos(\theta)$$

$$\sin(-\theta) = -\sin(\theta)$$

$$\tan(-\theta) = -\tan(\theta)$$

Definition

<u>Complementary Relationships</u>

- Consider the quadrant for signs.

  - First Quadrant $\left(\frac{\pi}{2} - \theta\right)$

$$\cos\left(\frac{\pi}{2} - \theta\right) = +\sin(\theta)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = +\cos(\theta)$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = +\frac{1}{\tan(\theta)}$$

  - Second Quadrant $\left(\frac{\pi}{2} + \theta\right)$

$$\sin\left(\frac{\pi}{2} + \theta\right) = +\cos(\theta)$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$$

$$\tan\left(\frac{\pi}{2}+\theta\right) = -\frac{1}{\tan(\theta)}$$

- Third Quadrant $\left(\frac{3\pi}{2}-\theta\right)$

$$\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos(\theta)$$

$$\cos\left(\frac{3\pi}{2}-\theta\right) = -\sin(\theta)$$

$$\tan\left(\frac{3\pi}{2}-\theta\right) = \frac{1}{\tan(\theta)}$$

- Fourth Quadrant $\left(\frac{3\pi}{2}+\theta\right)$

$$\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos(\theta)$$

$$\cos\left(\frac{3\pi}{2}+\theta\right) = +\sin(\theta)$$

$$\tan\left(\frac{3\pi}{2}+\theta\right) = -\frac{1}{\tan(\theta)}$$

Steps:

1. Note complementary relationship by identifying a vertical angle $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
2. Equate to the opposite trigonometric function $\cos$ /$\sin$ / $\frac{1}{\tan(\theta)}$.
3. Determine the sign $(\pm)$ by considering the quadrant.

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Definition

Supplementary v/s Complementary

Supplementary: $\boldsymbol{trig(Horizontal\ Angle \pm \theta)}$

Complementary: $\boldsymbol{trig(Vertical\ Angle \pm \theta)}$

Definition

Particular Solutions

- Solving trigonometric equations **for finite solutions.**
- **Steps:**

1. Make the trigonometric function the subject.
2. Find the necessary angle for one period.
3. Solve for x by equating the necessary angles to the inside of the trigonometric functions.
4. Add and subtract the period to find all other solutions in the domain.

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Section B: Warm Up (13 Marks)

INSTRUCTION:

- Regular: 13 Marks. 13 Minutes Writing.
- Extension: Skip.

Question 1 (7 marks)

a. Find $\left(\frac{3\pi}{4}\right)^c$ in degrees. (1 mark)

$135°$

b. Find $150°$ in radians. (1 mark)

$\frac{5\pi}{6}$

c. Determine the period of $\sin(3x)$. (1 mark)

$\frac{2\pi}{3}$

d. Determine the period of $\tan\left(\frac{\pi x}{3}\right)$. (1 mark)

3

e. Evaluate $\sin\left(\frac{3\pi}{2}\right)$. (1 mark)

-1

f. Evaluate $\cos\left(-\frac{\pi}{6}\right)$. (1 mark)

$\frac{\sqrt{3}}{2}$

g. Evaluate $\tan\left(\frac{7\pi}{4}\right)$. (1 mark)

-1

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Question 2 (6 marks)

Given that $\sin(x) = \frac{5}{13}$ and $\frac{\pi}{2} < x < \pi$, find:

a. $\cos(x)$. (2 marks)

$$-\frac{12}{13}$$

b. $\tan(x)$. (1 mark)

$$-\frac{5}{12}$$

c. $\cos\left(x + \frac{\pi}{2}\right)$. (1 mark)

$$-\frac{5}{13}$$

d. $\sin(2\pi - x)$. (1 mark)

$$-\frac{5}{13}$$

e. $\tan\left(\frac{\pi}{2}+x\right)$. (1 mark)

$$\frac{12}{5}$$

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Section C: Exam 1 Questions (17 Marks)

INSTRUCTION:

- Regular: 17 Marks. 5 Minutes Reading. 25 Minutes Writing.
- Extension: 17 Marks. 5 Minutes Reading. 17 Minutes Writing.

Question 3 (5 marks)

a. What value(s) can $\cos(x)$ take given that $\sin(x) = \frac{3}{5}$? (3 marks)

Pythagorean identity: $\sin^2(x) + \cos^2(x) = 1$. (1M)

Thus $\left(\frac{3}{5}\right)^2 + \cos^2(x) = 1 \implies \cos^2(x) = \frac{16}{25}$. (1M)

So $\cos(x) = \pm\frac{4}{5}$.

b. Hence, find the possible value(s) of $\tan(x)$. (2 marks)

$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{3/5}{\pm 4/5}$. (1M).

Thus $\tan(x) = \pm\frac{3}{4}$.

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Question 4 (2 marks)

Given that the period of the function $\tan(n^2x)$ is 2. Find the value(s) of n.

Period is 2 so $\frac{\pi}{n^2} = 2$. (1M)

Thus $n^2 = \frac{\pi}{2} \implies n = \pm\sqrt{\frac{\pi}{2}} = \pm\frac{\sqrt{2\pi}}{2}$. (1A)

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Question 5 (7 marks)

Given that $\sin(\alpha) = m$, and $\cos(\beta) = 0.2$.

a. Find the value of $\cos\left(\frac{\pi}{2} - \alpha\right)$. (2 marks)

$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha) = m$. (1M for m in answer, 1M correct sign)

b. Find the value of $\sin\left(\frac{3\pi}{2} - \beta\right)$. (2 marks)

$\sin\left(\frac{3\pi}{2} - \beta\right) = -\cos(\beta) = -0.2$. (1M for 0.2 in answer, 1M for correct sign)

c. Find the value(s) of $\tan(\alpha)$. (3 marks)

By Pythagorean identity we have $\cos(\alpha) = \pm\sqrt{1-m^2}$. (1M)

Thus $\tan(\alpha) = \pm\frac{m}{\sqrt{1-m^2}}$. (1M, correct fraction, 1A also include the $\pm$).

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Question 6 (3 marks)

Consider the functions:

$$f(x) = \sin(nx) \quad \text{and} \quad g(x) = \cos(nx)$$

For what integer value of n will $f(x) = g(x)$ have exactly 6 solutions for, $x \in [0, 2\pi]$? Justify your answer.

Intersect if $\sin(nx) = \cos(nx) \implies \tan(nx) = 1$. (1M)
$\tan(nx)$ has period $\frac{\pi}{n}$. (1M)
Exactly 6 solutions in $[0, 2\pi]$, 2 solutions per period so three full periods.
Thus period $= \frac{2\pi}{3}$. So $n = 3$. (1A)

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Section D: Tech Active Exam Skills

Calculator Commands: Solving Trigonometric Functions

TI	Casio	Mathematica
solve($trig(..) = a, x$) | domain restriction | is under control equal.	solve($trig(..) = a, x$) | domain restriction | is under maths 3.	Solve[$trig[] == a$ && domain restriction, x]

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Section E: Exam 2 Questions (26 Marks)

INSTRUCTION:

- Regular: 26 Marks. 5 Minutes Reading. 35 Minutes Writing.
- Extension: 26 Marks. 5 Minutes Reading. 26 Minutes Writing.

Question 7 (1 mark)

$\frac{\pi}{2}$ radians in degrees is given by:

A. $30°$

B. $90°$

C. $15°$

D. $60°$

Question 8 (1 mark)

For what values of k will $\sin(x+k) = \sin(x)$?

A. $2n\pi, n \in Z$

B. π

C. 3π

D. $\frac{\pi}{2}$

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Question 9 (1 mark)

Given that $\sin(\alpha) = \frac{3}{5}$ and $\cos(\beta) = \frac{5}{13}$, with $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\beta \in \left(\frac{3\pi}{2}, 2\pi\right)$. Evaluvate,

$$\frac{\sin(\beta)}{\cos(\alpha)}$$

A. $\frac{13}{15}$

B. $-\frac{15}{13}$

C. $\frac{39}{25}$

D. $-\frac{25}{39}$

Question 10 (1 mark)

What are the coordinates of the unit circle in terms of x?

A. $\left(x, \sqrt{1-x^2}\right)$

B. $\left(x, \pm\sqrt{1-x^2}\right)$

C. $\left(\sqrt{1-x^2}, x\right)$

D. $\left(\sqrt{1-x^2}, -\sqrt{1-x^2}\right)$

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Question 11 (1 mark)

The following equation has no real solutions:

$$\sin(n^2x) = \frac{\sqrt{5}}{2}, \qquad 0 < x < 2\pi$$

Which of the following is the best explanation for why this is the case?

A. We are not given the value of n.

B. There are real solutions but they are not in the domain $x \in (0, 2\pi)$.

C. The range of the sine function is $[-1, 1]$ but $\frac{\sqrt{5}}{2} > 1$.

D. $\frac{\sqrt{5}}{2}$ is inside the domain $x \in (0, 2\pi)$.

Question 12 (1 mark)

What is the range of $y = \tan(x)$?

A. R

B. R^+

C. $[-1,1]$

D. $R \setminus \left\{\frac{n\pi}{2}\right\}$

Question 13 (1 mark)

Solve the following equation for the given domain:

$$\sqrt{3}\tan\left(x - \frac{\pi}{2}\right) = 1, \qquad x \in [0, \pi]$$

A. $\frac{\pi}{3}$

B. $\frac{\pi}{6}$

C. $\frac{5\pi}{6}$

D. $\frac{2\pi}{3}$

Question 14 (1 mark)

Why does $\tan(x)$ have a period of π?

A. It is asymptotic.

B. Its values repeat every π radians.

C. $\cos(x)$ and $\sin(x)$ has a period of π.

D. Its period is not π.

Question 15 (1 mark)

Which of the following equations is true?

A. $\sin\left(\frac{\pi}{2} - x\right) = -\cos(x)$

B. $\sin\left(\frac{3\pi}{2} - x\right) = \cos(x)$

C. $\cos\left(\frac{\pi}{2} + x\right) = -\sin(x)$

D. $\cos\left(\frac{\pi}{2} + x\right) = \sin(x)$

Question 16 (1 mark)

How many x-intercepts will $\sin(nx)$ have over $(0, n\pi]$?

A. $\frac{2\pi}{n}n$

B. n^2

C. n

D. $n^2 + 1$

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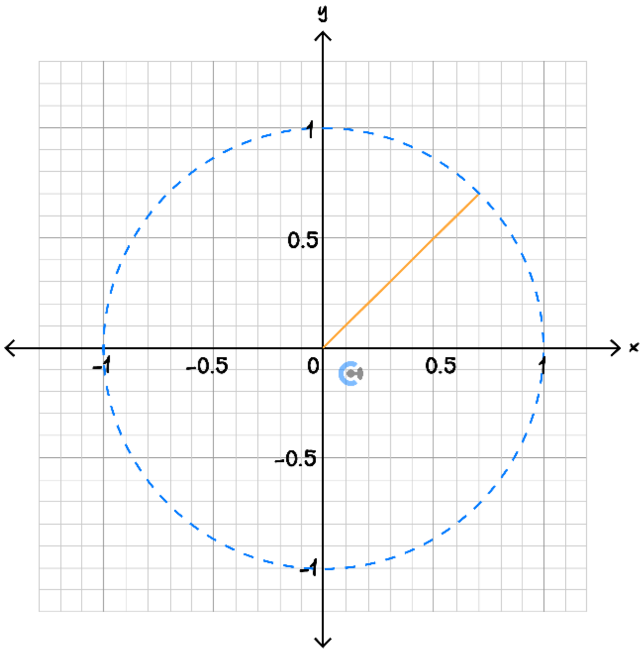
Question 17 (6 marks)

Consider the following unit circle:



a. If the line makes an angle of, θ, with the ***y*-axis**. Express the coordinates of the unit circle in terms of θ. (2 marks)

Makes an angle θ with the y-axis. Thus makes angle $\frac{\pi}{2} - \theta$ with the x-axis.
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$. (1M)
Thus coordinates are $(\sin(\theta), \cos(\theta))$. (1A)

b. Find the coordinates in terms of y, for $x > 0$. (2 marks)

$y = \cos(\theta)$ and $\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{1 - y^2}$. (1M)
Thus $\left(\sqrt{1 - y^2}, y\right)$. (1A)

c. Express $\tan(\theta)$ in terms of y, for $x > 0$. (2 marks)

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}. \text{ (1M)}$$

$$\text{Thus, } \tan(\theta) = \frac{\sqrt{1-y^2}}{y}. \text{ (1A)}$$

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Question 18 (10 marks)

The height of a point on the pump of an oil rig relative to the ground can be modelled using the following function:

$$f(t) = 2\sin(t) - \sqrt{2}, \text{ for } t \geq 0$$

where $y = 0$ is the ground level and t is measured in seconds.

a. How long does it take for the point to first return to its starting height? (1 mark)

It takes π seconds for the point to first return to its starting height. (1A)

b. What is the maximum, and minimum height of the point? (2 marks)
Hint: sin and cos can only be between -1 and 1.

Max $= 2 - \sqrt{2}$. (1A)
Min $= -2 - \sqrt{2}$. (1A)

c.

i. For what values of $t \in [0, 4\pi]$, will the point be level with the ground? (3 marks)

We must solve $f(t) = 0$.

Thus $\sin(t) = \frac{\sqrt{2}}{2}$. (1M)

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$. (1M any correct solution, 1A all four correct solutions).

ii. Hence, state the values of $t \in [0, 4\pi]$ for which the point is above the ground level. (2 marks)

We solve $f(t) > 0$.
$t \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{9\pi}{4}, \frac{11\pi}{4}\right)$. (1A correct intervals, 1A correct brackets used.)

d. The height of another point on the pump is modelled by $g(t) = \sin(t) - \sqrt{2}$ instead. Can this point reach the ground level? Justify. (2 marks)

The ground is at $y = 0$.
The range of $\sin(t)$ is $[-1, 1]$, thus the range of $g(t)$ is $[-1-\sqrt{2}, 1-\sqrt{2}]$. (1M)
Thus this point never reaches ground level. (1A)

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Section F: Extension Exam 1 (9 Marks)

INSTRUCTION:

- Regular: Skip.
- Extension: 9 Marks. 10 Minutes Writing.

Question 19 (9 marks)

Solve the following trigonometric equations, giving all solutions in the given domain.

a. Solve the equation $\sin(2x) = \frac{1}{2}$ for $0 \leq x \leq 2\pi$. (2 marks)

Let $2x = \theta$. Then solve $\sin(\theta) = \frac{1}{2} \implies \theta = \frac{\pi}{6}, \frac{5\pi}{6}$ (1M)

Then,

$$2x = \frac{\pi}{6}, \frac{5\pi}{6} \implies x = \frac{\pi}{12}, \frac{5\pi}{12}$$

Also, $2x \in [0, 4\pi]$, so repeat the angles in that interval:

$$\theta = \frac{13\pi}{6}, \frac{17\pi}{6} \implies x = \frac{13\pi}{12}, \frac{17\pi}{12}$$

Solutions: $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ (1A)

b. Solve the equation $\tan^2(x) = 3$ for $0 \leq x < 2\pi$. (2 marks)

Take square root:

$$\tan(x) = \pm\sqrt{3} \implies x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Solutions: $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$, (1M at least one correct solution, 1A all correct)

c. Solve the equation $\sin(x) = \cos(x)$ for $0 \leq x < 2\pi$. (2 marks)

Divide both sides: $\tan(x) = 1 \implies x = \frac{\pi}{4}, \frac{5\pi}{4}$ (1M, getting tan equation)

Solutions: $x = \frac{\pi}{4}, \frac{5\pi}{4}$ (1A)

d. Solve the equation $2\cos^2(x) - 3\cos(x) + 1 = 0$ for $-\pi \leq x \leq \pi$. (3 marks)

Let $y = \cos(x)$, then:

$$2y^2 - 3y + 1 = 0 \implies (2y-1)(y-1) = 0 \implies y = \frac{1}{2}, 1 \quad \text{(1M)}$$

So $\cos(x) = 1 \implies x = 0$

And $\cos(x) = \frac{1}{2} \implies x = -\frac{\pi}{3}, \frac{\pi}{3}$ (1M for two different cos equations)

Solutions: $x = 0, -\frac{\pi}{3}, \frac{\pi}{3}$ (1A)

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Section G: Extension Exam 2 (12 Marks)

INSTRUCTION:

- Regular: Skip.
- Extension: 12 Marks. 15 Minutes Writing.

Question 20 (5 marks)

Consider the following two functions:

$$g(x) = \sin(x) \text{ and } f(x) = \cos(x - k), 0 \leq x \leq 2\pi$$

a. For what value of k will $f(x)$ have three x-intercepts? For this value of k state the value(s) of x where $f(x)$ crosses the x-axis. Just provide one possible value for k. (2 marks)

$k = \frac{\pi}{2}$. (1M or any $k = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$)
$x = 0, \pi, 2\pi$. (1A)

b. Suppose $k \in [0, 2\pi]$. Provide a value of k for which $f(x) = g(x)$ has:

i. 3 solutions. (1 mark)

$k = \frac{3\pi}{2}$. (1A)

ii. 2 solutions. (1 mark)

$k = \pi$. (1A or any $k \in [0, 2\pi]$ except for $k = \frac{\pi}{2}, \frac{3\pi}{2}$)

iii. Infinitely many solutions. (1 mark)

$k = \frac{\pi}{2}$. (1A)

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Question 21 (7 marks)

The temperature $T(t)$ in degrees Celsius inside an office at time t hours after midnight is modelled by:

$$T(t) = 21 + 3\cos\left(\frac{\pi}{6}(t - 4)\right),$$

where $0 \leq t \leq 24$.

a. State the maximum and minimum temperatures in the office, and the times at which they occur. (2 marks)

The cosine function has range $-1 \leq \cos(\cdot) \leq 1$, so:

$$\text{Minimum: } T(t) = 21 - 3 = 18°\text{C}$$
$$\text{Maximum: } T(t) = 21 + 3 = 24°\text{C}$$

The cosine function reaches a maximum when its argument is 0, so:

$$\frac{\pi}{6}(t-4) = 0 \implies t = 4$$

The cosine reaches a minimum when its argument is π, so:

$$\frac{\pi}{6}(t-4) = \pi \implies t - 4 = 6 \implies t = 10$$

Now note that the function has two periods and period = 12. So,
Maximum temperature: 24°C at $t = 4, 16$ (4 am, 4pm) (1A)
Minimum temperature: 18°C at $t = 10, 22$ (10 am, 10pm) (1A)

b. Find the **exact** value of t for which the temperature is first 23°C. What time of day, to the nearest minute, does this t correspond to? (3 marks)

We solve:

$$21 + 3\cos\left(\frac{\pi}{6}(t-4)\right) = 23 \implies \cos\left(\frac{\pi}{6}(t-4)\right) = \frac{2}{3} \quad \text{(1M)}$$

Now solve:

$$\frac{\pi}{6}(t-4) = \pm\cos^{-1}\left(\frac{2}{3}\right) \implies t = 4 \pm \frac{6}{\pi}\cos^{-1}\left(\frac{2}{3}\right)$$

Earliest time: $t = 4 - \frac{6}{\pi}\cos^{-1}\left(\frac{2}{3}\right)$ (1A)
This is $t \approx 2.3968$ so at 2 : 24 am. (1A)

c. What fraction of the day is the temperature above 22.5°C? (A day starts at midnight and ends at midnight 24 hours later.) (2 marks)

Solve:

$$21 + 3\cos\left(\frac{\pi}{6}(t-4)\right) = 22.5 \implies \cos\left(\frac{\pi}{6}(t-4)\right) = \frac{1}{2}$$

$$\frac{\pi}{6}(t-4) = \pm\frac{\pi}{3} \implies t-4 = \pm 2 \implies t = 2 \text{ and } t = 6 \quad (1M)$$

So temperature is above 22.5°C between $t = 2$ and $t = 6$. Since the period is 12, it also occurs between $t = 14$ and $t = 18$.
Total duration $= (6-2) + (18-14) = 8$ hours

$$\text{Fraction of the day} = \frac{8}{24} = \frac{1}{3} \quad (1A)$$

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